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Limiting Distributions of Skewness and Kurtosis Statistics in Error Component Models

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Abstract:

This paper investigates the limiting distributions of the estimated skewness and kurtosis statistics of unobserved error components in error component models (ECM). These statistics are crucial for normality testing procedures, which use basically the asymptotic properties of higher order sample moments. Based on the foundational framework of developed by old scholars. We provide a detailed asymptotic framework in which the empirical third and fourth central moments, also with the associated variance estimators. The analysis establishes that, under mild regularity conditions, the skewness and kurtosis statistics converge to asymptotically normal distributions, with variances determined by the underlying fourth and eighth order population moments. These results provide a formal foundation for constructing asymptotically valid tests of normality for unobserved components in panel structured environments

Introduction:

The statistical properties of higher order moment measures such as skewness and kurtosis play a central role in estimating variations from normality in statistical models. In many empirical applications, particularly those involving multiple error terms, reliable inference requires a clear and comprehensive understanding of the limiting behavior of these statistics. Classical normality tests, including those based on the Jarque – Bera framework, depend basically on the asymptotic distributions of the empirical third and fourth central moments. While some results are well established for independent and identically distributed (i.i.d) data, considerably less attention has been given to the problem of deriving limiting distribution for skewness and kurtosis associated with multiple error components embedded in multi – indexed models. This gap is important when dealing with data that has different index or structures, because the asymptotic framework should be able to handle those complexities correctly.

Early work in the area of higher-moment behavior in structured error component models focused on testing for normality of the error components in a one-way ECM. Gilbert (2002) proposed four tests to assess the normality of the error components in a one-way ECM. Two of the tests are based on evaluating the skewness and kurtosis of the first error component, individual effect and the other two assess the corresponding moments of the remainder error term. The result showed that the four tests have chi-square asymptotic distributions. Also the tests show acceptable finite-sample performance even for relatively small sample sizes. However, a limitation of the work is that it is limited to evaluate the normality of only the individual and remainder error terms and only in a one-way setup. This prevent assessment of the normality of the time-specific disturbances, or moving beyond a one-way setup to two-way models. As a result, the asymptotic distribution theory developed by Gilbert (2002) is not general enough to characterize non-normality in the presence of other sources of unobserved heterogeneity.

Additional work was made by Bai and Ng (2005) who studied the sampling distributions of the skewness, kurtosis, and joint normality statistics for dependent time-series observations. Their results stressed that serial correlation have a fixed impact on the limiting distribution of the empirical third- and four-moment estimators. In particular, they showed that consistent estimation of long run covariance matrices is needed for valid inference on symmetry or tail skewness. Monte Carlo results showed that tests for skewness and joint normality preserve good finite sample properties, while tests for kurtosis alone suffer from size distortions unless in thin tailed distribution shape. Although the Bai and Ng framework is useful, it is not suitable for layered random component settings and it thus does not yield the limiting distributions of skewness and kurtosis associated with error terms beyond simple time-series dependence.

A different approach was introduced by Meintanis (2011), who studied the properties of traditional goodness of fit tests (the Cramér–von Mises, Anderson–Darling, and Kolmogorov–Smirnov statistics) in a simulation framework where the data were structured. The author provided some support that these tests retained some desirable properties in certain settings, but the proposed methodology did not apply to multi - level error structures, and they did not derive analytic forms for the limiting skewness and kurtosis distributions for the error components. In addition, their results could not be extended to two-way or higher-dimensional models, and thus have limited relevance for an analysis of higher-order moments of the statistics of errors.

A major step toward filling this methodological gap was taken by Galvão et al. (2013), who proposed a comprehensive set of six tests for skewness, kurtosis, and joint normality of the error components in the one-way error structure. Tests were applied following model estimation with pooled estimators and resampling techniques were used to enhance finite sample inference. Galvão et al. (2013) showed that tests could be conducted on the skewness and kurtosis of the individual effect error and remainder error components separately or jointly, and that the simulation study results proved that the methodology had very good empirical performance. Nonetheless, the methodology still considered only a single source of unobserved heterogeneity. Furthermore, the derivation of the asymptotic distributions of the tests relied on the normalizing properties of the higher-moment estimators.

The objective of this paper is to provide a comprehensive review of the asymptotic distributions of normality tests in one-way error component models, with particular emphasis on the framework developed by Galvão et al. (2013). We systematically reconstruct the limiting distributions of the empirical skewness and kurtosis statistics for the unobserved components, presenting all derivations, expansions, and asymptotic arguments in full detail. Using Taylor-series methods, Lagrange-type remainders, and tools such as Slutsky's theorem and the Delta Method, the exposition clarifies how higher-order population moments enter the asymptotic variance expressions. By laying out each step explicitly, the paper serves as a detailed reference for the theoretical foundation underlying Galvão's normality tests.

Skewness and kurtosis in the one-way error component model

The model:

Consider the following Error Component Specification for a balanced panel with individuals $i = 1, \dots, N$, $t = 1, \dots, T$

$$y_{it} = \alpha + \sum_{k=1}^{p} \beta_k \, x_{kit} + \varepsilon_{it}$$

where the composite error term admits the decomposition

$$\varepsilon_{it} = \eta_i + \nu_{it}$$

Where, y_{it} denotes the i^{th} observation for t^{th} time period of the dependent variable. x_{kit} is the i^{th} observation for t^{th} time period of the independent variable k, β_k is a k vector of parameters and α is a scalar intercept. The composite error term ε_{it} is decomposed into an individual effect η_i component and a remainder error component ν_{it} both of which have mean zero. The standardized third moment quantities of interest for skewness,

$SK_\eta = \frac{\eta_3}{{\sigma_\eta}^3} = \frac{E[\eta^3]}{(E[\eta^2])^{\frac{3}{2}}}$ and $SK_\nu = \frac{\nu_3}{{\sigma_\nu}^3} = \frac{E[\nu^3]}{(E[\nu^2])^{\frac{3}{2}}}$

The standardized fourth moment quantities of interest for kurtosis,

$$KU_\eta = \frac{\eta_4}{\sigma_\eta^4} = \frac{E[\eta^4]}{[E(\eta^2)]^2} \text{ and } KU_\nu = \frac{\nu_4}{\sigma_\nu^4} = \frac{E[\nu^4]}{[E(\nu^2)]^2}$$

Following Galvão (2013), our analysis considers tests for skewness and kurtosis in the individual effect and remainder error components, assessed separately and jointly. Under a normally distributed error structure, the corresponding null and alternative hypotheses can be as follows:

$$H_0^{s_\eta}: s_\eta = 0 \qquad H_1^{s_\eta}: s_\eta \neq 0$$
$$H_0^{s_\nu}: s_\nu = 0 \qquad H_1^{s_\nu}: s_\nu \neq 0$$

And kurtosis

$$H_0^{k_\eta}: k_\eta = 3 \qquad H_1^{k_\eta}: k_\eta \neq 3$$
$$H_0^{k_\nu}: k_\nu = 3 \qquad H_1^{k_\nu}: k_\nu \neq 3$$

And jointly;

$$H_0^{s_\eta \& k_\eta}: s_\eta = 0 \qquad H_0^{s_\eta \& k_\eta}: k_\eta = 3$$
$$H_0^{s_\nu \& k_\nu}: s_\nu = 0 \qquad H_0^{s_\nu \& k_\nu}: k_\nu = 3$$

Skewness

To study skewness in each component of the one-way error structure, it is useful to express the relevant third order moments in terms of the between and within residuals. Following the between decomposition $\bar{\varepsilon}_i = \eta_i + \bar{\nu}_i$, consider the cube of it,

$$\bar{\varepsilon}_i^3 = \eta_i^3 + 3\eta_i^2\bar{\nu}_i + 3\eta_i\bar{\nu}_i^2 + \bar{\nu}_i^3$$

And after taking expectations and based on the following fact$\bar{\nu}_i = T^{-1}\sum_{t=1}^{T}\nu_{it}$

We obtain,

$$E[\bar{\varepsilon}_t^3] = \eta_3 + \frac{1}{T^2}\nu_3$$

Where, $\eta_3 = E[\eta^3]$ and $\nu_3 = E[\nu^3]$

A similar expansion for the within residuals $\tilde{\varepsilon}_{it} = \nu_{it} - \bar{\nu}_i$ yields

$$\tilde{\varepsilon}_{it}^3 = \nu_{it}^3 - \bar{\nu}_i^3 - 3\nu_{it}^2\bar{\nu}_i + 3\nu_{it}\bar{\nu}_i^2$$

And taking the expectations gives

$$E\left[\tilde{\varepsilon}_{it}^3\right] = \nu_3\left(1 - \frac{1}{T^2} - \frac{3}{T} + \frac{3}{T^2}\right)$$

These two equations can be solved for the unknown third moments of the remainder and individual components, leading to

$$\nu_3 = \frac{E\left[\tilde{\varepsilon}_{it}^3\right]}{\left(1 - \frac{1}{T^2} - \frac{3}{T} + \frac{3}{T^2}\right)} \qquad \eta_3 = E\left[\bar{\varepsilon}_i^3\right] - \frac{E\left[\tilde{\varepsilon}_{it}^3\right]}{(T^2 - 3T + 2)}$$

For practical implementation, we replace the population moments by their sample counterparts. Since,

$$\tilde{\varepsilon}_{it}^3 = \varepsilon_{it}^3 - \bar{\varepsilon}_i^3 - 3\varepsilon_{it}^2\bar{\varepsilon}_i + 3\varepsilon_{it}\bar{\varepsilon}_i^2$$

Then,

$$E\left[\tilde{\varepsilon}_{it}^3\right] = E\left[\overline{\varepsilon_{it}^3} - 3\bar{\varepsilon}_i\overline{\varepsilon_{it}^2} + 2\bar{\varepsilon}_i^3\right]$$

$$\nu_3 = \frac{E\left[\overline{\varepsilon_{it}^3} - 3\bar{\varepsilon}_i\overline{\varepsilon_{it}^2} + 2\bar{\varepsilon}_i^3\right]}{\left(1 - \frac{1}{T^2} - \frac{3}{T} + \frac{3}{T^2}\right)} \qquad \eta_3 = E\left[\bar{\varepsilon}_i^3\right] - \frac{E\left[\overline{\varepsilon_{it}^3} - 3\bar{\varepsilon}_i\overline{\varepsilon_{it}^2} + 2\bar{\varepsilon}_i^3\right]}{(T^2 - 3T + 2)}$$

Because the true disturbances are unobserved, the above expressions are evaluated using the pooled OLS residuals

$$\hat{\varepsilon}_{it} = y_{it} - \hat{\alpha} - x_{it}'\hat{\beta}$$

The corresponding estimators of the third moments are therefore

$$\hat{\nu}_3 = \frac{E\left[\overline{\hat{\varepsilon}_{it}^3} - 3\bar{\hat{\varepsilon}}_i\overline{\hat{\varepsilon}_{it}^2} + 2\bar{\hat{\varepsilon}}_i^3\right]}{\left(1 - \frac{1}{T^2} - \frac{3}{T} + \frac{3}{T^2}\right)} \qquad \hat{\eta}_3 = \frac{(T^2 - 3T)E\left[\bar{\hat{\varepsilon}}_i^3\right]}{T^2 - 3T + 2} - \frac{E\left[\overline{\hat{\varepsilon}_i^3} - 3\overline{\hat{\varepsilon}_i^2}\,\bar{\hat{\varepsilon}}_i\right]}{T^2 - 3T + 2}$$

With these components in hand, the sample skewness measures for the remainder and individual effects are defined as

$$\widehat{SK}_\nu = \frac{\hat{\nu}_3}{\hat{\sigma}_\nu^3} \qquad\qquad \widehat{SK}_\eta = \frac{\hat{\eta}_3}{\hat{\sigma}_\eta^3}$$

Kurtosis

To obtain statistics suitable for testing kurtosis in each unobserved component, we begin by examining the fourth power of the between residual,

$$\bar{\varepsilon}_i^4 = (\eta_i + \bar{\nu}_i)^4 = {\eta_i}^4 + 4{\eta_i}^3\bar{\nu}_i + 6{\eta_i}^2{\bar{\nu}_i}^2 + 4\eta_i{\bar{\nu}_i}^3 + {\bar{\nu}_i}^4$$

Taking expectations leads to

$$E[\bar{\varepsilon}_i^4] = \eta_4 + \frac{6}{T}\sigma_\eta^2\sigma_\nu^2 + \frac{1}{T^3}[\nu_4 + 6(T-1)(\sigma_\nu^4)]$$

Similarly, consider the fourth power of the within residual,

$$\tilde{\varepsilon}_{it}^4 = \nu_{it}^4 - 4\nu_{it}^3\bar{\nu}_i + 6\nu_{it}^2\bar{\nu}_i^2 - 4\nu_{it}{\bar{\nu}_i}^3 + {\bar{\nu}_i}^4$$

Taking expectations yields

$$E[\tilde{\varepsilon}_{it}^4] = \nu_4\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{4}{T^3} + \frac{1}{T^3}\right) + \sigma_\nu^4(T-1)\left[\frac{6}{T^2} + \frac{6}{T^3}\right]$$

we may solve the system for ν_4 and η_4. After rearrangement, this gives

$$\eta_4 = E[\bar{\varepsilon}_i^4] - \frac{6}{T}\sigma_\eta^2\sigma_\nu^2 - \frac{1}{T^3}[\nu_4 + 6(T-1)(\sigma_\nu^4)]$$

$$\nu_4 = \frac{E[\tilde{\varepsilon}_{it}^4]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{4}{T^3} + \frac{1}{T^3}\right)} - \frac{\sigma_\nu^4(T-1)\left[\frac{6}{T^2} + \frac{6}{T^3}\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{4}{T^3} + \frac{1}{T^3}\right)}$$

Equivalently, the above expression can be written in the normalized form

$$\eta_4 = E[\bar{\varepsilon}_i^4] - \frac{6}{T}\sigma_\eta^2\sigma_\nu^2 - \frac{1}{T^3}\left[\begin{array}{c}\dfrac{E\left[\overline{\varepsilon_i^4} - 4\bar{\varepsilon}_i\overline{\varepsilon_i^3} + 6\bar{\varepsilon}_i^2\overline{\varepsilon_i^2} - 3\bar{\varepsilon}_i^4\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)} - \dfrac{\sigma_\nu^4(T-1)\left[\frac{6}{T^2} + \frac{6}{T^3}\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)} \\ +3(T-1)(\sigma_\nu^4)\end{array}\right]$$

$$\nu_4 = \frac{E\left[\overline{\varepsilon_i^4} - 4\bar{\varepsilon}_i\overline{\varepsilon_i^3} + 6\bar{\varepsilon}_i^2\overline{\varepsilon_i^2} - 3\bar{\varepsilon}_i^4\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)} - \frac{\sigma_\nu^4(T-1)\left[\frac{6}{T^2} + \frac{6}{T^3}\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)}$$

As before, these expressions depend on population errors and must therefore be estimated using the OLS residuals $\hat{\varepsilon}_{it}$. Defining $\hat{\nu}_4$ and $\hat{\eta}_4$ as sample analogues to ν_4 and η_4, we obtain

$$\hat{\eta}_4 = E\left[\bar{\hat{\varepsilon}}_i^4\right] - \frac{6}{T}\hat{\sigma}_\eta^2\hat{\sigma}_\nu^2 - \frac{1}{T^3}\left[\begin{array}{c}\dfrac{E\left[\overline{\hat{\varepsilon}_i^4} - 4\bar{\hat{\varepsilon}}_i\overline{\hat{\varepsilon}_i^3} + 6\bar{\hat{\varepsilon}}_i^2\overline{\hat{\varepsilon}_i^2} - 3\bar{\hat{\varepsilon}}_i^4\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)} - \dfrac{\hat{\sigma}_\nu^4(T-1)\left[\frac{6}{T^2} + \frac{6}{T^3}\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)} \\ +3(T-1)(\hat{\sigma}_\nu^4)\end{array}\right]$$

$$\hat{\nu}_4 = \frac{E\left[\overline{\hat{\varepsilon}_i^4} - 4\bar{\hat{\varepsilon}}_i\overline{\hat{\varepsilon}_i^3} + 6\bar{\hat{\varepsilon}}_i^2\overline{\hat{\varepsilon}_i^2} - 3\bar{\hat{\varepsilon}}_i^4\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)} - \frac{\hat{\sigma}_\nu^4(T-1)\left[\frac{6}{T^2} + \frac{6}{T^3}\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)}$$

Finally, the standardized kurtosis statistics for the remainder and individual components are given by

$$\widehat{KU}_\nu = \frac{\hat{\nu}_4}{\hat{\sigma}_\nu^4} \qquad \widehat{KU}_\eta = \frac{\hat{\eta}_4}{\hat{\sigma}_\eta^4}$$

Asymptotic theory

To derive the limiting distributions of the empirical skewness and kurtosis measures for both the individual and remainder components, we rely on a set of standard assumptions concerning the sampling structure, the higher-order moments of the disturbances, and the behavior of the independent variable.

These conditions guarantee the existence of the necessary moments and permit the application of multivariate central limit arguments as $N \to \infty$ (with fixed T)

Assumption 1. The panel consists of N independent cross – sectional units observed over T periods. For each i, the sequence $\{(x_{it}, \nu_{it}): t = 1, \dots T\}$ is identically distributed across t. Moreover, the individual effect $\{\eta_i\}_{i=1}^{N}$ are $i.i.d$ across i and independent of $\{x_{it}, \nu_{it}\}$ for all t. In addition to that, the remainder error component $\{\nu_{it}\}$ are independent across i and serially uncorrelated across t

Assumption 2. The individual and remainder components satisfy

$$E[\eta_i] = 0 \qquad\qquad E[\nu_{it}] = 0$$

Also, Higher-order moments exist and are finite:

$$E[|\eta_i|^8] < \infty \qquad\qquad E[|\nu_{it}|^8] < \infty$$

The variance $\sigma_\eta^2 = E[\eta_i^2]$ and $\sigma_\nu^2 = E[\nu_{it}^2]$ are strictly positive.

These conditions ensure that population skewness and kurtosis of both components are well defined and that sample moments satisfy a law of large numbers and a central limit theorem.

Assumption 3. The independent variable satisfies $E[\|x_{it}\|^4] < \infty$ and the second matrix $\Sigma_x = E[x_{it}x'_{it}]$ is finite and positive definite

These conditions guarantee that the pooled OLS residuals admit the usual $\sqrt{N}$ asymptotic linearization under fixed T.

Theorem 1: under assumption 1,2 and 3 and for $N \to \infty$ (with fixed T)

i. $\sqrt{N}\left(\widehat{SK}_\nu - S_\nu\right) = \sqrt{N}\left(\frac{\hat{\nu}_3}{\hat{\sigma}_\nu^3} - \frac{\nu_3}{\sigma_\nu^3}\right) = \left(\frac{\sqrt{N}(\hat{\nu}_3-\nu_3)}{\hat{\sigma}_\nu^3} - \frac{3S_\nu\hat{\sigma}_\nu\sqrt{N}(\hat{\sigma}_\nu^2-\sigma_\nu^2)}{2\hat{\sigma}_\nu^3}\right)$

ii. $\sqrt{N}\left(\widehat{SK}_\eta - S_\eta\right) = \sqrt{N}\left(\frac{\hat{\eta}_3}{\hat{\sigma}_\eta^3} - \frac{\eta_3}{\sigma_\eta^3}\right) = \left(\frac{\sqrt{N}(\hat{\eta}_3-\eta_3)}{\hat{\sigma}_\eta^3} - \frac{3S_\eta\hat{\sigma}_\eta\sqrt{N}(\hat{\sigma}_\eta^2-\sigma_\eta^2)}{2\hat{\sigma}_\eta^3}\right)$

iii. $\sqrt{N}\left(\widehat{KU}_\nu - k_\nu\right) = \sqrt{N}\left(\frac{\hat{\nu}_4}{\hat{\sigma}_\nu^4} - \frac{\nu_4}{\sigma_\nu^4}\right) = \left(\frac{\sqrt{N}(\hat{\nu}_4-\nu_4)}{\hat{\sigma}_\nu^4} - \frac{3S_\nu\hat{\sigma}_\nu\sqrt{N}(\hat{\sigma}_\nu^2-\sigma_\nu^2)}{2\hat{\sigma}_\nu^4}\right)$

iv. $\sqrt{N}\left(\widehat{KU}_\eta - k_\eta\right) = \sqrt{N}\left(\frac{\hat{\eta}_4}{\hat{\sigma}_\eta^4} - \frac{\eta_4}{\sigma_\eta^4}\right) = \left(\frac{\sqrt{N}(\hat{\eta}_4-\eta_4)}{\hat{\sigma}_\eta^4} - \frac{3S_\eta\hat{\sigma}_\eta\sqrt{N}(\hat{\sigma}_\eta^2-\sigma_\eta^2)}{2\hat{\sigma}_\eta^4}\right)$

Where,

$$\sqrt{N}(\hat{\nu}_3-\nu_3) = \sqrt{T}\left(\frac{E\left(\overline{\hat{\varepsilon}_i^3} - 3\overline{\hat{\varepsilon}_i^2}\ \bar{\hat{\varepsilon}}_i + 2\bar{\hat{\varepsilon}}_i^{\,3}\right)}{\left(1-\frac{1}{T^2}-\frac{3}{T}+\frac{3}{T^2}\right)} - \frac{E\left(\overline{\varepsilon_i^3} - 3\overline{\varepsilon_i^2}\ \bar{\varepsilon}_i + 2\bar{\varepsilon}_i^3\right)}{\left(1-\frac{1}{T^2}-\frac{3}{T}+\frac{3}{T^2}\right)}\right)$$

$$= \left[\frac{1}{\left(1-\frac{1}{T^2}-\frac{3}{T}+\frac{3}{T^2}\right)}\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\hat{\varepsilon}_i^3} - \sqrt{N}E\left[\overline{\varepsilon_i^3}\right]\right) - 3\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{\hat{\varepsilon}}_i\overline{\hat{\varepsilon}_i^2} - \sqrt{N}E\left[\overline{\varepsilon_i^2}\ \bar{\varepsilon}_i\right]\right) + 2\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{\hat{\varepsilon}}_i^{\,3} - \sqrt{N}E\left[\bar{\varepsilon}_i^3\right]\right)\right]\right]$$

$$\sqrt{N}(\hat{\eta}_3-\eta_3) = \sqrt{N}\left[\frac{(T^2-3T)E\left[\bar{\hat{\varepsilon}}_i^{\,3}\right]}{T^2-3T+2} - \frac{E\left[\overline{\hat{\varepsilon}_i^3} - 3\overline{\hat{\varepsilon}_i^2}\ \bar{\hat{\varepsilon}}_i\right]}{T^2-3T+2} - \left(\frac{(T^2-3T)E\left[\bar{\varepsilon}_t^3\right]}{T^2-3T+2} - \frac{E\left[\overline{\varepsilon_t^3} - 3\overline{\varepsilon_t^2}\ \bar{\varepsilon}_t\right]}{T^2-3T+2}\right)\right]$$

$$= \frac{(T^2 - 3T)}{T^2 - 3T + 2}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{\hat{\varepsilon}}_i{}^3 - \sqrt{N}E\left(\bar{\varepsilon}_i^3\right)\right] - \frac{1}{N^2 - 3N + 2}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\hat{\varepsilon}_i^3} - 3\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\hat{\varepsilon}_i^2}\,\bar{\hat{\varepsilon}}_i - \sqrt{N}E\left(\overline{\varepsilon_i^3}\right) + 3\sqrt{N}E\left(\overline{\varepsilon_i^2}\,\bar{\varepsilon}_i\right)\right]$$

$$\sqrt{N}(\hat{\eta}_4 - \eta_4) = \sqrt{N}\left(\left[\frac{T^3 - 4T^2 + 6T}{T^3 - 4T^2 + 6T - 3}E\left[\bar{\hat{\varepsilon}}_i^4\right] - \frac{6}{T}\hat{\sigma}_\eta^2\hat{\sigma}_\nu^2 - \frac{E\left[\overline{\hat{\varepsilon}_i^4} - 4\bar{\hat{\varepsilon}}_i\overline{\hat{\varepsilon}_i^3} + 6\bar{\hat{\varepsilon}}_i^2\overline{\hat{\varepsilon}_i^2}\right]}{(T^3 - 4T^2 + 6T - 3)} - \frac{\hat{\sigma}_\nu^4(T-1)(3T^3 - 12T^2 + 12T + 3)}{(T^3 - 4T^2 + 6T - 3)T^3}\right] - \left[\frac{T^3 - 4T^2 + 6T}{T^3 - 4T^2 + 6T - 3}E\left[\bar{\varepsilon}_i^4\right] - \frac{6}{T}\sigma_\eta^2\sigma_\nu^2 - \frac{E\left[\overline{\varepsilon_i^4} - 4\bar{\varepsilon}_i\overline{\varepsilon_i^3} + 6\bar{\varepsilon}_i^2\overline{\varepsilon_i^2}\right]}{(T^3 - 4T^2 + 6T - 3)} - \frac{\sigma_\nu^4(T-1)(3T^3 - 12T^2 + 12T + 3)}{(T^3 - 4T^2 + 6T - 3)T^3}\right]\right)$$

$$= \frac{T^3 - 4T^2 + 6T}{T^3 - 4T^2 + 6T - 3}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^4 - \sqrt{N}E[\bar{\varepsilon}_i^4]\right]$$

$$- \frac{1}{(T^3 - 4T^2 + 6T - 3)}\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i^4} - \sqrt{N}E\left[\overline{\varepsilon_i^4}\right]\right.\right.$$

$$\left.- 4E\left[\varepsilon_{it}^3\right]\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i}\right)$$

$$- 4\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i^3}\,\bar{\varepsilon}_i - \sqrt{N}E\left[\bar{\varepsilon}_i\overline{\varepsilon_i^3}\right]\right.$$

$$\left.- \sqrt{N}\bar{\varepsilon}E\left[3\overline{\varepsilon_i^2}\bar{\varepsilon}_i + \varepsilon_{it}^3\right] - \sqrt{N}E\left[\bar{\varepsilon}_i\overline{\varepsilon_i^3}\right]\right)$$

$$+ 6\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i^2}\,\bar{\varepsilon}_i{}^2 - \sqrt{N}E\left[\bar{\varepsilon}_i^2\overline{\varepsilon_i^2}\right]\right.$$

$$\left.\left.- 2\sqrt{N}\bar{\varepsilon}E\left[\bar{\varepsilon}_i^3 + \overline{\varepsilon_i^2}\bar{\varepsilon}_i\right]\right)\right] + o_p(1)$$

$$\sqrt{N}(\hat{\nu}_4 - \nu_4) = \sqrt{N}\left(\left[\frac{E\left[\overline{\hat{\varepsilon}_i^4}\right] - 4E\left[\bar{\hat{\varepsilon}}_i\overline{\hat{\varepsilon}_i^3}\right] + 6E\left[\bar{\hat{\varepsilon}}_i^2\overline{\hat{\varepsilon}_i^2}\right] - 3E\left[\bar{\hat{\varepsilon}}_i^4\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)}\right.\right.$$

$$\left.- \frac{\hat{\sigma}_\nu^4(T-1)\left[\frac{6}{T^2} + \frac{6}{T^3}\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)}\right]$$

$$- \left[\frac{E\left[\overline{\varepsilon_{it}^4}\right] - 4E\left[\bar{\varepsilon}_i\overline{\varepsilon_{it}^3}\right] + 6E\left[\bar{\varepsilon}_i^2\overline{\varepsilon_{it}^2}\right] - 3[\bar{\varepsilon}_i^4]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)}\right.$$

$$\left.\left.- \frac{\sigma_\nu^4(T-1)\left[\frac{6}{T^2} + \frac{6}{T^3}\right]}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)}\right]\right)$$

$$
\begin{aligned}
= \frac{1}{\left(1 - \frac{4}{T} + \frac{6}{T^2} - \frac{3}{T^3}\right)} \Bigg[\Bigg(& \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \overline{\varepsilon_\iota^4} - \sqrt{T} E\left[\overline{\varepsilon_{\iota t}^4}\right] \\
& - 4E\left[\varepsilon_{it}^3\right] \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \overline{\varepsilon_\iota} \Bigg) \\
& - 4\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \overline{\varepsilon_\iota^3}\, \bar{\varepsilon}_i - \sqrt{N} E\left[\bar{\varepsilon}_i \overline{\varepsilon_{\iota t}^3}\right]\right. \\
& \left. - \sqrt{N} \bar{\varepsilon} E\left[3 \overline{\varepsilon_\iota^2} \bar{\varepsilon}_i + \varepsilon_{it}^3\right]\right) \\
& + 6\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \overline{\varepsilon_\iota^2}\, \bar{\varepsilon}_i{}^2 - \sqrt{N} E\left[\bar{\varepsilon}_i^2 \overline{\varepsilon_{\iota t}^2}\right]\right. \\
& \left. - 2\sqrt{N} \bar{u} E\left[\bar{\varepsilon}_i^3 + \overline{\varepsilon_\iota^2} \bar{\varepsilon}_i\right]\right) \\
& - 3\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\bar{\varepsilon}_i - \bar{\varepsilon})^4 - \sqrt{N}\left[\bar{\varepsilon}_i^4\right]\right) \Bigg]
\end{aligned}
$$

Proof. See Appendix A.

Conclusion

This paper reviews the theoretical foundations necessary for moment-based diagnostic testing in error-component models, offering a rigorous asymptotic framework for skewness and kurtosis statistics associated with unobserved components. By deriving the limiting behavior of the empirical third and fourth central moments, and the corresponding variance estimators this study establishes a mathematically coherent basis for conducting asymptotically valid tests of normality within general error-component structures.

The results not only support but also extend the existing literature on moment-based tests in panel data models, particularly the contributions of Galvão (2013). The asymptotic machinery developed here clarifies the distributional properties of skewness and kurtosis estimators for error components. Moreover, the theoretical insights provided by this analysis create a pathway for future methodological progress. They can support the construction of joint multivariate normality tests for error components, guide improvements in bootstrap-based inference, and facilitate extensions to more complex environments such as two-way and higher-order error-component models.

Appendix A

Lemma One. Under assumption 1,2 and 3, for $n = 2{,}3$ and 4 we have

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\hat{\varepsilon}_i^n} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i^n} - nE[\varepsilon_{it}^{n-1}]\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i} + o_p(1)$$

Proof. The proof is complete if the following equality holds.

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{i=1}^{N}(\varepsilon_{it} - \bar{\varepsilon}_i)^n = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{i=1}^{N}\hat{\varepsilon}_{it}^n - nE[\varepsilon_{it}^{n-1}]\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it} + o_p(1)$$

The proofs of the first and second equalities are modifications of the proofs of theorem 5 and Lemma A.1, respectively, of Bai and Ng to accommodate the one way error component model

Lemma Two. Under assumption 1,2 and 3, for $n = 2{,}3$ and 4 we have

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{\hat{\varepsilon}}_i^n = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{\varepsilon}_i^n - nE[\bar{\varepsilon}_i^{n-1}]\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{\varepsilon}_i + o_p(1)$$

Proof. if $\hat{\varepsilon}_{it} = (\varepsilon_{it} - \bar{\varepsilon}) - (x_{it} - \bar{x})'\left(\hat{\beta} - \beta\right)$and $\bar{\hat{\varepsilon}}_i = (\bar{\varepsilon}_i - \bar{\varepsilon}) - (\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)$ Also, we will use the fact that $\hat{\beta} - \beta = O_p\left(N^{-\frac{1}{2}}\right)$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^a\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^b = O_p\left(N^{-\frac{1}{2}}\right)$$

Where $a \geq 0$ and $b > 0$ are integers. In fact, whenever $a \geq 0$ and > 2 , if $E[(\bar{\varepsilon}_i - \bar{\varepsilon})^a \parallel \bar{x}_i - \bar{x} \parallel^b]$ exists, the expression equals $O_p\left(N^{\frac{1}{2}}\right) O_p\left(N^{-\frac{b}{2}}\right) = O_p\left(N^{\frac{(1-b)}{2}}\right) = O_p\left(N^{-\frac{1}{2}}\right)$

For $n = 2,$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{\hat{\varepsilon}}^2 = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[(\bar{\varepsilon}_i - \bar{\varepsilon}) - (\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^2$$

$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^2 + \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^2 - \frac{2}{\sqrt{N}}\sum_{i=1}^{N}\left[(\bar{\varepsilon}_i - \bar{\varepsilon})(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]$$

$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\bar{\varepsilon}_i^2 + \bar{\varepsilon}^2 - 2\bar{\varepsilon}_i\bar{\varepsilon}\right) + o_p(1)$$

For $n = 3,$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{\hat{u}}^3 = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[(\bar{\varepsilon}_i - \bar{\varepsilon}) - (\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^3$$

$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^3 - \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^3$$

$$-3\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^2\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]$$

$$+3\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^2$$

$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^3 - O_p\left(N^{\frac{1}{2}}\right) O_p\left(N^{\frac{-3}{2}}\right) - O_p(1)O_p\left(N^{\frac{1}{2}}\right) + O_p(1)O_p(N^{-1})$$

For $n = 4$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[(\bar{\varepsilon}_i - \bar{\varepsilon}) - (\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^4$$

$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^4 - \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^4$$

$$- 3\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^3\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]$$

$$+ 3\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^3$$

$$+ 6\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \bar{\varepsilon})^2\left[(\bar{x}_i - \bar{x})'\left(\hat{\beta} - \beta\right)\right]^2$$

Then,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\bar{\hat{\varepsilon}}^4 = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\bar{\varepsilon}_t - \bar{\varepsilon})^4 + O_p\left(N^{\frac{-1}{2}}\right)$$

Lemma Three. under assumptions 1, 2 and 3, the following equalities hold

Lemma 3 Part One:

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\hat{\varepsilon}_i^2}\,\bar{\hat{\varepsilon}}_i = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i^2}\,\bar{\varepsilon}_i - \sqrt{N}\bar{\varepsilon}E[\varepsilon_{it}^2 + 2\bar{\varepsilon}_i^2] + o_p(1)$$

Lemma 3 Part Two:

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\hat{\varepsilon}_i^3}\,\bar{\hat{\varepsilon}}_i = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i^3}\,\bar{\varepsilon}_i - \sqrt{N}\bar{\varepsilon}E\left[3\overline{\varepsilon_i^2}\bar{\varepsilon}_i\right] + o_p(1)$$

Lemma 3 Part three:

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\hat{\varepsilon}_i^2}\,\bar{\hat{\varepsilon}}_i^{\,2}=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{\varepsilon_i^2}\,\bar{\varepsilon}_i^{\,2}-2\sqrt{N}\bar{\varepsilon}E\left[\bar{\varepsilon}_i^3+\overline{\varepsilon_i^2}\bar{\varepsilon}_i\right]+o_p(1)$$

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