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## MATHEMATICAL STATISTICS

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# Parameter Estimation for the Mixture of Burr XII and Burr X Distribution Based on Complete Sample

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## Abstract

Burr Type XII and Burr Type X distributions are very important and extensively used in many practical applications. In this article, statistical properties of a mixture model of Burr XII and Burr X (BXIIBX) are studied including, reliability function, hazard rate, moments and incomplete moments. Estimation of the model parameters is considered based on complete sample. Two estimation procedures are considered, namely, maximum likelihood and moments. A simulation study is implemented for investigating the accuracy of estimators for different sample sizes.

**Keywords:** Burr Type XII, Burr Type X, Type II, Method of Moments, Maximum likelihood method.

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## 1. Introduction

A mixture of probability distributions appears in many situations when a population of interest comprises two or extra sub-populations. The finite mixtures of probability distributions play a vital role in many fields of applied sciences. A lot of studies about the mixture distributions have been considered by several authors, see for example; Cheng and Fu (1982), Jiang and Murthy(1995), McLachlan and Peel (2000), Zhang et al. (2001), Marin et al. (2005), Sultan et al. (2007), Ahmad et al. (2009), Abu-Zinadah (2010), Ateya and Alharthi (2014), Daniyal and Aleem (2014), and Karakoca et al. (2015), Elshahat and Mahmoud (2016), Al-Moisheer et al. (2017).

In 1942 Burr introduced twelve different forms of cumulative distribution functions for modeling lifetime data. Burr type XII and Burr X distributions are used in many applied fields such as business, engineering, quality control, life testing studies and reliability.

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The probability density function (pdf) and cumulative distribution function (cdf) of Burr Type XII distribution take the following form, respectively,

$$f_1(x; c, k) = c k x^{c-1} (1 + x^c)^{-(k+1)}, \quad x > 0, \quad c, k > 0, \quad (1)$$

and,

$$F_1(x; c, k) = 1 - (1 + x^c)^{-k}, \quad x > 0, \quad c, k > 0, \quad (2)$$

where,  $c$  and  $k$  are two shape parameters. Also, the pdf and cdf of Burr Type X distribution are given, respectively, as follows

$$f_2(x; \alpha, \lambda) = 2\lambda^2 \alpha x [1 - e^{-(\lambda x)^2}]^{\alpha-1} e^{-(\lambda x)^2}, \quad x > 0, \quad \alpha, \lambda > 0, \quad (3)$$

and,

$$F_2(x; \alpha, \lambda) = [1 - e^{-(\lambda x)^2}]^\alpha \quad x > 0, \quad \alpha, \lambda > 0, \quad (4)$$

where,  $\alpha$  and  $\lambda$  are shape and scale parameters, respectively, ( see Kundu and Raqab 2005).

This paper aims to study the properties for mixture model BXIIBX as well as obtain the estimate of the unknown parameters based on moments and maximum likelihood methods of estimation via a complete sample. This article will be organized as follows. Section 2 contains the formulation of the BXIIBX model. In the same section, some statistical properties including, reliability function, a general expression for its moments and incomplete moments are provided. Maximum likelihood and moments estimators of BXIIBX model are given in Section 3. Numerical study is conducted to examine the biases and mean square errors (MSEs) of BXIIBX parameters estimates in Section 4.

## 2. Formulation of BXIIBX Model

According to Titterington et al. (1985), the cumulative distribution function of a mixture distribution function with  $n$ - components is defined as follows,

$$F(x) = \sum_{i=1}^n p_i F_i(x),$$

where  $p_i > 0$  is proportion of the sub-mode and  $\sum_{i=1}^n p_i = 1$ . In particular, for  $n = 2$  the distribution function of two sub-population is defined as follows

$$F(x) = p_1 F_1(x) + p_2 F_2(x), \quad p_1 + p_2 = 1. \quad (5)$$

Now, we define a mixture of two BXIIBX distribution function by substituting cdf (2) and cdf (4) in cdf (5), setting  $p_1 = p$ ,  $p_2 = q$  and set of parameter  $\psi \equiv (c, k, \alpha, \lambda, p)$ , as follows

$$F(x; \underline{\psi}) = p[1 - (1 + x^c)^{-k}] + q[1 - e^{-(\lambda x)^2}]^\alpha; \quad c, k, \alpha, \lambda, x > 0, \quad (6)$$

where  $0 < p < 1$ ,  $p + q = 1$ ,  $p, q$  are the mixing proportions. The pdf of the BXIIBX distribution is given by

$$f(x; \underline{\psi}) = pck x^{c-1} (1 + x^c)^{-(k+1)} + 2q\alpha \lambda^2 x e^{-(\lambda x)^2} [1 - e^{-(\lambda x)^2}]^{\alpha-1}. \quad (7)$$

Additionally, the reliability and the hazard rate functions of BXIIBX can be expressed as,

$$R(x; \underline{\psi}) = p(1 + x^c)^{-k} + q(1 - [1 - e^{-(\lambda x)^2}]^\alpha). \quad (8)$$

The hazard rate function of BXIIBX can be expressed as,

$$h(x; \underline{\psi}) = \frac{pck x^{c-1} (1 + x^c)^{-(k+1)} + 2(1-p)\alpha e^{-(\lambda x)^2} [1 - e^{-(\lambda x)^2}]^{\alpha-1}}{p(1 + x^c)^{-k} + (1-p)(1 - [1 - e^{-(\lambda x)^2}]^\alpha)}$$

Plots of the BXIIBX hazard rate function for some parameter values are displayed in Figure1.

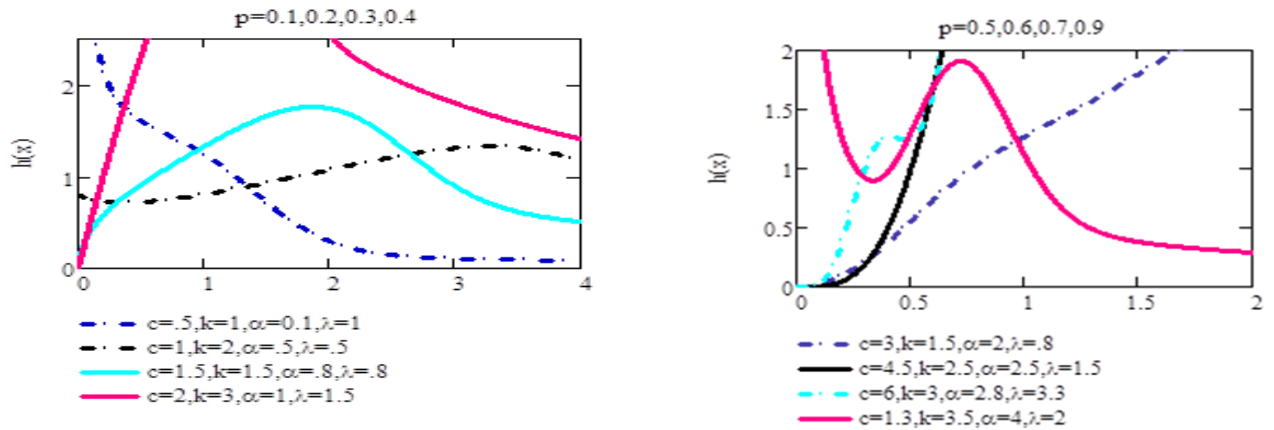


Figure 1: Hazard function for BXIIBX distribution

It is clear from Figure 1 that the hazard function takes different forms which are increasing, decreasing and bathtub according to different values of the parameters.

## 2.1 Moments

The  $r$ th moment of BXIIBX distribution about the origin can be obtained from (7) as follows

$$\mu'_r = E(X^r) = \int_0^\infty x^r \left\{ pck x^{c-1} (1+x^c)^{-(k+1)} + 2(1-p)\alpha \lambda^2 x e^{-(\lambda x)^2} \left[ 1 - e^{-(\lambda x)^2} \right]^{\alpha-1} \right\} dx.$$

After simplification, it can be expressed as follows

$$\mu'_r = p \left[ \frac{\Gamma\left(\frac{r}{c}+1\right) \Gamma\left(k-\frac{r}{c}\right)}{\Gamma(k)} \right] + (1-p) \left[ \frac{\alpha}{\lambda^r} \Gamma\left(\frac{r}{2}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j \binom{\alpha-1}{j}}{(j+1)^{\frac{r}{2}+1}} \right], k > \frac{r}{c}. \quad (9)$$

In particular, setting  $r = 1$  in (9), then the mean of BXIIBX distribution is as follows

$$\mu'_1 = p \left[ \frac{\Gamma\left(\frac{1}{c}+1\right) \Gamma\left(k-\frac{1}{c}\right)}{\Gamma(k)} \right] + (1-p) \left[ \frac{\alpha}{\lambda} \Gamma\left(\frac{3}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j \binom{\alpha-1}{j}}{(j+1)^{\frac{3}{2}}} \right], k > \frac{1}{c}.$$

Also, the variance of BXIIBX distribution takes the following form

$$V(X) = \left[ p \left[ \frac{\Gamma\left(\frac{2}{c}+1\right) \Gamma\left(k-\frac{2}{c}\right)}{\Gamma(k)} \right] + (1-p) \left[ \frac{\alpha}{\lambda^2} \sum_{j=0}^{\infty} \frac{(-1)^j \binom{\alpha-1}{j}}{(j+1)^2} \right] \right] - \left[ p \left[ \frac{\Gamma\left(\frac{1}{c}+1\right) \Gamma\left(k-\frac{1}{c}\right)}{\Gamma(k)} \right] + (1-p) \left[ \frac{\alpha\sqrt{\pi}}{2\lambda} \sum_{j=0}^{\infty} \frac{(-1)^j \binom{\alpha-1}{j}}{(j+1)^{\frac{3}{2}}} \right] \right]^2, k > \frac{2}{c}.$$

To examine how the mean, variance, skewness and kurtosis change for various values of parameters  $c, k$  and  $\alpha$ . Table 1 contains the mean and variance of the BXIIBX distribution. Also, Table 2 contains the skewness and kurtosis of the BXIIBX distribution

for various values of parameters  $c, k, \alpha$ . It can be notice that both the skewness and the kurtosis are decreasing and increasing functions for various values of  $c, k, \alpha$ . The following tables represent the mean and variance of the BXIIBX distribution for various values of  $c, k, \alpha$  and  $p$ .

**Table 1:** Mean and variance of the BXIIBX distribution for various values of  $c, k, \alpha$  and  $p$

| $c$ | $k$ | $\alpha$ | $\lambda = 1$ | $p = 0.5$ | $\lambda = 1$ | $p = 0.8$ |
|-----|-----|----------|---------------|-----------|---------------|-----------|
|     |     |          | $\mu'_1$      | $V(x)$    | $\mu'_1$      | $V(x)$    |
| 3   | 2.5 | 2        | 0.936         | 0.193     | 0.811         | 0.154     |
|     | 2.5 | 2.5      | 0.977         | 0.206     | 0.827         | 0.163     |
|     | 2.5 | 3        | 1.009         | 0.219     | 0.84          | 0.173     |
| 3.5 | 3   | 2        | 0.924         | 0.177     | 0.791         | 0.125     |
|     | 3   | 2.5      | 0.965         | 0.192     | 0.808         | 0.135     |
|     | 3.5 | 2.5      | 0.946         | 0.196     | 0.778         | 0.133     |
| 4   | 3   | 2.5      | 0.978         | 0.179     | 0.828         | 0.121     |
|     | 3.5 | 3        | 0.993         | 0.198     | 0.814         | 0.130     |
|     | 3   | 3        | 1.01          | 0.192     | 0.841         | 0.130     |
| 5   | 1.5 | 3        | 1.11          | 0.161     | 1.002         | 0.125     |
|     | 1.5 | 2        | 1.038         | 0.149     | 0.973         | 0.115     |
|     | 1.5 | 3.5      | 1.137         | 0.167     | 1.013         | 0.130     |

The following table represent skewness and kurtosis of the BXIIBX distribution for various values of  $c, k, \alpha$  and  $p$

**Table 2:** Skewness and kurtosis of the BXIIBX distribution for various values of  $c, k, \alpha$  and  $p$

| $c$ | $k$ | $\alpha$ | $\lambda = 1$ | $p = 0.5$  | $\lambda = 1$ | $p = 0.8$  |
|-----|-----|----------|---------------|------------|---------------|------------|
|     |     |          | $\gamma_1$    | $\gamma_2$ | $\gamma_1$    | $\gamma_2$ |
| 3   | 2.5 | 2        | 0.778         | 3.709      | 1.069         | 4.994      |
|     | 2.5 | 2.5      | 0.689         | 3.449      | 1.043         | 4.757      |
|     | 2.5 | 3        | 0.612         | 3.239      | 1.022         | 4.559      |
| 3.5 | 3   | 2        | 0.882         | 3.817      | 1.174         | 5.261      |
|     | 3   | 2.5      | 0.795         | 3.528      | 1.173         | 5.078      |
|     | 3.5 | 2.5      | 0.826         | 3.516      | 1.288         | 5.432      |
| 4   | 3   | 2.5      | 0.875         | 3.733      | 1.264         | 5.566      |
|     | 3.5 | 3        | 0.798         | 3.477      | 1.381         | 5.940      |
|     | 3   | 3        | 0.798         | 3.477      | 1.260         | 5.372      |
| 5   | 1.5 | 3        | 0.821         | 3.976      | 1.086         | 5.445      |
|     | 1.5 | 2        | 0.863         | 4.172      | 1.078         | 5.762      |
|     | 1.5 | 3.5      | 0.772         | 3.802      | 1.076         | 5.275      |

## 2.2 Incomplete moments

The answers for too many important questions in economics require more than just knowing the mean of a distribution, but its shape as well. Incomplete moments of the income distribution form natural building blocks for measuring inequality: for example, income quintiles, the Lorenz curve, Pietra and Gini measures of inequality all depend upon the incomplete moments of the income distribution (see Butler and McDonald (1989)).

The  $s^{th}$  incomplete moment, say  $\mu_s''$  is defined by

$$\mu_s'' = E(X^s) = \int_{-\infty}^t x^s f(x) dx. \quad (10)$$

Hence, the  $s^{th}$  incomplete moment for BXIIBX mixture model is derived by inserting (7) in (10) as follows,

$$\mu_s'' = E(X^s) = \int_0^t x^s \left\{ pck x^{c-1} (1+x^c)^{-(k+1)} + 2(1-p)\alpha \lambda^2 x e^{-(\lambda x)^2} \left[ 1 - e^{-(\lambda x)^2} \right]^{\alpha-1} \right\} dx.$$

After, simplification, the  $s^{th}$  incomplete moment is given by

$$\mu_s'' = p k B\left(t^c, \frac{s}{c} + 1, k - \frac{s}{c}\right) + (1-p)\alpha \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \frac{1}{\lambda^s (j+1)^{s/2}} \gamma\left(\frac{s}{2} + 1, (j+1)(\lambda t)^2\right), \quad (11)$$

where  $\gamma\left(\frac{s}{2} + 1, (j+1)(\lambda t)^2\right)$ , is the lower incomplete gamma function and

$B\left(t^c, \frac{s}{c} + 1, k - \frac{s}{c}\right)$  is incomplete beta.

The first incomplete moments of the BXIIBX mixture model can be obtained by putting  $s = 1$  in (11), as follows

$$\mu_1'' = p k B\left(t^c, \frac{1}{c} + 1, k - \frac{1}{c}\right) + (1-p)\alpha \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \frac{1}{\lambda (j+1)^{1/2}} \gamma\left(\frac{3}{2}, (j+1)(\lambda t)^2\right).$$

Bonferroni and Lorenz curves are useful applications to first incomplete moments. They have been widely used not only in the fields to study poverty and income but also in

reliability, medicine, economics, demography and insurance. The Lorenz and Bonferroni curves are given, respectively, as follows

$$L_F(t) = \frac{1}{\mu} \int_0^t x^s f(x) dx,$$

$$L_F(t) = \frac{p k B\left(t^c, \frac{1}{c} + 1, k - \frac{1}{c}\right) + (1-p) \alpha \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \frac{1}{\lambda (j+1)^{1/2}} \gamma\left(\frac{3}{2}, (j+1)(\lambda t)^2\right)}{p \left[ \frac{\Gamma\left(\frac{1}{c} + 1\right) \Gamma\left(k - \frac{1}{c}\right)}{\Gamma(k)} \right] + (1-p) \left[ \frac{\alpha}{\lambda} \Gamma\left(\frac{3}{2}\right) \sum_{s=0}^{\infty} \frac{(-1)^s \binom{\alpha-1}{s}}{(s+1)^{3/2}} \right]},$$

and

$$B_F(t) = \frac{L_F(t)}{F(t)} = \frac{p k B\left(t^c, \frac{1}{c} + 1, k - \frac{1}{c}\right) + (1-p) \alpha \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \frac{1}{\lambda (j+1)^{1/2}} \gamma\left(\frac{3}{2}, (j+1)(\lambda t)^2\right)}{p \left[ \frac{\Gamma\left(\frac{1}{c} + 1\right) \Gamma\left(k - \frac{1}{c}\right)}{\Gamma(k)} \right] + (1-p) \left[ \frac{\alpha}{\lambda} \Gamma\left(\frac{3}{2}\right) \sum_{s=0}^{\infty} \frac{(-1)^s \binom{\alpha-1}{s}}{(s+1)^{3/2}} \right] \left[ p \left[ 1 - (1+t^c)^{-k} \right] + (1-p) \left[ 1 - e^{-(\lambda t)^2} \right]^\alpha \right]}.$$

### 3. Estimation of Parameters based on Complete Sample

In this section, the estimators of the unknown parameters  $c, k, \alpha, \lambda, p$  of BXIIBX model are obtained by using the maximum likelihood and moments methods of estimation based on complete samples.

#### 3.1 Moments estimators

Method of moments is the oldest method for generating estimates of the unknown parameters from a sample data. The method of moments consists of equating a suitable number of sample moments to the corresponding population moments.

Let  $X$  be random variable from BXIIBX mixture model, then the first five population moments are given respectively by substituting  $r = 1, 2, 3, 4, 5$  in (9).

Let  $x_1, x_2, \dots, x_n$  be a random sample of size  $n$  drawn from mixture BXIIBX distribution, the first five sample moments are given by

$$m'_1 = \frac{1}{n} \sum_{i=1}^n x_i, \quad m'_2 = \frac{1}{n} \sum_{i=1}^n x_i^2, \quad \text{and} \quad m'_3 = \frac{1}{n} \sum_{i=1}^n x_i^3$$

$$m'_4 = \frac{1}{n} \sum_{i=1}^n x_i^4, \quad m'_5 = \frac{1}{n} \sum_{i=1}^n x_i^5.$$

The moment estimators of  $c, k, \alpha, p$  and  $\lambda$ , denoted by  $\bar{c}, \bar{k}, \bar{\alpha}, \bar{p}$  and  $\bar{\lambda}$  are obtained by equating the sample moments with corresponding populations moments. Numerical iterative technique is used to obtain  $\bar{c}, \bar{k}, \bar{\alpha}, \bar{p}$  and  $\bar{\lambda}$  respectively.

### 3.2 Maximum likelihood estimators

The maximum likelihood estimators of unknown parameters  $c, k, \alpha, p$  and  $\lambda$  of BXIIBX distribution from complete samples are derived. Let  $x_1, x_2, \dots, x_n$  be random sample from BXIIBX distribution with observed values  $x_1, x_2, \dots, x_n$ . The likelihood function of (7) is obtained as follows

$$L(x; \underline{\psi}) = \prod_{i=1}^n p c k x_i^{c-1} (1 + x_i^c)^{-(k+1)} + 2(1-p) \alpha \lambda^2 x_i e^{-(\lambda x_i)^2} \left[ 1 - e^{-(\lambda x_i)^2} \right]^{\alpha-1}.$$

For simplicity, let  $\ln L \equiv \ln L(x; \underline{\psi})$ . Therefore, the natural logarithm of the likelihood function is obtained as follows

$$\ln L = \sum_{i=1}^n \ln \left[ p c k x_i^{c-1} (1 + x_i^c)^{-(k+1)} + 2(1-p) \alpha \lambda^2 x_i e^{-(\lambda x_i)^2} \left[ 1 - e^{-(\lambda x_i)^2} \right]^{\alpha-1} \right]. \quad (12)$$

Partially differentiating (12) with respect to  $c, k, \alpha, p$  and  $\lambda$ , so the maximum likelihood estimators  $\hat{c}, \hat{k}, \hat{\alpha}, \hat{\lambda}$  and  $\hat{p}$ ; are the solution of the system of equations obtained by setting

the first partial derivative equal to zero. For simplicity, let  $d_i = (1 + x_i^c)$  and

$$l_i = (1 - e^{-(\lambda x_i)^2})$$

$$\frac{\partial \ln L}{\partial c} = \sum_{i=1}^n \frac{p k}{f(x_i; \underline{\psi})} \left\{ x_i^{c-1} d_i^{-(k+1)} + c x_i^{c-1} d_i^{-(k+1)} \ln(x_i) - c x_i^{2c-1} (k+1) d_i^{-(k+2)} \ln(x_i) \right\}, \quad (13)$$

$$\frac{\partial \ln L}{\partial k} = \sum_{i=1}^n \frac{p c x_i^{c-1}}{f(x_i; \underline{\psi})} \left\{ d_i^{-(k+1)} - k d_i^{-(k+1)} \ln d_i \right\}, \quad (14)$$

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^n \frac{2 \lambda^2 (1-p) x_i e^{-(\lambda x_i)^2}}{f(x_i; \underline{\psi})} \left\{ l_i^{\alpha-1} + \alpha l_i^{\alpha-1} \ln l_i \right\}, \quad (15)$$

$$\frac{\partial \ln L}{\partial p} = \sum_{i=1}^n \frac{1}{f(x_i; \underline{\psi})} \left\{ c k x_i^{c-1} d_i^{-(k+1)} - 2 \alpha \lambda^2 x_i e^{-(\lambda x_i)^2} l_i^{\alpha-1} \right\}, \quad (16)$$

and,

$$\begin{aligned} \frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^n \frac{2 \alpha x_i (1-p)}{f(x_i; \underline{\psi})} \left\{ 2 \lambda e^{-(\lambda x_i)^2} l_i^{\alpha-1} - 2 \lambda^3 x_i^2 e^{-(\lambda x_i)^2} l_i^{\alpha-1} \right. \\ \left. + 2 \lambda^3 x_i^2 e^{-2(\lambda x_i)^2} (\alpha-1) l_i^{\alpha-2} \right\}, \end{aligned} \quad (17)$$

Therefore, the maximum likelihood estimates of the parameters  $c, k, \alpha, p$  and  $\lambda$ , denoted by  $\hat{c}, \hat{k}, \hat{\alpha}, \hat{p}$  and  $\hat{\lambda}$  are obtained by setting the partially derivatives to be zero and solving them numerically.

#### 4. Numerical Study

In any estimation problems, it is required to study the properties of the derived estimators. As seen in the previous subsections, the derived expressions for the estimators are too complicated to study analytically. Therefore, an extensive numerical study is carried out to compare the performances of the estimates, mainly, with respect to their

biases and MSEs, for different sample sizes and for different set of parameters. The numerical process are described through the following steps

**Step (1):** A random sample  $x_1, x_2, \dots, x_n$  of sizes 20, 30, 40, 50 and 60, are generated from the mixture of Burr Type XII and Burr Type X distribution as follows

- (i) Generate  $U_1$  and  $U_2$  from the uniform distribution  $U(0,1)$ .
- (ii) If  $U_1 \leq p$

Generate from  $F_1(x, c, k)$  using  $U_2$ , otherwise

Generate from  $F_2(x, \alpha, \lambda)$  using  $U_2$  (see Ateya (2014)).

**Step (2):** Assuming the scale parameter,  $\lambda$  to be known and equal one throughout the experiment. Different sets of parameters values  $(c, k, \alpha, p)$  are selected as Set I  $\equiv (c = 2, k = 2.5, \alpha = 0.5, p = 0.5)$ , Set II  $\equiv (c = 1.5, k = 3, \alpha = 0.5, p = 0.5)$ , Set III  $\equiv (c = 1.5, k = 3, \alpha = 0.25, p = 0.5)$  and Set IV  $\equiv (c = 2, k = 2.5, \alpha = 0.5, p = 0.7)$ .

**Step (3):** The moment's estimators  $\bar{c}, \bar{k}, \bar{\alpha}$  and  $\bar{\lambda}$  are obtained by solving nonlinear equations. The maximum likelihood estimators  $\hat{c}, \hat{k}, \hat{\alpha}$  and  $\hat{p}$  are obtained by solving the nonlinear Equations (13) to (16).

**Step (4):** Steps from 1 to 3 are replicated  $N = 1000$  for each sample size and for selected sets of the parameters. Then the biases and mean square errors of estimates are computed as follows

$$\text{Biases} = \frac{1}{N} \sum_{i=1}^N (\hat{\psi} - \psi),$$

and,

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^N (\hat{\psi} - \psi)^2.$$

The numerical results about the properties of estimates are listed in Tables 3 and 4 and represented through some Figures from 2 to 7. The following observations about the performance of estimates from these tables are summarized as follows.

- 1- The MSE of the moments estimates  $\bar{c}, \bar{k}, \bar{\alpha}$  and  $\bar{p}$  decreases as the sample size increases in approximately different selected sets of parameters (for examples see Figure 2).

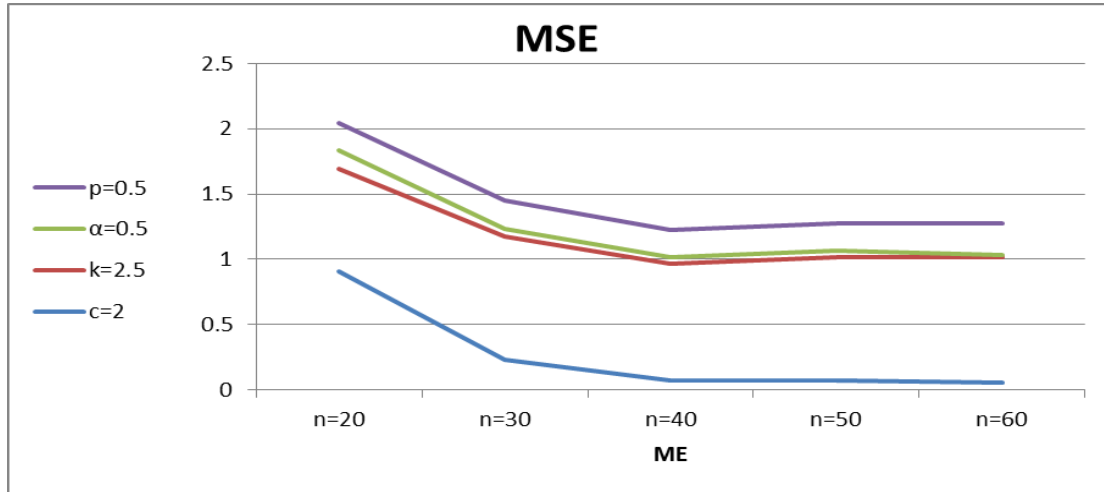


**Figure 2:** MSE for moment estimates for the Set I

- 2- The MSE of the maximum likelihood estimates  $\hat{c}, \hat{k}, \hat{\alpha}$  and  $\hat{p}$  decreases as sample size increases for all set of parameters (see for example Figure 3 for Set I ( $c = 2, k = 2.5, \alpha = 0.5, p = 0.5$ )).

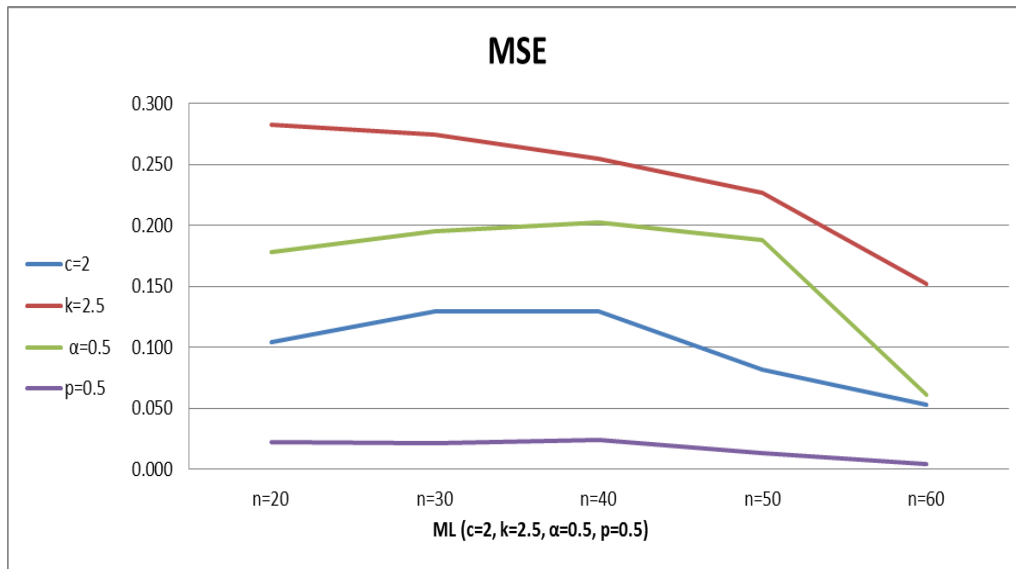


**Figure 3:** MSE of ML estimates for the Set I

3- It seems from Figure 4 that MSE of ML estimates for  $\hat{c}$  takes smallest value corresponding to the MSE of moment estimates;  $\bar{c}$ , for the same sample size  $n=60$ . Also, it seems from Figures 5 and 7 respectively that MSE of  $\hat{k}, \hat{\alpha}$  and  $\hat{p}$  takes the smallest value corresponding to the MSE of  $\bar{k}, \bar{\alpha}$  and  $\bar{p}$  for the same sample size  $n=60$  and for all sets of parameters.

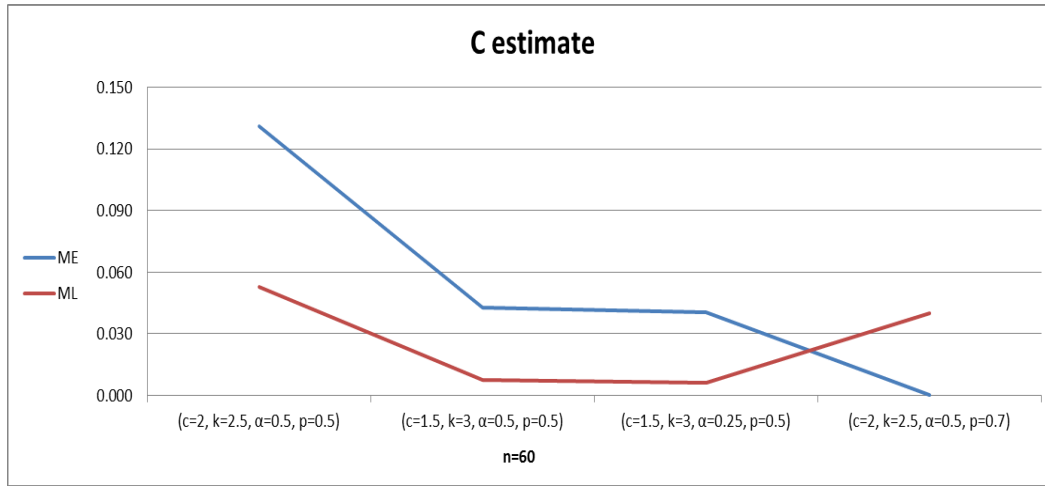


**Figure 4:** MSE of  $\hat{c}$  and  $\bar{c}$  for different set of parameters at  $n=60$

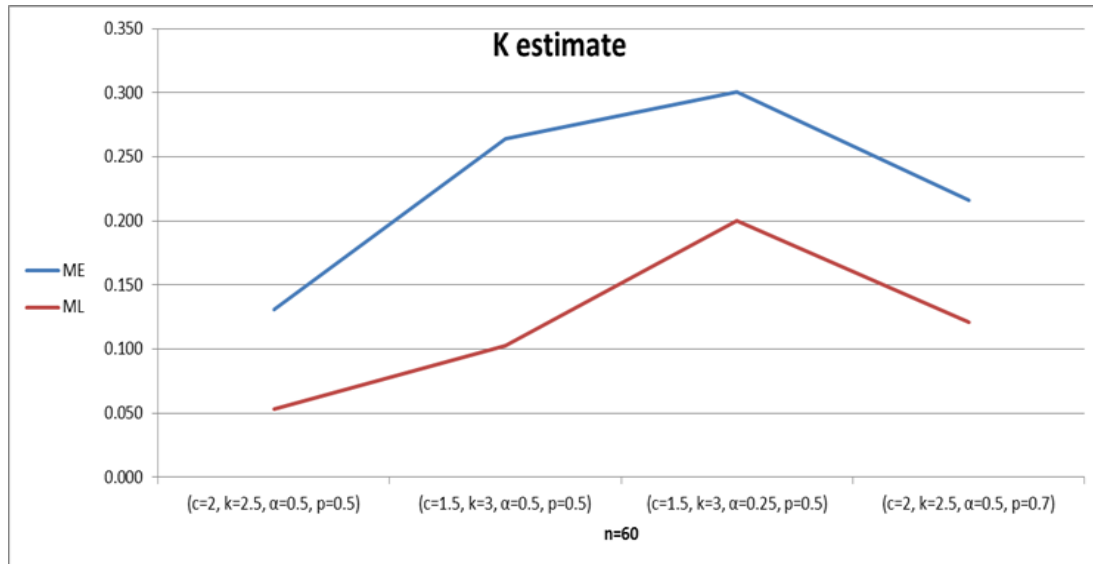


**Figure 5:** MSE of  $\hat{k}$  and  $\bar{k}$  for different set of parameters at  $n=60$

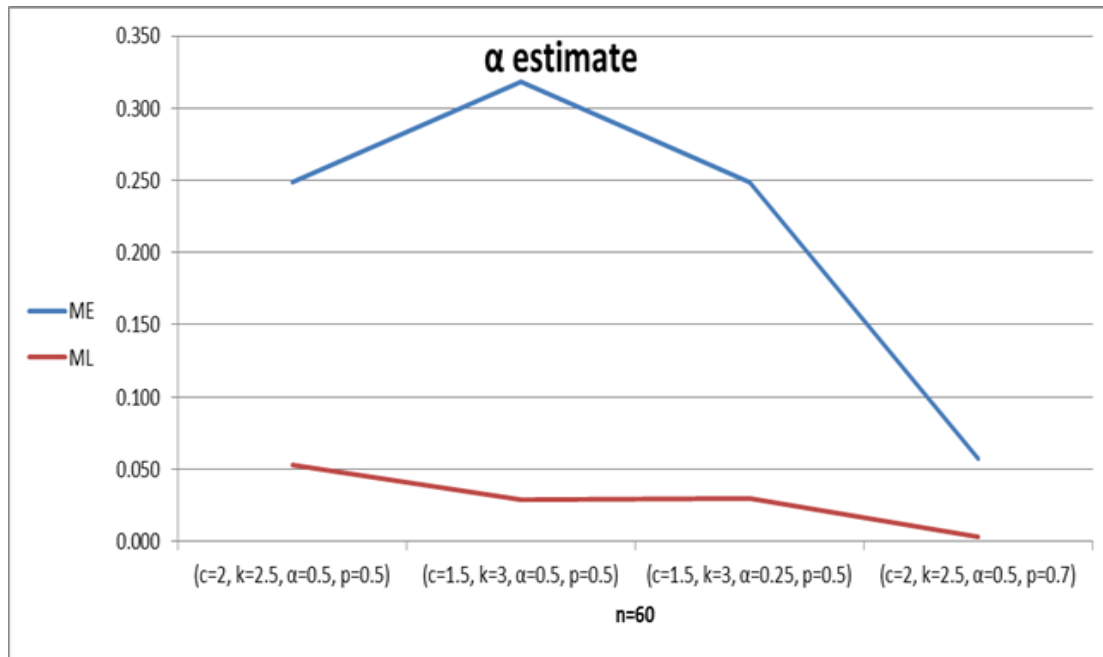


**Figure 6:** MSE of  $\hat{\alpha}$  and  $\bar{\alpha}$  for different set of parameters at n=60

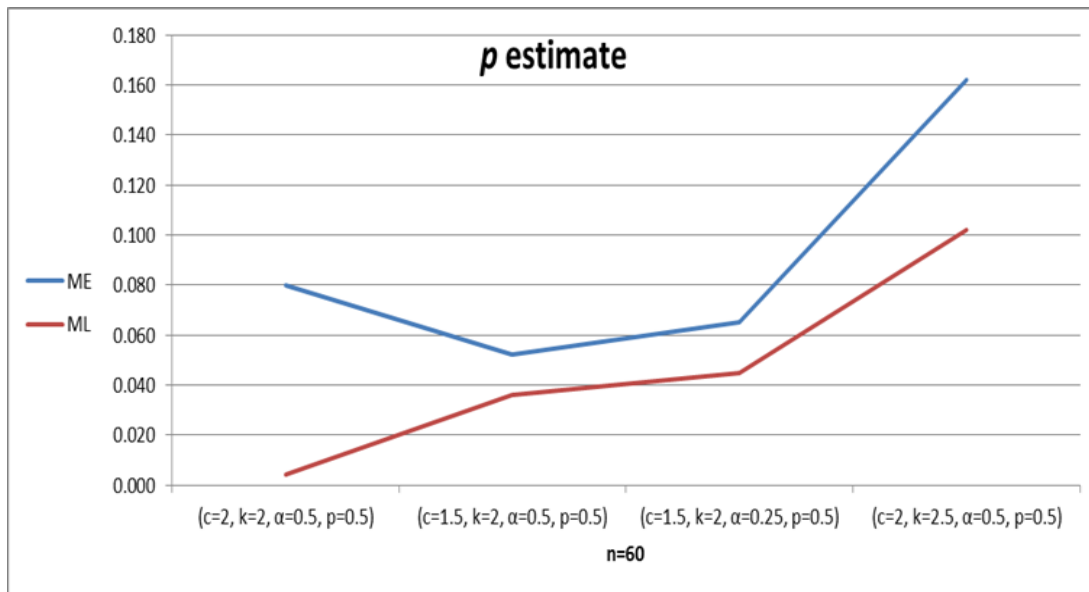


**Figure 7:** MSE of  $\hat{p}$  and  $\bar{p}$  for different set of parameters at n=60

4-The MSE of  $\hat{c}, \hat{\alpha}$  for the Set IV is smaller than the MSE of  $\bar{c}, \bar{\alpha}$  in approximately all sample sizes (see Figures 4 and 6). 5-The MSE of  $\hat{p}, \hat{k}$  for the Set I is smaller than MSE of  $\bar{p}, \bar{k}$  in approximately all sample sizes (see Figures 5 and 7).

6- In general it is observed that the biases and MSEs of all estimates decrease as sample size increases in almost all the cases. Furthermore, the MSEs for the ML estimates are smaller than the MSEs for the moment estimates in almost all the cases.

**Table 3:** Biases and MSEs of estimates for Set I and Set II of BXIIBX distribution in complete sample

| set of parameters |        |            |                                    |        |       |        |                                    |        |       |       |
|-------------------|--------|------------|------------------------------------|--------|-------|--------|------------------------------------|--------|-------|-------|
| sample size       | method | Properties | (c=2, k=2.5, $\alpha=0.5$ , p=0.5) |        |       |        | (c=1.5, k=3, $\alpha=0.5$ , p=0.5) |        |       |       |
| 20                | ML     | Bias       | -0.282                             | 0.461  | 0.372 | -0.052 | 0.025                              | 0.336  | 0.330 | 0.118 |
|                   |        | MSE        | 0.105                              | 0.283  | 0.178 | 0.022  | 0.038                              | 0.179  | 0.207 | 0.045 |
|                   | ME     | Bias       | 0.518                              | -0.851 | 0.333 | -0.445 | 0.091                              | -0.511 | 0.521 | 1.171 |
|                   |        | MSE        | 0.911                              | 0.779  | 0.141 | 0.214  | 0.009                              | 0.263  | 0.475 | 0.177 |
| 30                | ML     | Bias       | -0.294                             | 0.471  | 0.266 | 0.017  | -0.011                             | 0.399  | 0.266 | 0.068 |
|                   |        | MSE        | 0.130                              | 0.274  | 0.195 | 0.021  | 0.030                              | 0.251  | 0.117 | 0.066 |
|                   | ME     | Bias       | 0.297                              | -0.966 | 0.224 | -0.464 | 0.095                              | -0.520 | 0.322 | 0.481 |
|                   |        | MSE        | 0.233                              | 0.941  | 0.059 | 0.220  | 0.010                              | 0.272  | 0.414 | 0.197 |
| 40                | ML     | Bias       | -0.390                             | 0.562  | 0.203 | -0.006 | -0.007                             | 0.287  | 0.231 | 0.125 |
|                   |        | MSE        | 0.130                              | 0.258  | 0.173 | 0.024  | 0.023                              | 0.148  | 0.073 | 0.034 |
|                   | ME     | Bias       | 0.110                              | -0.944 | 0.199 | -0.456 | 0.089                              | -0.513 | 0.354 | 0.567 |
|                   |        | MSE        | 0.070                              | 0.897  | 0.045 | 0.211  | 0.008                              | 0.264  | 0.447 | 0.139 |
| 50                | ML     | Bias       | -0.407                             | 0.523  | 0.362 | -0.042 | -0.037                             | 0.173  | 0.105 | 0.023 |
|                   |        | MSE        | 0.082                              | 0.227  | 0.188 | 0.014  | 0.033                              | 0.073  | 0.015 | 0.017 |
|                   | ME     | Bias       | -0.011                             | -0.966 | 0.204 | -0.454 | 0.085                              | -0.507 | 0.305 | 0.319 |
|                   |        | MSE        | 0.068                              | 0.944  | 0.052 | 0.212  | 0.007                              | 0.258  | 0.365 | 0.135 |
| 60                | ML     | Bias       | -0.052                             | 0.381  | 0.238 | -0.046 | 0.086                              | 0.243  | 0.145 | 0.185 |
|                   |        | MSE        | 0.053                              | 0.152  | 0.061 | 0.004  | 0.008                              | 0.102  | 0.029 | 0.045 |
|                   | ME     | Bias       | -0.357                             | -1.047 | 0.125 | -0.300 | 0.044                              | -0.513 | 0.236 | 0.106 |
|                   |        | MSE        | 0.131                              | 0.809  | 0.019 | 0.080  | 0.043                              | 0.264  | 0.318 | 0.052 |

**Table 4:** Biases and MSEs of estimates for Set III and Set IV of BXIIBX distribution in complete sample

| sample size | method | Properties | set of parameters                   |        |       |        |                                    |        |       |        |
|-------------|--------|------------|-------------------------------------|--------|-------|--------|------------------------------------|--------|-------|--------|
|             |        |            | (c=1.5, k=3, $\alpha=0.25$ , p=0.5) |        |       |        | (c=2, k=2.5, $\alpha=0.5$ , p=0.7) |        |       |        |
| 20          | ML     | Bias       | 0.091                               | 0.793  | 0.309 | 0.132  | -0.691                             | 0.428  | 0.273 | 0.496  |
|             |        | MSE        | 0.015                               | 0.521  | 0.167 | 0.112  | 0.478                              | 0.204  | 0.076 | 0.287  |
|             | ME     | Bias       | -0.334                              | -0.539 | 0.507 | -0.332 | 0.086                              | -0.489 | 0.186 | -0.413 |
|             |        | MSE        | 0.127                               | 0.300  | 0.269 | 0.119  | 0.578                              | 0.910  | 0.041 | 0.165  |
| 30          | ML     | Bias       | 0.083                               | 0.497  | 0.151 | -0.098 | -0.578                             | 0.596  | 0.206 | 0.207  |
|             |        | MSE        | 0.012                               | 0.441  | 0.030 | 0.032  | 0.335                              | 0.373  | 0.043 | 0.067  |
|             | ME     | Bias       | -0.367                              | -0.563 | 0.473 | -0.408 | 0.602                              | -0.625 | 0.250 | 0.091  |
|             |        | MSE        | 0.151                               | 0.330  | 0.227 | 0.189  | 0.073                              | 0.621  | 0.057 | 0.201  |
| 40          | ML     | Bias       | 0.117                               | 0.583  | 0.179 | -0.048 | -0.120                             | 0.492  | 0.088 | -0.255 |
|             |        | MSE        | 0.016                               | 0.402  | 0.051 | 0.025  | 0.069                              | 0.347  | 0.011 | 0.121  |
|             | ME     | Bias       | -0.418                              | -0.627 | 0.404 | -0.287 | 0.295                              | -0.754 | 0.250 | -0.100 |
|             |        | MSE        | 0.188                               | 0.399  | 0.314 | 0.371  | 0.099                              | 0.575  | 0.063 | 0.058  |
| 50          | ML     | Bias       | 0.100                               | 0.532  | 0.113 | -0.212 | -0.343                             | -0.454 | 0.268 | 0.435  |
|             |        | MSE        | 0.014                               | 0.324  | 0.015 | 0.072  | 0.121                              | 0.302  | 0.072 | 0.121  |
|             | ME     | Bias       | -0.407                              | -0.595 | 0.439 | -0.348 | 0.020                              | -0.448 | 0.162 | -0.461 |
|             |        | MSE        | 0.117                               | 0.364  | 0.271 | 0.280  | 0.009                              | 0.202  | 0.028 | 0.262  |
| 60          | ML     | Bias       | 0.074                               | 0.600  | 0.160 | -0.100 | -0.064                             | 0.317  | 0.041 | -0.365 |
|             |        | MSE        | 0.006                               | 0.200  | 0.030 | 0.035  | 0.040                              | 0.121  | 0.003 | 0.102  |
|             | ME     | Bias       | -0.449                              | -0.546 | 0.496 | -0.318 | 0.009                              | -0.465 | 0.239 | -0.517 |
|             |        | MSE        | 0.047                               | 0.301  | 0.248 | 0.205  | 0.000                              | 0.216  | 0.057 | 0.162  |

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## Bayesian Inference for Mixture of Burr XII and Burr X Distribution based on Type I Censored Sample Via Different Priors

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### Abstract

Mixtures of distributions provide an important tool in modeling wide range of observed phenomena, which do not normally yield for modelling from classical distributions like normal, gamma, Poisson, binomial, etc. A mixed model occurs when two or more different causes of failure are presented each with parametric form of life distributions with unknown proportions. Burr Type XII and Burr Type X distributions are of great importance and extensively used in many practical applications. This study is concerned with Bayesian approach using the non-informative and informative priors for mixture of Burr XII and Burr X (BXIIBX) distribution. Based on Type I censored samples, the parameters of a mixture model are estimated. The Bayesian estimators are computed using the idea of Markov Chain Monte Carlo (MCMC) method. A simulation study is implemented for investigating the accuracy of estimators for different sample sizes.

**Keywords:** Burr Type XII, Burr Type X, Type I censored sample, Bayesian approach, non-informative, informative priors.

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### 1. Introduction

In practical situations, it often happens that the data from the population is not homogenous, but is come from sub-populations with unknown proportions. Finite mixture models receive increasing attention over the recent years in many practical applications. For excellnt survey of estimation techniques, discussion and applications, see Everitt and Hand (1981), Titterington et al. (1985), Maclachlan and Basford (1988), Lindsay (1995), AL-Hussaini (1999), McLachlan and Peel (2000), Sultan et al. (2007), Abu-Zinadah (2010), AL-Hussaini and Hussein (2012), Kazmi et al. (2012), Ateya and Alharthi (2014), Ali et al.(2012)

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Saieed (2014), Ali et al. (2014), Ali (2015), Karaoke et al. (2015) and Tahir et al. (2016), Sultana et al. (2017), Al-Moisheer et al. (2017).

Burr (1942) developed a family of twelve cumulative distribution functions, based on generating the Pearson differential equation, for modelling lifetime data. From the family of Burr distributions, the Burr XII and Burr X distributions are widely used models and most desirable for statistical modelling. Different forms of Burr distributions are related to other by simple transformation. As indicated by Johnson et al. (1995), the Burr Type III distribution is obtained from Burr Type II distribution by replacing  $x$  with  $\ln(x)$ . Burr and Cislak (1968) mentioned that Burr Type XII distribution is the inverse distribution from Burr Type III. The Burr Type I family is more commonly known as the uniform distribution. The probability density function (pdf) of Burr XII distribution with shape parameters  $c$  and  $k$  is given by

$$g_1(x; c, k) = c k x^{c-1} (1+x^c)^{-(k+1)}, \quad x > 0, \quad c, k > 0, \quad (1)$$

Further, the pdf of Burr X distribution with shape parameter  $\alpha$  and scale parameter  $\lambda$  is given by

$$g_2(x; \alpha, \lambda) = 2\lambda^2 \alpha x \left[1 - e^{-(\lambda x)^2}\right]^{\alpha-1} e^{-(\lambda x)^2}, \quad x > 0, \quad \alpha, \lambda > 0. \quad (2)$$

A finite mixture density function with the 2 - component densities of specified parametric form with unknown mixing weights  $p_i$  is defined as follows

$$f(x) = \sum_{i=1}^2 p_i g_i(x), \quad \text{for } i = 1, 2$$

where, the mixing proportions  $p_i \geq 0$  and  $\sum_{i=1}^2 p_i = 1$ . A finite mixture of two component with mixed proportions  $p, (1-p)$ . Hence, the probability density function of 2-component mixture of Burr Type-XII and Type X distributions, denoted by BXIIBX, is defined mathematically as follows

$$f(x; \underline{\psi}) = p c k x^{(c-1)} (1+x^c)^{-(k+1)} + 2(1-p) \alpha \lambda^2 x e^{-(\lambda x)^2} [1 - e^{-(\lambda x)^2}]^{(\alpha-1)}, \quad c, k, \alpha, \lambda, x > 0, \quad (3)$$

where,  $0 < p < 1$ , and  $\underline{\psi} \equiv (c, k, \alpha, \lambda, p)$  is the parameter vector of the BXIIBX model. The cumulative distribution function (cdf) and the reliability function of mixture in Equation 3 denoted by  $F(x; \underline{\psi})$  and  $R(x; \underline{\psi})$  are given, respectively, by

$$F(x; \underline{\psi}) = p \left[ 1 - (1 + x^c)^{-k} \right] + (1 - p) \left[ 1 - e^{-(\lambda x)^2} \right]^\alpha, \quad (4)$$

and,

$$R(x; \underline{\psi}) = p (1 + x^c)^{-k} + (1 - p) \left( 1 - \left[ 1 - e^{-(\lambda x)^2} \right]^\alpha \right). \quad (5)$$

In Type-I censoring scheme, the experiment terminated after a pre-determined time of censoring. In Type-II censoring scheme, the experiment is usually terminated when  $r^{th}$  failure is observed.

In this study, we concern with the estimation and evaluation of BXIIBX mixture distribution in a Bayesian approach, based on Type I censored samples. Bayesian estimates are obtained via MCMC algorithm. This article is organized as follows. In Section 2, sampling scheme and maximum likelihood method of estimation is performed based on Type I censored samples. In Section 3, the posterior distributions are derived under different priors. Section 4 presents a numerical study to examine Bayes estimates of BXIIBX mixture distribution using MCMC algorithm. A conclusion is summarized at last section.

## 2. Sampling Scheme and Likelihood Function

In this section, the likelihood function of the parameters for BXIIBX distribution is obtained based on Type I censored samples. In Type I censoring scheme, the experiment of  $n$  items is placed on a test with fixed test termination time  $T$ . The experiment is performed and its observed  $r$  out of  $n$  units failed up to time  $T$  instead of continuing experiments until all  $n$  items have failed. The failed objects in many real life situations can be recognized as belong to subpopulation 1 or subpopulation 2, so depending on the cause of failure it may be observed  $r_1$  and  $r_2$ . The number of uncensored observations is  $r = r_1 + r_2$ , where  $r_1$  failure units from subpopulation (1),  $r_2$  failure units from subpopulation (2), and the remaining  $n - r$  observations are censored giving no information about to which subpopulation they belong. Furthermore, let  $x_{ij}$  be the failure time of the  $j^{th}$  unit to the  $i^{th}$  subpopulation, where  $j = 1, 2, \dots, r_i, i = 1, 2; 0 \leq x_{1j}, x_{2j} \leq T$ . The likelihood function for two-component BXIIBX mixture model with  $n$  item under study in which  $r_1$  will fail due to cause 1,  $r_2$  will fail due to cause 2 and the remaining  $(n - r_1 - r_2)$  will survive at time  $T$ , (See Mendenhall and Hadar (1958)), is given as follows:

$$L(\psi|x) \propto \prod_{j=1}^{r_1} p g_1(x_{1j}) \prod_{j=1}^{r_2} (1-p) g_2(x_{2j}) [1-F(T)]^{n-r} . \quad (6)$$

The likelihood function (6) can be written as follows

$$L(\psi|x) \propto \prod_{j=1}^{r_1} p c k x_{1j}^{c-1} (1+x_{1j}^c)^{-(k+1)} \prod_{j=1}^{r_2} 2(1-p) \lambda^2 \alpha x_{2j} \left[ 1 - e^{-(\lambda x_{2j})^2} \right]^{\alpha-1} e^{-(\lambda x_{2j})^2} \left[ p (1+T^c)^{-k} + (1-p) \left( 1 - \left[ 1 - e^{-(\lambda T)^2} \right]^\alpha \right) \right]^{n-r} . \quad (7)$$

A simplified form for the likelihood function (7) is as follows

$$L(\psi|x) \propto 2^{r_2} p^{n-r-\delta_1+r_1} (1-p)^{r_2+\delta_1-\delta_2} c^{r_1} k^{r_1} \lambda^{2r_2} \alpha^{r_2} \exp \left[ \sum_{j=1}^{r_1} (c-1) \ln x_{1j} - (k+1) \ln (1+x_{1j}^c) \right] \exp \left[ \sum_{j=1}^{r_2} \ln x_{2j} - \sum_{j=1}^{r_2} (\lambda x_{2j})^2 + (\alpha-1) \ln \left[ 1 - e^{-(\lambda x_{2j})^2} \right] \right] \sum_{\delta_1=0}^{n-r} \sum_{\delta_2=0}^{\delta_1} \binom{n-r}{\delta_1} \binom{\delta_1}{\delta_2} (-1)^{\delta_1} \left[ (1+T^c)^{-k} \right]^{\delta_2} \left[ 1 - \left[ 1 - e^{-(\lambda T)^2} \right]^\alpha \right]^{\delta_1-\delta_2} , \quad (8)$$

where,  $x = (x_{11}, x_{12}, \dots, x_{1r_1}, x_{21}, x_{22}, \dots, x_{2r_2})$  is the observed failure times for the non-censored observations.

### 3. Prior and Posterior Distributions

Here, we present the priors distributions of the parameters when the non-informative and informative priors are available for making Bayesian inference.

#### 3.1 Posterior distributions under non-informative prior

In this subsection, we consider the Bayesian estimators of the unknown parameter(s) in case of non-informative prior (NIP). In many practical situations, the information about the parameters is available in an independent manner. Assuming that the NIP of parameters  $c, k, \alpha$  and  $\lambda$  has the following uniform distribution

$$\pi_1(c) \propto \frac{1}{c}, \quad \pi_2(k) \propto \frac{1}{k}, \quad \pi_3(\alpha) \propto \frac{1}{\alpha}, \quad \pi_4(\lambda) \propto \frac{1}{\lambda}.$$

Also, it is assumed that the proportion parameter  $p$  has the uniform distribution over the interval  $(0, 1)$ , that is  $\pi_5(p) \propto 1$ . Assuming independence of parameters, the joint prior distribution of parameters  $c, k, \alpha, \lambda$  and  $p$  is as follows

$$\pi(\psi) \propto \frac{1}{c k \alpha \lambda}.$$

The joint posterior distribution of parameters;  $c, k, \lambda, \alpha$  and  $p$  is defined as follows

$$\Pi_1(\psi|x) = \frac{\pi(\psi)L(\psi|x)}{\int_{\psi} \pi(\psi)L(\psi|x) d\psi}. \quad (9)$$

Hence, the posterior distribution can be written as follows

$$\begin{aligned} \Pi_1(\psi|x) = & \frac{1}{H_1} 2^{r_2} p^{n-r-\delta_1+r_1} (1-p)^{r_2+\delta_1-\delta_2} c^{r_1-1} k^{r_1-1} \lambda^{2r_2-1} \alpha^{r_2-1} \exp \left[ \sum_{j=1}^{r_1} (c-1) \ln x_{1j} - \sum_{j=1}^{r_1} (k+1) \ln(1+x_{1j}^c) \right] \\ & \exp \left[ \sum_{j=1}^{r_2} \ln x_{2j} - \sum_{j=1}^{r_2} (\lambda x_{2j})^2 + (\alpha-1) \ln \left[ 1 - e^{-(\lambda x_{2j})^2} \right] \right] \sum_{\delta_1=0}^{n-r} \sum_{\delta_2=0}^{\delta_1} \binom{n-r}{\delta_1} \binom{\delta_1}{\delta_2} (-1)^{\delta_1} \left[ (1+T^c)^{-k} \right]^{\delta_2} \\ & \left[ \left[ 1 - \left[ 1 - e^{-(\lambda T)^2} \right]^\alpha \right]^{\delta_1-\delta_2} \right], \end{aligned}$$

where,

$$\begin{aligned} H_1 = & 2^{r_2} B(n-r-\delta_1+r_1+1, r_2+\delta_1-\delta_2+1) \exp \left( \sum_{j=1}^{r_2} \ln x_{2j} - \sum_{j=1}^{r_1} \ln x_{1j} \right) \sum_{\delta_1=0}^{n-r} \sum_{\delta_2=0}^{\delta_1} \binom{n-r}{\delta_1} \binom{\delta_1}{\delta_2} (-1)^{\delta_1} I_1, \\ I_1 = & \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 c^{r_1-1} k^{r_1-1} \lambda^{2r_2-1} \alpha^{r_2-1} \exp \left[ \sum_{j=1}^{r_1} (c \ln x_{1j} - \sum_{j=1}^{r_2} (\lambda x_{2j})^2 + (\alpha-1) \ln \left[ 1 - e^{-(\lambda x_{2j})^2} \right] - \sum_{j=1}^{r_1} k \ln(1+x_{1j}^c) \right] \\ & \left[ \left[ 1 - \left[ 1 - e^{-(\lambda T)^2} \right]^\alpha \right]^{\delta_1-\delta_2} \right] dc dk d\alpha d\lambda dp, \end{aligned}$$

and  $B(.,.)$  is the beta function.

The Bayesian estimators of the parameters  $c, k, \alpha, \lambda$  and  $p$  denoted by  $\tilde{c}, \tilde{k}, \tilde{\alpha}, \tilde{\lambda}$  and  $\tilde{p}$ , can be obtained respectively as follows (see Mood et al. (1974))

$$\tilde{c} \propto \int_0^\infty c \Pi_1(\psi|x) dc, \quad \tilde{k} \propto \int_0^\infty k \Pi_1(\psi|x) dk, \quad \tilde{\alpha} \propto \int_0^\infty \alpha \Pi_1(\psi|x) d\alpha, \quad \tilde{\lambda} \propto \int_0^\infty \lambda \Pi_1(\psi|x) d\lambda$$

$$\text{and } \tilde{p} \propto \int_0^1 p \Pi_1(\psi|x) dp.$$

Generally, it is impossible for  $\tilde{c}, \tilde{k}, \tilde{\alpha}, \tilde{\lambda}$  and  $\tilde{p}$  to have a closed form solution. Therefore, the Markov Chain Monte Carlo approach is used to approximate evaluation of the above integrals.

### 3.2 Posterior distributions under informative prior

Again, in this subsection, we also assumed that the prior distribution for  $p$  has uniform distribution over the interval (0,1). Further, assuming that the prior of parameter  $c$

has a gamma distribution with parameters  $(a, b)$ , while  $k, \alpha$  and  $\lambda$  have exponential distribution with parameters  $\beta_1, \beta_2$  and  $\beta_3$  respectively. Hence, assuming independence of parameters, the joint prior distribution of parameters is as follows,

$$\pi_i(\psi) \propto \frac{b^a}{\beta_1 \beta_2 \beta_3 \Gamma(a)} c^{a-1} e^{-\left(\frac{k}{\beta_1} + \frac{\alpha}{\beta_2} + \frac{\lambda}{\beta_3}\right) - b}, c, k, \lambda, \alpha, a, b, \beta_1, \beta_2, \beta_3 > 0.$$

The posterior distribution of  $c, k, \lambda, \alpha$  and  $p$  for BXIIBX under Type I censored takes the following form

$$\begin{aligned} \Pi_2(\psi|x) &= \frac{1}{H_2} \frac{b^a e^{-b} 2^{r_2}}{\beta_1 \beta_2 \beta_3 \Gamma(a)} p^{n-r-\delta_1+r_1} (1-p)^{r_2+\delta_1-\delta_2} c^{a+r_1-1} k^{r_1} \lambda^{2r_2} \alpha^{r_2} \\ &\quad \exp \left[ \sum_{j=1}^{r_1} (c-1) \ln x_{1j} - \frac{k}{\beta_1} - \sum_{j=1}^{r_1} (k+1) \ln(1+x_{1j}^c) \right] \\ &\quad \exp \left[ \sum_{j=1}^{r_2} \ln x_{2j} - \sum_{j=1}^{r_2} (\lambda x_{2j})^2 - \frac{\alpha}{\beta_2} - \frac{\lambda}{\beta_3} + (\alpha-1) \ln \left[ 1 - e^{-(\lambda x_{2j})^2} \right] \right] \\ &\quad \sum_{\delta_1=0}^{n-r} \sum_{\delta_2=0}^{\delta_1} \binom{n-r}{\delta_1} \binom{\delta_1}{\delta_2} (-1)^{\delta_1} \left[ (1+T^c)^{-k} \right] \left[ \left[ 1 - \left[ 1 - e^{-(\lambda T)^2} \right]^\alpha \right] \right]^{\delta_1-\delta_2}, \\ H_2 &= \frac{b^a e^{-b} 2^{r_2}}{\beta_1 \beta_2 \beta_3 \Gamma(a)} B(n-r-\delta_1+r_1+1, r_2+\delta_1-\delta_2+1) \exp \left( \sum_{j=1}^{r_2} \ln x_{2j} - \sum_{j=1}^{r_1} \ln x_{1j} \right) \\ &\quad \sum_{\delta_1=0}^{n-r} \sum_{\delta_2=0}^{\delta_1} \binom{n-r}{\delta_1} \binom{\delta_1}{\delta_2} (-1)^{\delta_1} I_2, \\ I_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 c^{a+r_1-1} k^{r_1} \lambda^{2r_2} \alpha^{r_2} \\ &\quad \exp \left[ \sum_{j=1}^{r_1} c \ln x_{1j} - \frac{k}{\beta_1} - \frac{\alpha}{\beta_2} - \frac{\lambda}{\beta_3} - \sum_{j=1}^{r_2} (\lambda x_{2j})^2 + (\alpha-1) \ln \left[ 1 - e^{-(\lambda x_{2j})^2} \right] - \sum_{j=1}^{r_1} k \ln(1+x_{1j}^c) \right] \\ &\quad \left[ \left[ 1 - \left[ 1 - e^{-(\lambda T)^2} \right]^\alpha \right] \right]^{\delta_1-\delta_2} dc dk d\alpha d\lambda dp. \end{aligned}$$

The Bayesian estimators of the parameters  $c, k, \lambda, \alpha, p$ , denoted by  $\hat{c}, \hat{k}, \hat{\alpha}, \hat{\lambda}$  and  $\hat{p}$ , are obtained respectively as follows

$$\begin{aligned} \hat{c} &\propto \int_0^\infty \Pi_2(\psi|x) dc, \quad \hat{k} \propto \int_0^\infty k \Pi_2(\psi|x) dk, \quad \hat{\alpha} \propto \int_0^\infty \alpha \Pi_2(\psi|x) d\alpha, \quad \hat{\lambda} \propto \int_0^\infty \lambda \Pi_2(\psi|x) d\lambda \text{ and} \\ \hat{p} &\propto \int_0^1 p \Pi_2(\psi|x) dp. \end{aligned}$$

Generally, it is difficult for  $\hat{c}, \hat{k}, \hat{\alpha}, \hat{\lambda}$  and  $\hat{p}$  to have explicit solution. Therefore; as previously mentioned the MCMC approach is applied to approximate evaluation for the previous integrals depending on Open BUGS software.

#### 4. Numerical Study

In this section, numerical study is performed to investigate the performance of Bayesian estimators for  $c, k, \lambda, \alpha$  and  $p$  under Type I censored samples. The Bayes estimates are obtained using non- informative and informative priors. The MCMC sampling algorithms carried out using the Open BUGS software, a specialized software package for implementing MCMC simulation and Gibbs sampling. The numerical study is carried out as follows

**Step (1):** Generate random samples of sizes 5, 15, 25, 30, 50 and 100, from the BXIIBX mixture distribution.

**Step (2):** The posterior density functions for the unknown parameters  $c, k, \lambda, \alpha$  and  $p$  are determined.

**Step (3):** Selected two chains with different initials for non-informative and informative priors, respectively, as follows:

(i)  $\{(c = 3, k = 2, \lambda = 5, \alpha = 3.5, p = 0.2), (c = 3.5, k = 2.5, \lambda = 3.5, \alpha = 4, p = 0.5)\}$ .

(ii)  $\{(c = 5, k = 5, \lambda = 3, \alpha = 3, p = 0.2), (c = 5.5, k = 5.5, \lambda = 3.5, \alpha = 4, p = 0.7)\}$ .

**Step (3):** Type I censoring samples are implemented using a fixed censoring time. All the observations which are greater than  $T$  are declared as censored ones. Different fixed censoring times are chosen as  $T = 0.7$  in case of non-informative prior and  $T = 0.5$  in case of informative prior.

**Step (4):** For each of the 10000 samples, the Bayes estimates are obtained using MCMC procedure using Open BUGS software.

The posterior descriptive summaries results are displayed in Tables 1 and 2. These tables contain posterior mean, standard deviation (S.D), MC error( the computational accuracy of the mean), median, 2.5% (an approximation of the lower endpoint of 95% credible interval), and 97.5% ( an approximation of the upper endpoint of the 95% credible interval). Also, the numerical results can be represented through some Figures from 1 to 8. The following conclusions can be observed on the properties of estimated parameters from the BXIIBX distribution for non-informative and informative priors.

**(i) Firstly in case of non-informative priors**

History plot is one of the oldest methods of qualitatively assessing MCMC sampler performance. As an example for  $n = 15$ ; history plots for unknown parameters  $c, k, \alpha, \lambda$  and  $p$  from two chains are represented in Figure 1. The plots look like a horizontal band with no long upward or downward trends and two chains are well mixed so these are indicators to convergence.

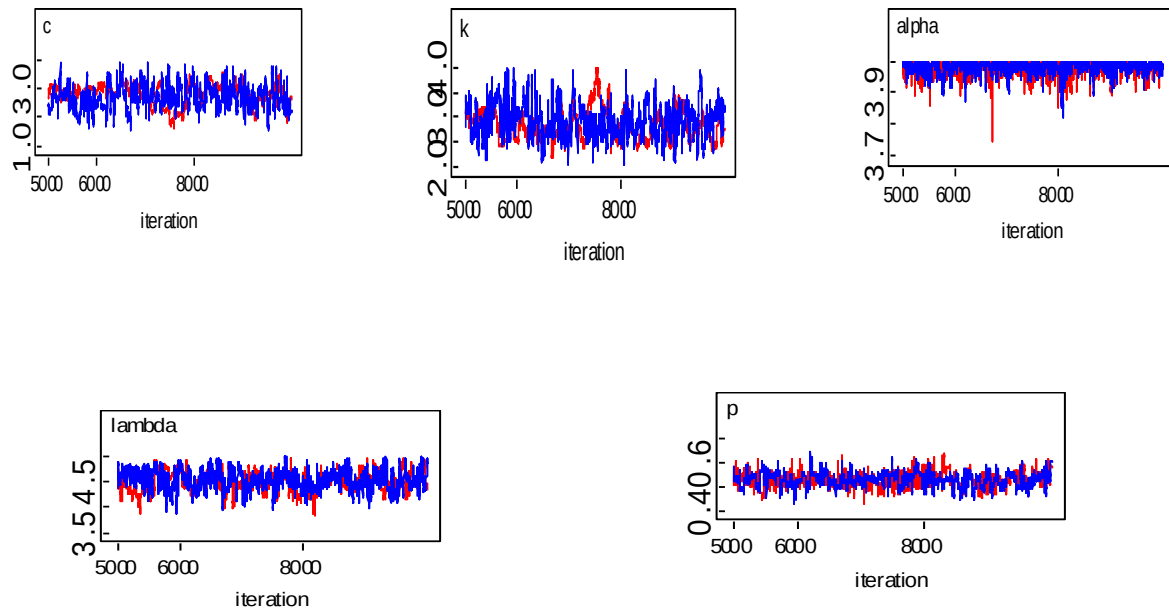
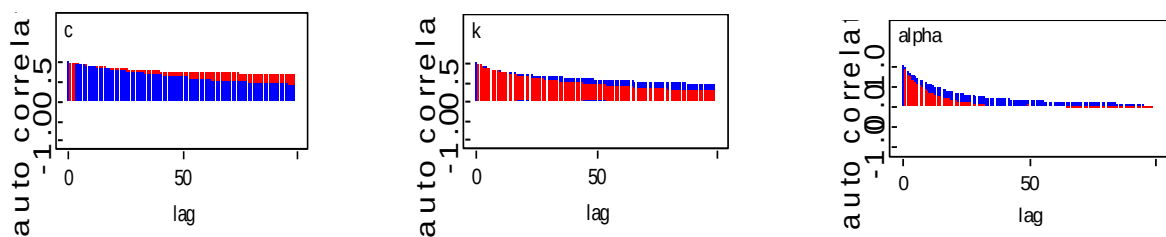


**Figure 1:** History plot for  $c, k, \alpha, \lambda$  and  $p$  at  $n=15$

The autocorrelation plots for two chains are represented in Figure 2. Usually in MCMC output, the lag 1 autocorrelation is positive, and the autocorrelation decreases as lag increases, that are indicating that the samples are independent. It's clear from Figure 2 that the autocorrelation of parameters  $\alpha$  and  $p$  are approximately ignored, in the sense that, the samples are independent. While autocorrelation of parameters  $c, k, \lambda$  are highly, i.e. the samples are dependent.



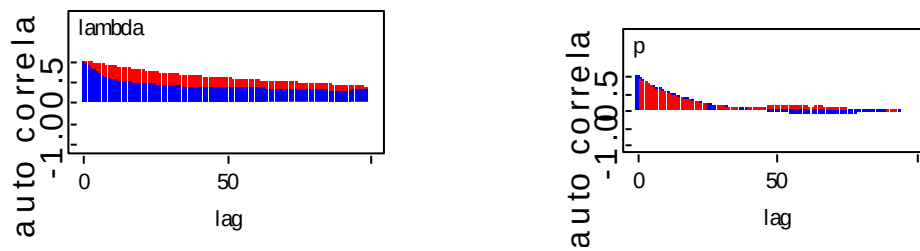


**Figure 2:** Autocorrelation plot for  $c, k, \alpha, \lambda$  and  $p$  at  $n = 15$

The ratio for Gelman –Rubin convergence is normalized to equal one and if Rubin is not equal one it means failure to convergence. It is clear from Figure 3 that the plots of chains for parameters  $c, k, \alpha, \lambda$  and  $p$  approach to 1, this is indicative of convergence.

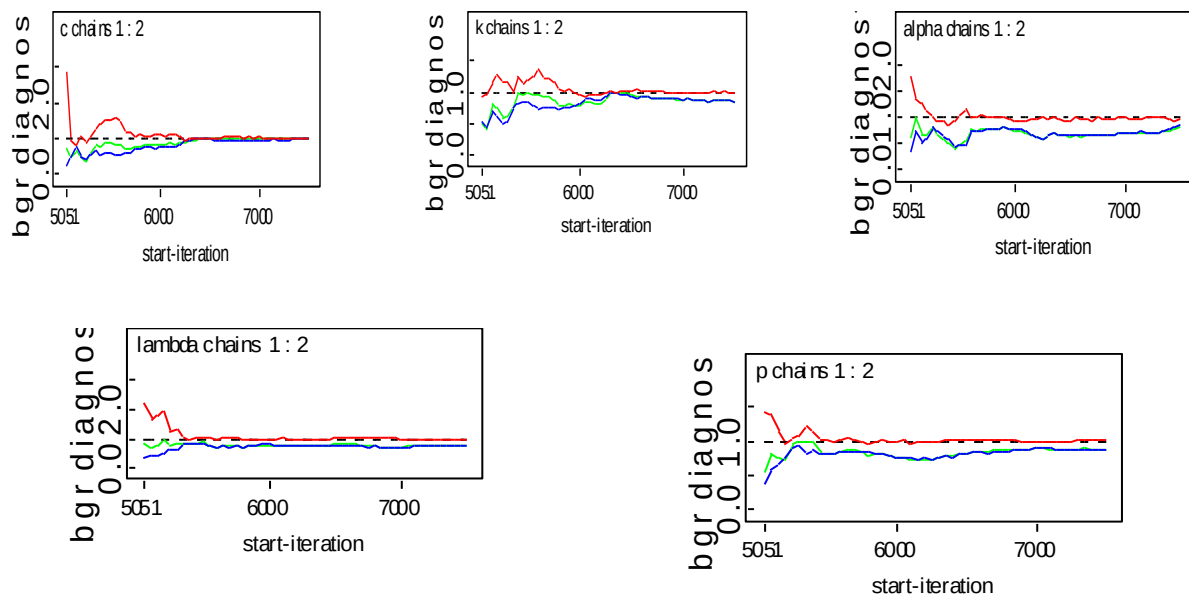
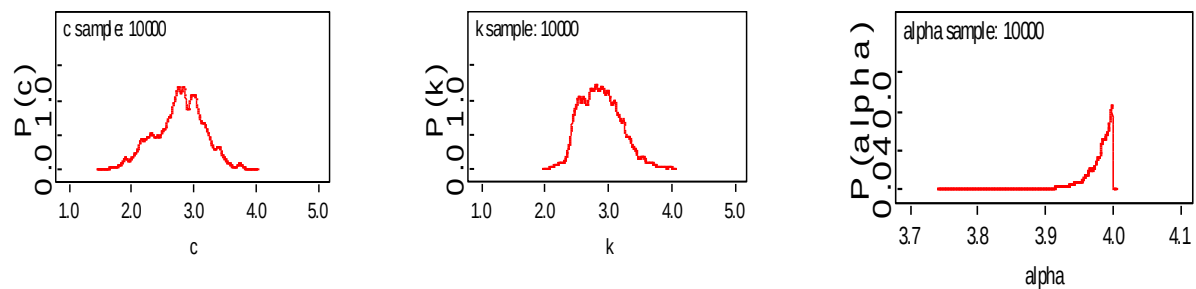


**Figure 3:** Gelman-Rubin convergence plot of  $c, k, \alpha, \lambda$  and  $p$  at  $n = 15$

The approximate marginal posterior density of each parameter can be displayed in Figure 4.



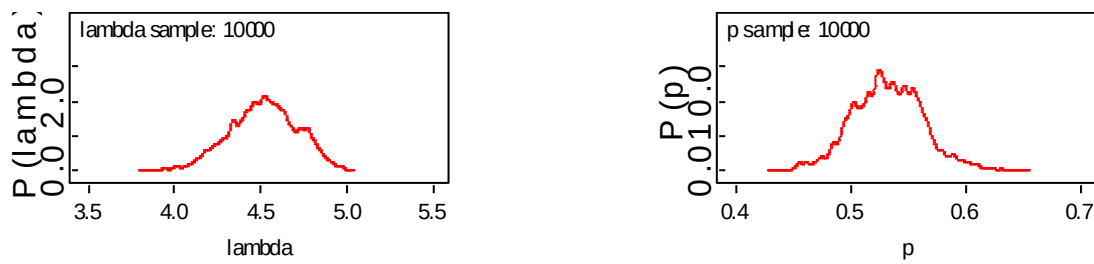


**Figure 4:** Estimated posterior density plot of  $c, k, \alpha, \lambda$  and  $p$  at  $n=15$

### **(ii) Secondly in case of informative priors**

As an example for  $n = 15$ ; history plots for unknown parameters  $c, k, \alpha, \lambda$  and  $p$  are displayed in Figure 5. As seen from Figure 5, the plots look like a horizontal band with no long upward or downward trends and two chains are well mixed so these are indicators to convergence.

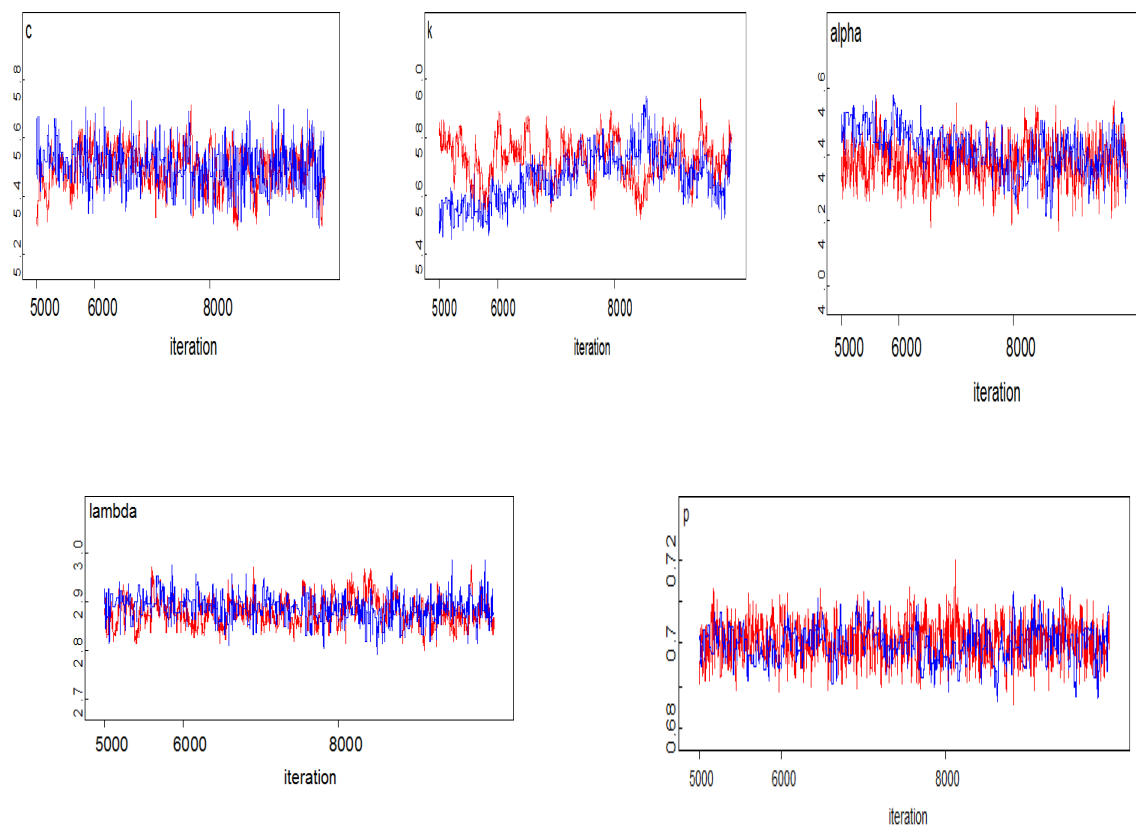


**Figure 5:** History plot for  $c, k, \alpha, \lambda$  and  $p$  at  $n= 100$  for first simulated data in case of informative prior

The auto-correlation for parameters  $c, k, \alpha, \lambda$  and  $p$  in Figure 6 displays that the correlation is approximately omitted, in the sense that, the samples are independent.

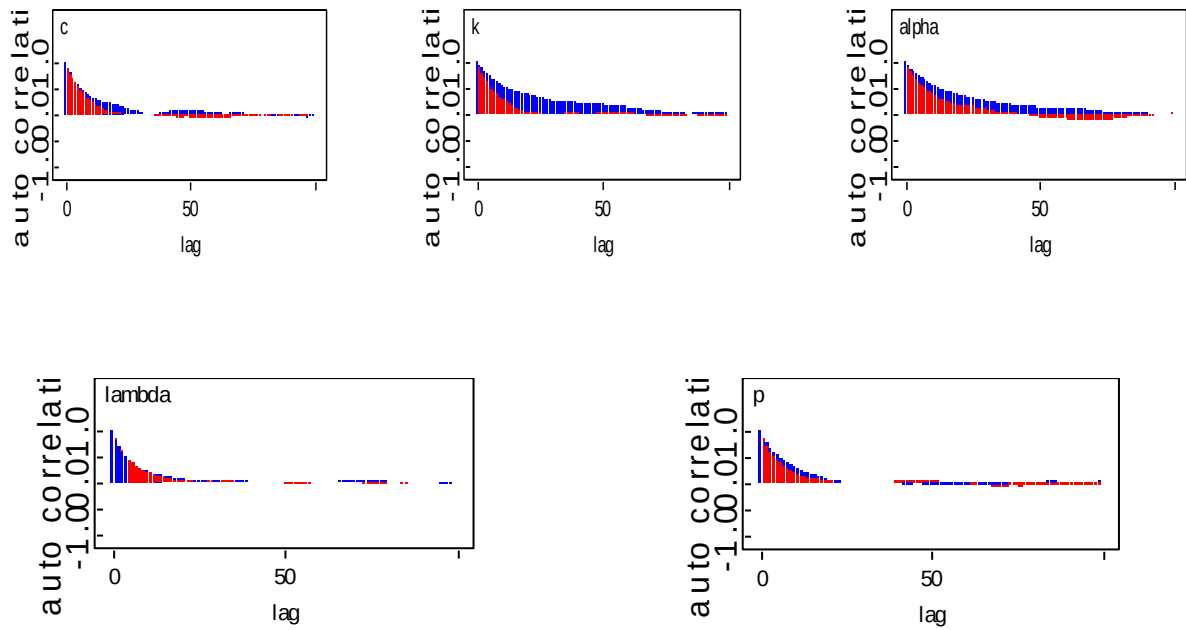


**Figure 6:** Autocorrelation plots for  $c, k, \alpha, \lambda$  and  $p$  at  $n=100$  for simulated data in case of informative prior

Figure 7 gives plots of the ratio for Gelman–Rubin convergence for parameters  $c, k, \alpha, \lambda$  and  $p$  of the two chains. Plots show that the chains approximate to equal one, i.e convergence is available.

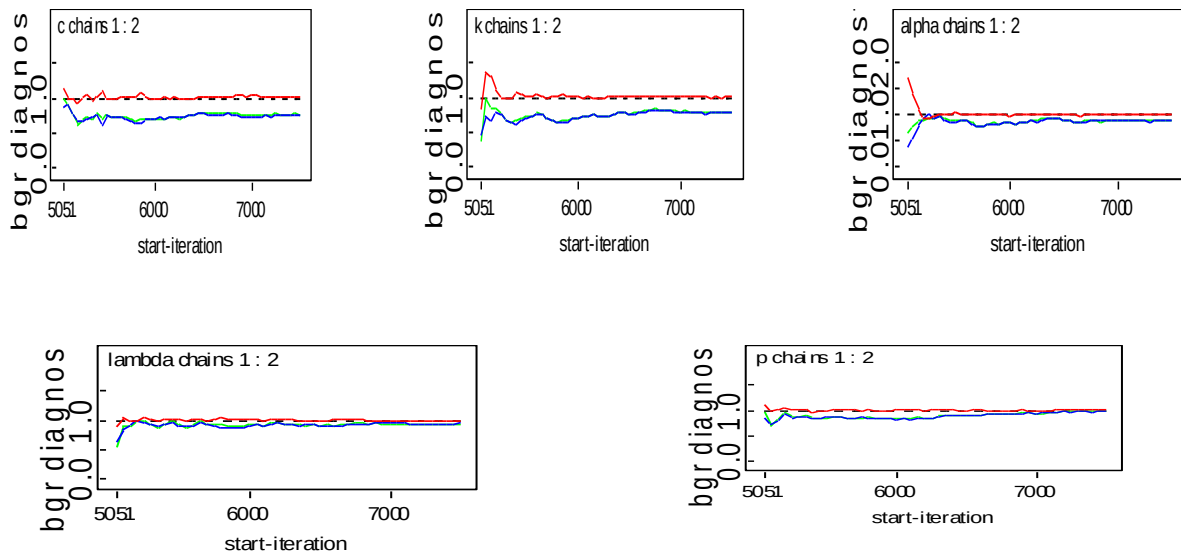


**Figure 7:** Gelman–Rubin convergence plot for  $c, k, \alpha, \lambda$  and  $p$  at  $n=100$  for simulated data in case of informative prior

The approximate marginal posterior density of each parameter can be displayed in Figure 8

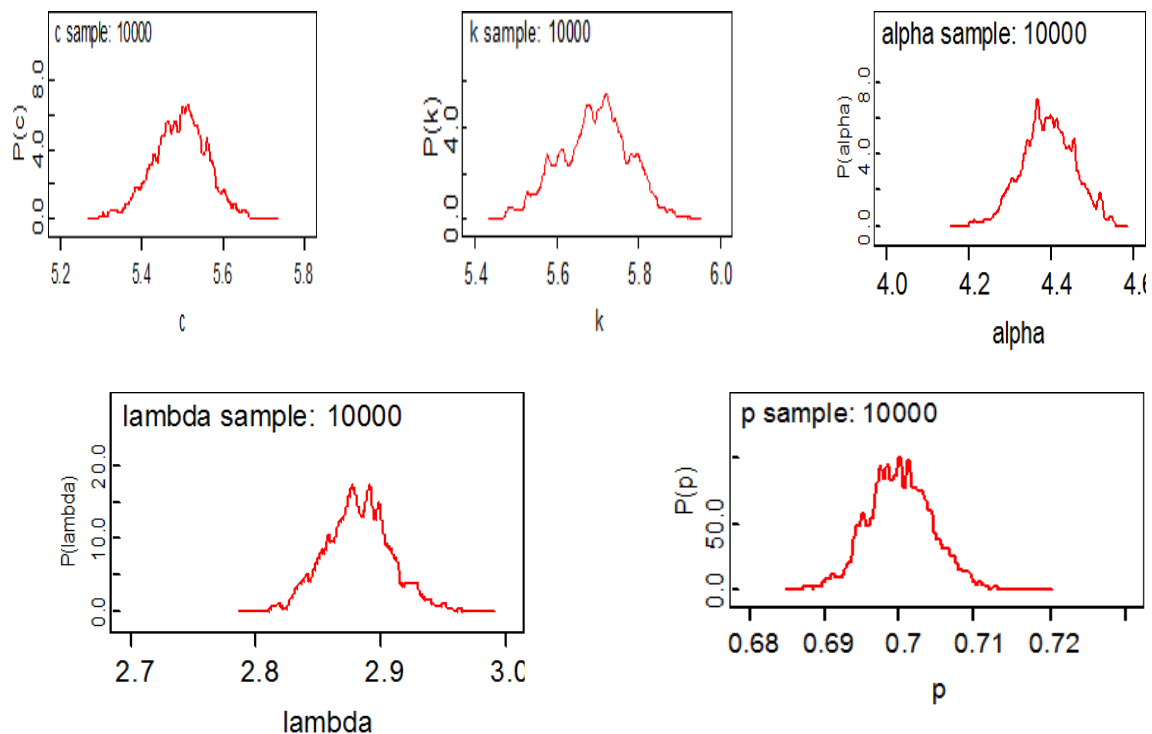


**Figure 8:** Estimated posterior density plot of  $c, k, \alpha, \lambda$  and  $p$  at  $n=100$  for simulated data in case of informative prior

## 5. Concluding Remarks

In this article, Bayesian analysis of the mixed Burr Type XII and Burr Type X distribution was developed under Type-I censored samples. The simulation study is performed for two chains with different initial values in case of non-informative and informative priors. Bayesian estimates for unknown parameters  $c, k, \alpha, \lambda$  and  $p$  of BXIIBX model are calculated numerically using Open BUGS software. Bayesian analysis is conducted to estimate posterior mean, standard deviation, median, MC error, 95% asymptotic confidence, history iteration, posterior density, Gelman-Rubin convergence and autocorrelation for unknown parameters  $c, k, \alpha, \lambda$  and  $p$  for different sample sizes. The Bayesian estimators for unknown parameters  $c, k, \alpha, \lambda$  and  $p$  have good statistical properties in each of case (non-informative and informative priors). Further, the MC error is less than 5% of the sample standard deviation.

Table 1: Posterior summaries in case of non-informative priors

| Sample size | $T$ | parameter         | Posterior mean | S.D     | MC error | Median  | 2.50%   | 97.50%  |
|-------------|-----|-------------------|----------------|---------|----------|---------|---------|---------|
| 5           |     | $\tilde{c}$       | 1.16300        | 0.13020 | 0.00888  | 1.13000 | 1.00500 | 1.49600 |
|             |     | $\tilde{k}$       | 2.25100        | 0.20520 | 0.01377  | 2.20300 | 2.00800 | 2.72200 |
|             |     | $\tilde{\alpha}$  | 3.77800        | 0.19500 | 0.00987  | 3.83300 | 3.26800 | 3.99400 |
|             |     | $\tilde{\lambda}$ | 3.09000        | 0.08689 | 0.00458  | 3.06400 | 3.00300 | 3.32000 |
|             |     | $\tilde{p}$       | 0.58770        | 0.08661 | 0.00517  | 0.58570 | 0.42750 | 0.75380 |
| 15          |     | $\tilde{c}$       | 2.76500        | 0.39730 | 0.02552  | 2.79200 | 1.93500 | 3.49300 |
|             |     | $\tilde{k}$       | 2.87500        | 0.32260 | 0.02041  | 2.85600 | 2.34800 | 3.56800 |
|             |     | $\tilde{\alpha}$  | 3.97900        | 0.02179 | 0.00083  | 3.98500 | 3.92100 | 3.99900 |
|             |     | $\tilde{\lambda}$ | 4.51300        | 0.19760 | 0.01130  | 4.51800 | 4.11600 | 4.87200 |
|             |     | $\tilde{p}$       | 0.53210        | 0.03078 | 0.00156  | 0.53160 | 0.47210 | 0.59700 |
| 25          |     | $\tilde{c}$       | 3.94700        | 0.08772 | 0.00715  | 3.97400 | 3.56300 | 3.99900 |
|             |     | $\tilde{k}$       | 2.93900        | 0.13650 | 0.01140  | 2.94900 | 2.45800 | 3.17300 |
|             |     | $\tilde{\alpha}$  | 3.99200        | 0.00783 | 0.00042  | 3.99400 | 3.97000 | 4.00000 |
|             |     | $\tilde{\lambda}$ | 4.94000        | 0.06343 | 0.00484  | 4.96500 | 4.78300 | 4.99800 |
|             |     | $\tilde{p}$       | 0.51790        | 0.01997 | 0.00150  | 0.51570 | 0.48280 | 0.56060 |
| 30          | 0.7 | $\tilde{c}$       | 3.70500        | 0.26440 | 0.02218  | 3.84100 | 3.39800 | 3.99800 |
|             |     | $\tilde{k}$       | 2.78100        | 0.18320 | 0.00047  | 2.69800 | 2.58100 | 3.18200 |
|             |     | $\tilde{\alpha}$  | 3.99200        | 0.00622 | 0.00038  | 3.99400 | 3.97600 | 4.00000 |
|             |     | $\tilde{\lambda}$ | 4.94300        | 0.04635 | 0.00318  | 4.95500 | 4.83400 | 4.99800 |
|             |     | $\tilde{p}$       | 0.53790        | 0.01755 | 0.00131  | 0.53860 | 0.50000 | 0.57520 |
| 50          |     | $\tilde{c}$       | 3.99700        | 0.00252 | 0.00021  | 3.99700 | 3.99200 | 4.00000 |
|             |     | $\tilde{k}$       | 3.99700        | 0.00188 | 0.00016  | 3.99700 | 3.99400 | 4.00000 |
|             |     | $\tilde{\alpha}$  | 3.97800        | 0.02646 | 0.00222  | 3.98800 | 3.92000 | 3.99500 |
|             |     | $\tilde{\lambda}$ | 4.23300        | 0.31070 | 0.02610  | 4.54000 | 3.82500 | 4.54000 |
|             |     | $\tilde{p}$       | 0.51770        | 0.22450 | 0.01883  | 0.65000 | 0.28170 | 0.83980 |
| 100         |     | $\tilde{c}$       | 3.99700        | 0.00252 | 0.00021  | 3.99700 | 3.99200 | 4.00000 |
|             |     | $\tilde{k}$       | 3.99700        | 0.00177 | 0.00015  | 3.99800 | 3.99400 | 3.99900 |
|             |     | $\tilde{\alpha}$  | 3.80800        | 0.06530 | 0.00548  | 3.86600 | 3.74100 | 3.88000 |
|             |     | $\tilde{\lambda}$ | 4.52200        | 0.45090 | 0.03786  | 4.86200 | 4.07200 | 4.99700 |
|             |     | $\tilde{p}$       | 0.65530        | 0.03853 | 0.00322  | 0.65100 | 0.61460 | 0.71460 |

**Table2: Posterior summaries in case of informative priors**

| Sample size | $T$ | parameter       | Posterior mean | S.D     | MC error | Median  | 2.50%   | 97.50%  |
|-------------|-----|-----------------|----------------|---------|----------|---------|---------|---------|
| 5           |     | $\hat{c}$       | 0.04890        | 0.00624 | 0.00036  | 0.04860 | 0.03759 | 0.06193 |
|             |     | $\hat{k}$       | 0.00552        | 0.00229 | 0.00009  | 0.00517 | 0.00197 | 0.01114 |
|             |     | $\hat{\alpha}$  | 0.03190        | 0.00525 | 0.00032  | 0.03168 | 0.02277 | 0.04205 |
|             |     | $\hat{\lambda}$ | 0.15110        | 0.02923 | 0.00199  | 0.15240 | 0.09355 | 0.20660 |
|             |     | $\hat{p}$       | 0.26060        | 0.11160 | 0.00846  | 0.22240 | 0.11200 | 0.51920 |
| 15          |     | $\hat{c}$       | 0.20610        | 0.01299 | 0.00064  | 0.20580 | 0.18130 | 0.23190 |
|             |     | $\hat{k}$       | 0.16240        | 0.01252 | 0.00062  | 0.16210 | 0.13890 | 0.18820 |
|             |     | $\hat{\alpha}$  | 0.05732        | 0.00756 | 0.00036  | 0.05715 | 0.04321 | 0.07299 |
|             |     | $\hat{\lambda}$ | 0.21510        | 0.03616 | 0.00197  | 0.21500 | 0.14640 | 0.28810 |
|             |     | $\hat{p}$       | 0.85840        | 0.02386 | 0.00114  | 0.85960 | 0.80720 | 0.90240 |
| 25          |     | $\hat{c}$       | 0.36830        | 0.01862 | 0.00096  | 0.36760 | 0.33400 | 0.40730 |
|             |     | $\hat{k}$       | 0.33710        | 0.01778 | 0.00082  | 0.33700 | 0.30280 | 0.37170 |
|             |     | $\hat{\alpha}$  | 0.33520        | 0.01777 | 0.00095  | 0.33510 | 0.30190 | 0.36900 |
|             |     | $\hat{\lambda}$ | 1.68800        | 0.08833 | 0.00363  | 1.68700 | 1.51600 | 1.86900 |
|             |     | $\hat{p}$       | 0.70840        | 0.01926 | 0.00096  | 0.70850 | 0.67170 | 0.74700 |
| 30          | 0.5 | $\hat{c}$       | 0.40550        | 0.02045 | 0.00104  | 0.40480 | 0.36700 | 0.44880 |
|             |     | $\hat{k}$       | 0.37010        | 0.01928 | 0.00090  | 0.36950 | 0.33350 | 0.40800 |
|             |     | $\hat{\alpha}$  | 0.34840        | 0.02199 | 0.00102  | 0.34750 | 0.30510 | 0.39400 |
|             |     | $\hat{\lambda}$ | 1.03100        | 0.07057 | 0.00378  | 1.03000 | 0.89570 | 1.16900 |
|             |     | $\hat{p}$       | 0.82790        | 0.01460 | 0.00058  | 0.82800 | 0.79850 | 0.85560 |
| 50          |     | $\hat{c}$       | 1.14300        | 0.02747 | 0.00156  | 1.14400 | 1.08800 | 1.19500 |
|             |     | $\hat{k}$       | 1.11500        | 0.02733 | 0.00122  | 1.11500 | 1.06300 | 1.17100 |
|             |     | $\hat{\alpha}$  | 0.97370        | 0.02914 | 0.00144  | 0.97290 | 0.91760 | 1.03100 |
|             |     | $\hat{\lambda}$ | 3.47700        | 0.09848 | 0.00433  | 3.47800 | 3.28100 | 3.66700 |
|             |     | $\hat{p}$       | 0.79590        | 0.00812 | 0.00035  | 0.79600 | 0.78000 | 0.81140 |
| 100         |     | $\hat{c}$       | 5.49500        | 0.06656 | 0.00337  | 5.49900 | 5.35900 | 5.62400 |
|             |     | $\hat{k}$       | 5.69300        | 0.08422 | 0.00627  | 5.69800 | 5.52600 | 5.84400 |
|             |     | $\hat{\alpha}$  | 4.39400        | 0.06479 | 0.00384  | 4.39300 | 4.27000 | 4.52300 |
|             |     | $\hat{\lambda}$ | 2.88300        | 0.02671 | 0.00129  | 2.88200 | 2.83200 | 2.93800 |
|             |     | $\hat{p}$       | 0.70010        | 0.00429 | 0.00019  | 0.70010 | 0.69170 | 0.70880 |

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## On Stationary Distributions of a TPM Arising from NHMC-RW

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**Abstract.** This study seeks to apply the method of state reduction to determine closed form expressions for the steady state probabilities of a general regular TPM arising from NHMC-RW, on the non-negative integers, between two segregating barriers. At the barriers, the particle is either staying on or reflected back to the system with different probabilities. Several qualitative explicit formulas for the results, depending on the relationship between transition and segregating probabilities, are obtained. An efficient algorithm for computing steady state probability vector of the NHMC-RW is given. We illustrate the results with simple examples.

**Keywords.** Markov chain, random walk, segregating barriers, regular MC, state reduction, steady state probability.

AMS subject classifications (2010) 60G40, 60G50, 60J10, 60J80.

### 1. Introduction

A non -homogeneous Markov chain random walk (NHMC-RW) model,

$\{X_k^{(N+1)}, k = 0, 1, \dots\}$  on state space  $S_{N+1} \equiv \{0, 1, \dots, N\}$ , with one- step transition

Probabilities  $\{p_{ij} = p(X_{k+1}^{(N+1)} = j | X_k^{(N+1)} = i)\}$ ,  $i, j \in S_{N+1}$  and transition probabi-

lity matrix (TPM)  $\mathbf{M} = (p_{ij})$ ,

$$\mathbf{M} = \begin{pmatrix} p_{00} & p_{01} & 0 & \cdots & 0 & 0 \\ p_{10} & p_{11} & p_{12} & 0 & \cdots & 0 \\ 0 & p_{21} & p_{22} & p_{23} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & p_{N-1, N-2} & p_{N-1, N-1} & p_{N-1, N} \\ 0 & 0 & \cdots & 0 & p_{NN-1} & p_{NN} \end{pmatrix} \equiv \mathbf{M}_{N+1}, \quad (\text{I})$$

is one of the most well-known and studied stochastic processes as well as a common tool in applied mathematics. A Markov chain random walk (MC-RW) model is a process moves from state to state in a manner that depends only on

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the last state and an associated set of transition probabilities from that state to the other states. MC-RW model received considerable attention from the scientific society since it found a wide range of distinct applications in various theoretical and applied fields such as in medicine, in biology "where partially absorbing barriers give a natural model for a wide variety of phenomena", in physics "as a simplified model of Brownian motion", in economics "to model share prices and their derivatives", in ecology "to describe individual animal movements and population dynamics", in statistics "to analyze sequential test procedures", and in computer science "to estimate the size of the world wide web using randomized algorithms" to name only few fields of applications cf., as references to this material we recommend books (Parzen (1962), Iosifescu (1980), and Ibe (2009)), and articles (Bhat (1961), Percus (1985), Grimmett (1987), et al, El-Shehawey (1994), Lengyel (2009), and Lorek (2017)), and the references therein. Despite its long history, novel aspects continue to surface. In particular, a number of very interesting problems of our everyday life such as stock market, queues and reliability theories can be expressed by means of MC-RW with two barriers. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. For comprehensive treatments of the MC-RW models and their applications, we refer to Asmussen (2003), Bini et al, (2005), Ching and Ng (2006), and Beichelt (2016).

As shown in (I), the NHMC-RW proceeds according to the following assumptions:

- a- A particle starting from the initial state  $i$ ,  $i \in S_{N+1} = \{0, 1, \dots, N\}$ .
- b- The transition probabilities between the states are dependent on the particle position (changing with each time unit).
- c- From the current state  $k$ ,  $k \in S_{N+1} \setminus \{0, N\}$  the one-step transition can only allowed to one of the states  $k-1$ ,  $k$  or  $k+1$  with respective probabilities  $p_{kk-1}$ ,  $p_{kk}$  and  $p_{kk+1}$ , with  $p_{kk} = 1 - p_{kk+1} - p_{kk-1}$ .
- d- If the particle reaches to the boundary states  $k=0$  or  $k=N$ , at the next move it is segregated or reflects (to the states  $k=1$  and  $k=N-1$ , respectively) with respective probabilities  $p_{00}$  and  $p_{01} = 1 - p_{00}$  at 0, and  $p_{NN}$ ,  $p_{NN-1} = 1 - p_{NN}$  at  $N$ ,  $0 \leq p_{00}, p_{NN} \leq 1$ .

An interpretation of NHMC-RW is a birth and death process with a bounded population. The term birth and death process results from the application of these processes to describing the development in time of biological populations. In this context, assuming that  $x_n$  is the number of individuals of a population at time  $n$ , and the population does not increase or decrease by more than one individual per unit time. If  $x_n = k$ , then  $x_{n+1}$  is either  $k-1$  (death),  $k$  (no birth or death) or  $k+1$  (birth), except for the boundary state  $k=0$  (or  $k=N$ ), where only birth (death) can occur. When  $x_n = k < N$ , the probability of a birth is  $p_{kk+1}$ .

A death occurs with probability  $p_{kk-1}$ , and  $p_{kk} = 1 - p_{kk+1} - p_{kk-1}$  is the probability that the process stays for a time unit at state  $k$ . A birth and death cannot occur simultaneously. Here,  $x_n$  is a random variable represents the state of a particle at time  $n$ .

Physically, this corresponds to the situation where upon reaching the boundary state 0 (or  $N$ ), the particle is either remains at 0 (or  $N$ ) with partially-segregating probability  $p_{00}$  (or  $p_{NN}$ ), or turned back to the system with partially-reflecting probability  $p_{01}$  (or  $p_{NN-1}$ ), and reduces to the classical NHMC-RW with segregating, reflecting and combination of segregating and reflecting barriers. Such model is a generalization of many particular MC-RW problems, depending on appropriate choices of the transition and the segregating boundary probabilities.

There is a one-to-one relationship between NHMC-RW and the TPM  $\mathbf{M} = (p_{ij})$ . Thus, the calculation of various characteristics of the NHMC-RW model was specified by the TPM  $\mathbf{M}$ . These characteristics include the distribution of a NHMC-RW at the moment of the first visit to a subset of its state space, the mean time spent at given states until such visit, the invariant distribution, the fundamental matrices for both transient and regular NHMC-RW, the covariance matrix, and many others.

One of the key properties of a NHMC-RW is its stationary distribution. The idea of stationary (steady state) distribution is that we have reached (or converging to) a point in the process where the distributions will no longer change. In recent years, widespread attention has been given to the determination of stationary distributions of Markov chains. A variety of methods have been suggested and implemented. Paige et al, (1975) presented a comprehensive survey of eight different algorithms involving a variety of procedures including the use of generalized inverses, rank reduction, least squares and power methods. Their recommendation was a direct method that involved transforming the singular set of stationary equations into a non-singular system using a rank one modification followed by Gaussian elimination with row pivoting.

Assume that the transition probabilities  $\{p_{ii-1} \text{ for } 0 < i \leq N, p_{ii+1} \text{ for } 0 \leq i < N\}$  are positive. Then the NHMC-RW is irreducible. We refer to Asmussen (2003), and Brémaud (2017) for basic concepts related to Markov chains with a finite state space: irreducibility, aperiodicity, regularity, steady state, steady state probability, limiting probability, stationary distribution, etc.

Iterative techniques and approximation methods have been surveyed by Koury et al, (1984). Sheskin (1985), presented a numerical algorithm to find the stationary probabilities of a finite irreducible MC. An equivalent related algorithm was developed by Grassmann et al, (1985). Both of the two

techniques appear to be, in effect, modifications of Gaussian elimination. Kohlas (1986), developed the numerical algorithm for finding the mean first passage times (MFPTs) (or absorption probabilities) in Markov renewal processes. It is known that the **absorption probability** of given chain can be related to the stationary distribution of some other ergodic chain. The relation is given via so-called **Siegmund duality** (cf., Lorek (2017)).

The main thrust of the present work is to use the method of state reduction to explore closed form expressions for the steady state probabilities of a general regular TPM  $\mathbf{m}_{N+1}$  arising from the NHMC-RW  $\{X_k^{(N+1)}, k = 0, 1, \dots\}$  on the state space  $\{0, 1, \dots, N\}$ , governed by two un-symmetric imperfect segregating barriers. Relationship between the steady state probability vectors of two NHMC-RW models in the two state spaces,  $\{0, 1, \dots, N\}$  and  $\{0, 1, \dots, N-1\}$  are obtained as well as some very simple forms have been immediately provided leading to explicit formulas for some special cases. Computational algorithm for computing steady state (stationary) probabilities for the NHMC-RW is introduced. There is not, to our knowledge, the derivation apparently covered by the literature.

## 2. Computation Steady State Distribution Vector

Kemeny and Snell (1976) (see also Hunter (1991) and (2016)) showed that a partitioned MC can be reduced in size by performing matrix operations to produce a smaller MC.

### 2.1 General procedure for computing the stationary probabilities

In this subsection we introduce a general procedure for computing the steady state probability vector. State reduction has two steps (cf. Sheskin (2011)).

#### ➤ First step: Matrix reduction

The matrix reduction step successively partitions the TPM  $\mathbf{m}_{N+1}$  for a NHMC-RW model. Each partition of  $\mathbf{m}_{N+1}$  produces four sub-matrices: a single element in the upper left hand corner, a row vector, a square matrix, and a column vector.

- 1- When combined, the four sub-matrices create a reduced TPM that also represents a regular NHMC-RW model. The number of states in the reduced matrix is one less than the number in the original matrix.
- 2- The reduced matrix equals the sum of the square matrix and the product of the column vector, the reciprocal of one minus the single element, and the row vector.
- 3- Each reduced matrix is partitioned in the same manner as the original matrix, creating a sequence of successively smaller reduced matrices. Each

reduced matrix overwrites its predecessor, beginning with the diagonal element in the upper left hand corner.

- 4- We terminate the matrix reduction when the last reduced matrix has two states (i.e., the reduction process continues until we reach the state space with two states).

➤ **Second step: Back substitution**

- 1- The back substitution begins with the last column vector from the final two states reduced matrix to express the steady state probability for the second state as a constant times of the steady state probability for the first state.
- 2- Use the last column vector from the next larger reduced matrix to express the steady state probability for the third state as a function of the steady state probabilities for the first two states. In this equation, we substitute the second state's steady state probability, expressed in terms of the steady state probability for the first state.
- 3- We then solve the equation to find the steady state probability for the third state as a constant times of the steady state probability for the first.
- 4- We repeat this recursive procedure, using the last column vector from each reduced matrix next larger in size, until the steady state probabilities for all states have been expressed as constants times of the steady state probability for the first state.
- 5- Solve the normalizing equation for the original NHMC-RW to find the steady state probability for the first state.
- 6- All other steady state probabilities are obtained by multiplying the constants previously found by the steady state probability for the first state.

**Remark.** To improve numerical accuracy, subtractions may be eliminated by replacing one minus the single element by the sum of the remaining elements in the top row, i.e., round off error can be reduced when subtractions are avoided.

## 2.2 Computation of the stationary probabilities for NHMC-RW on $S_{N+1}$

To find the stationary distributions of a regular NHMC-RW,  $\{X_k^{(N+1)}, k = 0, 1, \dots\}$  with state space  $S_{N+1} \equiv \{0, 1, \dots, N\}$ , and transition probabilities  $p_{ij} = p(X_{k+1} = j | X_k = i)$ , we start with the TPM  $\mathbf{m}_{N+1} = (p_{ij})$ , associated with the NHMC-RW and reduce the state space by one state at a time. Thus, in steps the state space is partitioned as: the first step is  $S_N \equiv \{0, 1, \dots, N\} = S_N \cup \{N\}$ , the second step is  $S_N = S_{N-1} \cup \{N-1\}$ , ..., and continue until we reach the state space with two states  $S_2 = \{0, 1\}$ . Once we get to two states we expand the state space one state at a time until we return to the final set of  $N+1$  states,  $S_{N+1} \equiv \{0, 1, \dots, N\}$ .

We concentrate on the sequential state reduction process at first by starting with  $N + 1$  states  $0, 1, \dots, N$  and initially reducing the state space to  $0, 1, \dots, N - 1$ .

We can partition  $\mathbf{M}_{N+1}$  into four sub-matrices called  $\mathbf{T}_N^{(N+1)}$ ,  $\mathbf{U}_N^{(N+1)}$ ,  $\mathbf{V}_N^{(N+1)}$  and  $\mathbf{J}$ , as shown below:

$$\mathbf{M}_{N+1} = \left( \begin{array}{cccc|c} p_{00} & p_{01} & 0 & \cdots & 0 \\ p_{10} & p_{11} & p_{12} & 0 & \cdots \\ 0 & p_{21} & p_{22} & p_{23} & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots \\ 0 & \vdots & \ddots & p_{N-1N-2} & p_{N-1N-1} \\ \hline 0 & 0 & \cdots & 0 & p_{NN-1} \end{array} \middle| \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ p_{NN} \end{array} \right) = \left( \begin{array}{c|c} \mathbf{T}_N^{(N+1)} & \mathbf{V}_N^{(N+1)} \\ \hline \mathbf{U}_N^{(N+1)} & \mathbf{J} \end{array} \right), \quad (2.1)$$

where,  $\mathbf{J} = p_{NN}$  is the single element in the lower right corner of  $\mathbf{M}_{N+1}$ ,

$$\mathbf{U}_N^{(N+1)} = (0 \quad \cdots \quad 0 \quad p_{NN-1}), \quad \mathbf{V}_N^{(N+1)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_{N-1N} \end{pmatrix}, \quad \text{with the superscript } N+1 \text{ denoting that}$$

they are from the  $(N+1)^{\text{th}}$  row or  $(N+1)^{\text{th}}$  column of the matrix  $\mathbf{M}_{N+1}$  and the subscript  $N$  that they are vectors of length  $N$ . Similarly, we use the superscript  $(N+1)$  in the sub-matrices  $\mathbf{T}_N^{(N+1)}$  to denote that they are sub-matrices of the TPM  $\mathbf{M}_{N+1}$  associated with an  $N+1$ -state NHMC-RW and the subscript  $N$  to denote that the matrix is of order  $N \times N$ :

$$\mathbf{T}_N^{(N+1)} = \begin{pmatrix} p_{00} & p_{01} & 0 & 0 \\ p_{10} & p_{11} & p_{12} & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & p_{N-1N-2} & p_{N-1N-1} \end{pmatrix} \equiv \mathbf{M}_N.$$

Assume that the stationary probability vector of the  $(N+1)$ -state NHMC-RW with state space  $S_{N+1} = \{0, 1, \dots, N\}$  and TPM  $\mathbf{M}_{N+1}$ , is denoted by

$$\boldsymbol{\pi} \equiv \boldsymbol{\pi}_{N+1}^{(N+1)} = \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \cdots \quad \pi_{N-1}^{(N+1)} \quad \pi_N^{(N+1)} \right) \equiv (\pi_0 \quad \pi_1 \quad \cdots \quad \pi_{N-1} \quad \pi_N),$$

where the limiting probabilities  $\{\pi_j\}$  are defined as

$$\pi_j = \lim_{n \rightarrow \infty} p \left( X_n^{(N+1)} = j \middle| X_0^{(N+1)} = i \right) = \lim_{n \rightarrow \infty} p_{ij}^{(n)} \quad \text{for } j \in S_{N+1}.$$

Partition the stationary probability vector

$$\boldsymbol{\pi}_{N+1}^{(N+1)} = \left( \boldsymbol{\omega}_N^{(N)} \quad \pi_N^{(N+1)} \right),$$

where,

$$\omega_N^{(N)} = \begin{pmatrix} \omega_0^{(N)} & \omega_1^{(N)} & \dots & \omega_{N-1}^{(N)} \end{pmatrix} = \begin{pmatrix} \pi_0^{(N+1)} & \pi_1^{(N+1)} & \dots & \pi_{N-1}^{(N+1)} \end{pmatrix}.$$

From the stationary equations

$$\pi_{N+1}^{(N+1)} = \pi_{N+1}^{(N+1)} \mathbf{M}_{N+1}, \quad (2.2)$$

we have

$$\omega_N^{(N)} \mathbf{T}_N^{(N+1)} + \pi_N^{(N+1)} \mathbf{U}_N^{(N+1)} = \omega_N^{(N)}, \quad (2.3)$$

and

$$\omega_N^{(N)} \mathbf{V}_N^{(N+1)} + \pi_N^{(N+1)} p_{NN} = \pi_N^{(N+1)}. \quad (2.4)$$

From equation (2.4), we have  $\pi_N^{(N+1)} (1 - p_{NN}) = \omega_N^{(N)} \mathbf{V}_N^{(N+1)}$  or

$$\pi_N^{(N+1)} = \frac{\omega_N^{(N)} \mathbf{V}_N^{(N+1)}}{1 - p_{NN}} = \frac{\omega_N^{(N)} \mathbf{V}_N^{(N+1)}}{\mathbf{U}_N^{(N+1)} \mathbf{e}_N}. \quad (2.5)$$

Thus, we can express  $\pi_N^{(N+1)}$  in terms of  $\omega_{N-1}^{(N)}$  and the transition probabilities associated with  $\mathbf{M}_{N+1}$  as

$$\pi_N^{(N+1)} = \frac{p_{N-1N}}{p_{NN-1}} \omega_{N-1}^{(N)} = \frac{p_{N-1N}}{1 - p_{NN}} \omega_{N-1}^{(N)}. \quad (2.6)$$

From (2.2) in element form  $\pi_j^{(N+1)} = \sum_{i=0}^N \pi_i^{(N+1)} p_{ij}$  for  $j \in S_{N+1}$ , we can express

$\pi_N^{(N+1)}$  in terms of  $\pi_{N-1}^{(N+1)}$ , as

$$\pi_N^{(N+1)} = \frac{\sum_{i=0}^{N-1} \pi_i^{(N+1)} p_{iN}}{\sum_{j=0}^{N-1} p_{Nj}} = \frac{\sum_{i=0}^{N-1} \pi_i^{(N+1)} p_{iN}}{1 - p_{NN}} = \frac{p_{N-1N}}{1 - p_{NN}} \pi_{N-1}^{(N+1)}, \quad (2.7)$$

and eliminate  $\pi_N^{(N+1)}$  from the stationary equations. Further, from equation (2.3), and (2.5), we get

$$\omega_N^{(N)} \left( \mathbf{I}_N - \mathbf{T}_N^{(N+1)} \right) = \pi_N^{(N+1)} \mathbf{U}_N^{(N+1)} = \omega_N^{(N)} \frac{\mathbf{V}_N^{(N+1)} \mathbf{U}_N^{(N+1)}}{\mathbf{U}_N^{(N+1)} \mathbf{e}_N}, \text{ or}$$

$$\omega_N^{(N)} \left( \mathbf{I}_N - \mathbf{T}_N^{(N+1)} - \frac{\mathbf{V}_N^{(N+1)} \mathbf{U}_N^{(N+1)}}{\mathbf{U}_N^{(N+1)} \mathbf{e}_N} \right) = \mathbf{0}, \quad (2.8)$$

or

$$\omega_N^{(N)} (\mathbf{I}_N - \mathbf{M}'_N) = \mathbf{0}, \quad (2.9)$$

where,

$$\mathbf{M}'_N = \mathbf{T}_N^{(N+1)} + \frac{\mathbf{V}_N^{(N+1)} \mathbf{U}_N^{(N+1)}}{p_{NN-1}}. \quad (2.10)$$

Note that  $\mathbf{V}_N^{(N+1)} \mathbf{U}_N^{(N+1)}$  is an  $N \times N$  matrix whose  $ij^{th}$  element is zero, except the  $(N, N)^{th}$  element is  $p_{N-1N} p_{NN-1}$ .

Note also that  $\mathbf{M}'_N$  is a stochastic matrix with  $N$  state, since from (2.10),

$$\begin{aligned} \mathbf{M}'_N \mathbf{e}_N &= \left( \mathbf{T}_N^{(N+1)} + \frac{\mathbf{V}_N^{(N+1)} \mathbf{U}_N^{(N+1)}}{\mathbf{U}_N^{(N+1)} \mathbf{e}_N} \right) \mathbf{e}_N = \mathbf{T}_N^{(N+1)} \mathbf{e}_N + \frac{\mathbf{V}_N^{(N+1)} \mathbf{U}_N^{(N+1)} \mathbf{e}_N}{\mathbf{U}_N^{(N+1)} \mathbf{e}_N} \\ &= \mathbf{T}_N^{(N+1)} \mathbf{e}_N + \mathbf{V}_N^{(N+1)} = \mathbf{e}_N, \end{aligned}$$

where,  $\mathbf{e}_N$  is  $N \times 1$  column vector with all elements equal one,

$$\mathbf{M}_{N+1} \mathbf{e}_{N+1} = \left( \begin{array}{c|c} \mathbf{T}_N^{(N+1)} & \mathbf{V}_N^{(N+1)} \\ \hline \mathbf{U}_N^{(N+1)} & p_{NN} \end{array} \right) \begin{pmatrix} \mathbf{e}_N \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_N^{(N+1)} \mathbf{e}_N + \mathbf{V}_N^{(N+1)} \\ \mathbf{U}_N^{(N+1)} \mathbf{e}_N + p_{NN} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_N \\ 1 \end{pmatrix}.$$

So that  $\omega_N^{(N)}$  satisfies the property required for a stationary probability vector of the regular NHMC-RW,  $\{X_k^{(N)}, k = 0, 1, \dots\}$  on the state space  $s_N$ , with irreducible stochastic TPM  $\mathbf{M}'_N$ . Hence  $\omega_N^{(N)}$  is a scaled stationary probability vector of this  $N$  state:

$$\omega_N^{(N)} = \alpha_N \pi_N^{(N)}, \quad (2.11)$$

where,  $\pi_N^{(N)} \mathbf{e}_N = 1$ . Thus  $\omega_N^{(N)} \mathbf{e}_N = \alpha_N \pi_N^{(N)} \mathbf{e}_N = \alpha_N$ , i.e.,  $\alpha_N = \omega_N^{(N)} \mathbf{e}_N$ .

Note that,  $\alpha_N = \sum_{i=0}^{N-1} \pi_i^{(N+1)} = 1 - \pi_N^{(N+1)}$ .

Therefore,  $\omega_N^{(N)} = \left( \omega_0^{(N)} \quad \omega_1^{(N)} \quad \dots \quad \omega_{N-1}^{(N)} \right) = \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \dots \quad \pi_{N-1}^{(N+1)} \right)$

is a scaled stationary probability vector of this  $N$  state NHMC-RW:

$$\begin{aligned} \pi_N^{(N)} &= \left( \pi_0^{(N)} \quad \pi_1^{(N)} \quad \dots \quad \pi_{N-1}^{(N)} \right) = \frac{\omega_N^{(N)}}{\omega_N^{(N)} \mathbf{e}_N} = \frac{\omega_N^{(N)}}{\alpha_N} = \frac{\omega_N^{(N)}}{1 - \pi_N^{(N+1)}} \\ &= \frac{1}{\alpha_N} \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \dots \quad \pi_{N-1}^{(N+1)} \right). \end{aligned}$$

Thus, for  $0 \leq i \leq N-1$  and  $\alpha_N = 1 - \pi_N^{(N+1)}$ ,

$$\pi_i^{(N)} = \frac{\pi_i^{(N+1)}}{1 - \pi_N^{(N+1)}} = \frac{\pi_i^{(N+1)}}{\sum_{j=0}^{N-1} \pi_j^{(N+1)}} = \frac{\pi_i^{(N+1)}}{\alpha_N}. \quad (2.12)$$

Thus we have reduced the state space from  $N + 1$  to  $N$  with the resulting NHMC-RW  $\{X_k^{(N)}, k = 0, 1, \dots\}$  having a stationary distribution  $\{\pi_i^{(N)}\}$  that is a scaled version of the first  $N$  components of the stationary distribution of the NHMC-RW with  $N + 1$  state. Therefore, the stationary probability vector of the  $(N + 1)$ -states NHMC-RW with state space  $s_{N+1} = \{0, 1, \dots, N\}$  and TPM  $\mathbf{m}_{N+1}$ , is

$$\begin{aligned} \boldsymbol{\pi} \equiv \boldsymbol{\pi}_{N+1}^{(N+1)} &= \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \dots \quad \pi_{N-1}^{(N+1)} \quad \pi_N^{(N+1)} \right) \equiv (\pi_0 \quad \pi_1 \quad \dots \quad \pi_{N-1} \quad \pi_N) \\ &= \alpha_N \left( \pi_0^{(N)} \quad \pi_1^{(N)} \quad \pi_2^{(N)} \quad \dots \quad \pi_{N-1}^{(N)} \quad \frac{\pi_N^{(N+1)}}{1 - \pi_N^{(N+1)}} \right). \end{aligned} \quad (2.13)$$

So that the first  $N$  stationary probabilities of the  $N + 1$ -state NHMC-RW are scaled versions of the  $N$  state NHMC-RW, i.e. the first  $N$  terms of  $\pi_i^{(N+1)}$  are a multiple of  $\pi_i^{(N)}$ . This leads to the relationship between the steady-state probability vectors of the NHMC-RW in the two state spaces  $s_{N+1}$  and  $s_N$  as in the following theorem.

**Theorem 2.1** The connection between the steady-state probability vectors of the regular NHMC-RW in the two state spaces  $s_{N+1}$  and  $s_N$  is given by (2.13),

$$\text{where } \alpha_N = \sum_{i=0}^{N-1} \pi_i^{(N+1)} = 1 - \pi_N^{(N+1)}.$$

Therefore, if the steady-state probabilities for the original NHMC-RW model are known, then the steady-state probabilities for the smaller NHMC-RW model are proportional to those for the corresponding states of the original model.

**Lemma 2.1** A closed form formula for the solution of the non-homogeneous first order difference equation

$$D(r+2) = p_{r+1r} D(r+1) + \prod_{i=0}^r p_{ii+1}, \quad \text{for } r = 1, 2, \dots, N-1 \quad (2.14)$$

with  $D(2) = p_{01} + p_{10}$ , and  $D(1) = 1$ , is given by

$$D(r+2) = \sum_{s=0}^{r+1} \left( \prod_{i=0}^{s-1} p_{ii+1} \right) \left( \prod_{j=s+1}^{r+1} p_{jj-1} \right), \quad \text{for } r = 0, 1, 2, \dots, N-1 \quad (2.15)$$

For details proof of lemma 2.1, see Appendix (a).

**Corollary 2.1**  $D(N+1)$  can be written as

$$D(N+1) = \sum_{s=0}^N \left( \prod_{i=0}^{s-1} p_{ii+1} \right) \left( \prod_{j=s+1}^N p_{jj-1} \right). \quad (2.16)$$

In the next theorem a closed form expression for the stationary probability vector

$$\pi \equiv \pi_{N+1}^{(N+1)} = \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \dots \quad \pi_{N-1}^{(N+1)} \quad \pi_N^{(N+1)} \right) \equiv (\pi_0 \quad \pi_1 \quad \dots \quad \pi_{N-1} \quad \pi_N)$$

of the  $(N+1)$ -states regular NHMC-RW with state space  $S_{N+1} = \{0, 1, \dots, N\}$  and TPM  $\mathbf{m}_{N+1}$ , is formally established.

**Theorem 2.2** Closed form expression for the steady-state probabilities of the TPM  $\mathbf{m}_{N+1}$ , is uniquely determined by

$$\pi_k^{(N+1)} = \frac{1}{D(N+1)} \left( \prod_{i=0}^{k-1} p_{ii+1} \right) \left( \prod_{j=k+1}^N p_{jj-1} \right), \quad \text{for all } k = 0, 1, \dots, N \quad (2.17)$$

where  $D(N+1)$  is given as in (2.16).

Proof. Following Mathematical induction. The case  $N = 1$ , is introduced in the following lemma, it may be found for example in Kemeny and Snell's (1976) (see also Iosifescu (1980)).

**Lemma 2.2** Closed form expression for the steady-state probability vector

$$\pi_2^{(2)} = \left( \pi_0^{(2)} \quad \pi_1^{(2)} \right) \equiv (\pi_0 \quad \pi_1) \text{ of the two states NHMC-RW with state space}$$

$$S_2 = \{0, 1\} \text{ and TPM, } \mathbf{m}_2 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}, \text{ is given as}$$

$$\pi_2^{(2)} = (\pi_0 \quad \pi_1) = \left( \frac{p_{10}}{p_{01} + p_{10}} \quad \frac{p_{01}}{p_{01} + p_{10}} \right).$$

Note that, the second stationary equation is  $\pi_1^{(2)} = p_{01}\pi_0^{(2)} + p_{11}\pi_1^{(2)}$ , implying,

$$\pi_1 \equiv \pi_1^{(2)} = 1 - \pi_0^{(2)} \equiv 1 - \pi_0 = \frac{p_{01}}{p_{01} + p_{10}} = \frac{1}{D(2)} \left( \prod_{i=0}^{1-1} p_{ii+1} \right) \left( \prod_{j=1+1}^1 p_{jj-1} \right),$$

$$\text{where } D(2) = p_{01} + p_{10} = \sum_{s=0}^1 \left( \prod_{i=0}^{s-1} p_{ii+1} \right) \left( \prod_{j=s+1}^1 p_{jj-1} \right).$$

Formula (2.17) holds for  $N = 1$ .

We proceed with increasing the state space in the case  $N = 2$ , the steady-state

probability vector  $\pi_3^{(3)} = \begin{pmatrix} \pi_0^{(3)} & \pi_1^{(3)} & \pi_2^{(3)} \end{pmatrix}$  of the regular NHMC-RW in this case is given as in the following lemma.

**Lemma 2.3** Closed form expression for the steady-state probabilities of the three states NHMC-RW with state space  $s_3 = \{0, 1, 2\}$  and TPM,

$$\mathbf{M}_3 = \begin{pmatrix} p_{00} & p_{01} & 0 \\ p_{10} & p_{11} & p_{12} \\ 0 & p_{21} & p_{22} \end{pmatrix},$$

is given by

$$\pi_k^{(3)} = \frac{1}{D(3)} \left( \prod_{i=0}^{k-1} p_{ii+1} \right) \left( \prod_{j=k+1}^2 p_{jj-1} \right), \text{ for } k = 0, 1, 2$$

where  $D(N+1)$  is given as in (2.16) with  $N = 2$ .

Proof. For details proof, using the method of state reduction, see Appendix (b). Lemma 2.3 shows that formula (2.17) holds for  $N = 2$ .

Assume that formula (2.17) holds for  $N = L - 1$ , i.e., the steady-state probabilities of the  $L$ -state NHMC-RW with state space  $s_L = \{0, 1, \dots, L - 1\}$  is given as

$$\pi_k^{(L)} = \frac{1}{D(L)} \left( \prod_{i=0}^{k-1} p_{ii+1} \right) \left( \prod_{j=k+1}^{L-1} p_{jj-1} \right), \text{ for } k = 0, 1, \dots, L - 1, \quad (2.18)$$

where  $D(N+1)$  is given as in (2.16) with  $N = L - 1$ .

Using theorem 2.1, and corollary 2.1, with  $N = L$  and state space  $s_{L+1} = \{0, 1, \dots, L\}$ , we have

$$\begin{aligned} \pi_{L+1}^{(L+1)} &= \begin{pmatrix} \pi_0^{(L+1)} & \pi_1^{(L+1)} & \dots & \pi_{L-1}^{(L+1)} & \pi_L^{(L+1)} \end{pmatrix} = \\ &= \alpha_L \begin{pmatrix} \pi_0^{(L)} & \pi_1^{(L)} & \pi_2^{(L)} & \dots & \pi_{L-1}^{(L)} & \frac{\pi_L^{(L+1)}}{1 - \pi_L^{(L+1)}} \end{pmatrix}, \text{ or} \\ \pi_k^{(L+1)} &= \alpha_L \pi_k^{(L)} \text{ for } k = 0, 1, \dots, L - 1, \end{aligned} \quad (2.19)$$

where  $\alpha_L = 1 - \pi_L^{(L+1)}$ .

We can eliminate  $\pi_L^{(L+1)}$  from the stationary equations  $\pi_{L+1}^{(L+1)} = \pi_{L+1}^{(L+1)} \mathbf{M}_{L+1}$ , or in element form

$$(p_{kk+1} + p_{kk-1}) \pi_k^{(L+1)} = p_{k-1k} \pi_{k-1}^{(L+1)} + p_{k+1k} \pi_{k+1}^{(L+1)}, \quad 0 \leq k \leq L \quad (2.20)$$

After simplification, we immediately obtain  $\pi_L^{(L+1)} = \frac{1}{D(L+1)} \left( \prod_{i=0}^{L-1} p_{ii+1} \right)$ .

Thus, we have

$$\alpha_L = 1 - \pi_L^{(L+1)} = \frac{p_{LL-1} D(L)}{D(L+1)}, \quad (2.21)$$

where  $D(N+1)$  is given as in (2.16) with  $N = L-1$ , and  $N = L$ .

Using (2.18), (2.19) and (2.21), we obtain the steady-state probabilities of the  $(L+1)$ -state NHMC-RW with state space  $s_{L+1} = \{0, 1, \dots, L\}$ ,

$$\begin{aligned} \pi_k^{(L+1)} &= \alpha_L \pi_k^{(L)} = \frac{p_{LL-1} D(L)}{D(L+1)} \times \frac{1}{D(L)} \left( \prod_{i=0}^{k-1} p_{ii+1} \right) \left( \prod_{j=k+1}^{L-1} p_{jj-1} \right) \\ &= \frac{1}{D(L+1)} \left( \prod_{i=0}^{k-1} p_{ii+1} \right) \left( \prod_{j=k+1}^L p_{jj-1} \right) \text{ for } k = 0, 1, \dots, L, \end{aligned} \quad (2.22)$$

then formula (2.17) holds for  $N = L$ . Therefore formula (2.17) holds for any  $L \in s_{N+1}$  and the proof of the theorem follows.

**Corollary 2.2** Closed form expression for the steady-state probability vector of the  $(N+1)$ -states regular NHMC-RW with state space  $s_{N+1}$  and TPM  $\mathbf{m}_{N+1}$ , is given as

$$\begin{aligned} \boldsymbol{\pi}_{N+1}^{(N+1)} &= \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \dots \quad \pi_{N-1}^{(N+1)} \quad \pi_N^{(N+1)} \right) \\ &= \frac{1}{D(N+1)} \left( \prod_{j=1}^N p_{jj-1} \quad p_{01} \prod_{j=2}^N p_{jj-1} \quad \dots \quad p_{NN-1} \prod_{i=0}^{N-2} p_{ii+1} \quad \prod_{i=0}^{N-1} p_{ii+1} \right), \end{aligned}$$

where  $D(N+1)$  is given as in (2.16).

An alternative formula for the steady-state probabilities of the  $(N+1)$ -states regular NHMC-RW with state space  $s_{N+1}$  and TPM  $\mathbf{m}_{N+1}$ , in terms of  $\pi_0^{(N+1)}$  is given in the following theorem.

**Theorem 2.3** Closed form expression for the steady-state probabilities,  $\pi_k^{(N+1)}$  for  $k = 1, 2, \dots, N$ , in term of  $\pi_0^{(N+1)}$ , of the  $(N+1)$ -states NHMC-RW with state space  $s_{N+1}$  and TPM,  $\mathbf{m}_{N+1}$ , can be rewritten as

$$\pi_k^{(N+1)} = \left( \prod_{i=1}^k \frac{p_{i-1i}}{p_{ii-1}} \right) \pi_0^{(N+1)}, \quad \text{for all } k = 1, 2, \dots, N \quad (2.23)$$

where  $\pi_0^{(N+1)} = \left( \sum_{j=0}^N \prod_{i=1}^j \frac{p_{i-1i}}{p_{ii-1}} \right)^{-1}$ .

Proof. Using theorem 2.2, the steady-state probabilities of the TPM,  $\mathbf{m}_{N+1}$ , can be rewritten (after multiplying the numerator and denominator by  $\prod_{j=1}^N p_{jj-1}$ ) as

$$\pi_k^{(N+1)} = \frac{1}{\prod_{j=1}^k p_{jj-1} \prod_{j=k+1}^N p_{jj-1}} \left( \prod_{i=0}^{k-1} p_{ii+1} \right) \left( \prod_{j=k+1}^N p_{jj-1} \right) \pi_0^{(N+1)} = \left( \prod_{i=1}^k \frac{p_{i-1i}}{p_{ii-1}} \right) \pi_0^{(N+1)},$$

$$\pi_0^{(N+1)} = \frac{1}{D(N+1)} \prod_{j=1}^N p_{jj-1} = \frac{1}{\frac{1}{N} D(N+1)} = \left( \frac{D(N+1)}{N} \right)^{-1} = \left( \sum_{j=0}^N \prod_{i=1}^j \frac{p_{i-1i}}{p_{ii-1}} \right)^{-1},$$

where  $D(N+1)$  is given as in (2.16). Thus the proof is completed.

**Corollary 2.3** The steady-state probability vector of the  $(N+1)$ -state regular NHMC-RW with state space  $s_{N+1}$  and TPM  $\mathbf{m}_{N+1}$ , in terms of  $\pi_0^{(N+1)}$  is given by

$$\pi_{N+1}^{(N+1)} = \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \dots \quad \pi_{N-1}^{(N+1)} \quad \pi_N^{(N+1)} \right)$$

$$= \pi_0^{(N+1)} \begin{pmatrix} 1 & \frac{p_{01}}{p_{10}} & \frac{p_{01}p_{12}}{p_{10}p_{21}} & \dots & \frac{p_{01}p_{12}\dots p_{N-1N}}{p_{10}p_{21}\dots p_{NN-1}} \end{pmatrix}, \quad (2.24)$$

with  $\pi_0^{(N+1)} = \left( \sum_{j=0}^N \prod_{i=1}^j \frac{p_{i-1i}}{p_{ii-1}} \right)^{-1}$ .

Proof. Using theorem 2.3, then formula (2.24) immediately follows.

### 3. Particular Cases

Many interesting explicit results can be introduced from the present results through an appropriate choice of the transition probabilities and the boundary probabilities, with particular state spaces. Two examples are listed below.

### Example 3.1

The major interest of this example is that the transition probabilities are

$$p_{ii+1} = p \left( X_{k+1}^{(N+1)} = i+1 \mid X_k^{(N+1)} = i \right) = p_i \quad \text{and} \quad p_{ii-1} = p \left( X_{k+1}^{(N+1)} = i-1 \mid X_k^{(N+1)} = i \right) = q_i$$

with  $p_{ii} = 1 - p_i - q_i$ ,  $i \in S_{N+1}$ ;  $\{p_i \text{ for } 0 \leq i < N\}$  and  $\{q_i \text{ for } 0 < i \leq N\}$  are positive.

In this case closed form expression for the steady-state probabilities is

$$\pi_k^{(N+1)} = \frac{1}{D(N+1)} \left( \prod_{i=0}^{k-1} p_i \right) \left( \prod_{j=k+1}^N q_j \right), \quad \text{for all } k = 0, 1, \dots, N \quad (3.1)$$

$$= \pi_0^{(N+1)} \prod_{i=1}^k \left( \frac{p_{i-1}}{q_i} \right), \quad \text{for all } k = 1, 2, \dots, N \quad (3.2)$$

$$\text{where } D(N+1) = \sum_{s=0}^N \left( \prod_{i=0}^{s-1} p_i \right) \left( \prod_{j=s+1}^N q_j \right), \text{ and } \pi_0^{(N+1)} = \left( \sum_{j=0}^N \prod_{i=1}^j \frac{p_{i-1}}{q_i} \right)^{-1}.$$

### Example 3.2

The major interest of this example is the spatial homogeneity Markov chain random walk (HMC-RW) models,  $\{X_k^{(N+1)}, k = 0, 1, \dots\}$  on state space

$S_{N+1} = \{0, 1, \dots, N\}$ , with one- step transition probabilities are

$$p_{ij} = p \left( X_{k+1}^{(N+1)} = j \mid X_k^{(N+1)} = i \right) = \begin{cases} p & \text{for } j = i+1 \\ 1 - p - q & \text{for } j = i \\ q & \text{for } j = i-1 \end{cases} \quad (3.3)$$

with  $p$  and  $q$  are positive,  $p_{01} = p_0 \equiv \rho$  and  $p_{NN-1} = q_N \equiv \omega$ . In this case closed form expression for the steady-state probabilities can be easily obtained from (3.1) and (3.3) in the following form

**case 1:**  $p \neq q$

$$\pi_k^{(N+1)} = \frac{1}{D(N+1)} \begin{cases} \omega q^{N-1}, & k = 0 \\ p^k q^{N-k}, & k = 1, 2, \dots, N-1 \\ \rho p^{N-1}, & k = N \end{cases} \quad (3.4)$$

$$\text{where } D(N+1) = \rho \omega \left( q^{N+1} - p^{N+1} \right) \left( q^2 p - p^2 q \right)^{-1}.$$

**case 2:**  $p = q$

$$\pi_k^{(N+1)} = \frac{1}{(N+1)\rho\omega p^{N-2}} \begin{cases} \omega p^{N-1}, & k = 0 \\ p^N, & k = 1, 2, \dots, N-1 \\ \rho p^{N-1}, & k = N \end{cases} \quad (3.5)$$

We can obtain closed form expression for the steady-state probabilities in special case  $p_{01} = p$  and  $p_{NN-1} = q$ , after simplification, yield the following formula

$$\pi_k^{(N+1)} = \begin{cases} \frac{1 - (p/q)}{1 - (p/q)^{N+1}} (p/q)^k, & p \neq q \\ \frac{1}{N+1}, & p = q \end{cases} \quad \text{for all } k = 0, 1, \dots, N. \quad (3.6)$$

Note that,  $\sum_{s=0}^N p^s q^{N-s} = (q^{N+1} - p^{N+1})(q - p)^{-1}$ .

We see that with the appropriate change of notation formula (3.5) in the special case  $k = 0$  agrees with that of El-Shehawey ((2000) and (2009)) (see also Percus (1985), and Lefebvre (2008)).

#### 4. Algorithm

An efficient computational algorithm for computing the steady state probability vector  $\pi_{N+1}^{(N+1)} = \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \pi_2^{(N+1)} \quad \dots \quad \pi_{N-1}^{(N+1)} \quad \pi_N^{(N+1)} \right)$  of the TPM  $\mathbf{m}_{N+1}$ , as shown in (I), is developed as:

**INPUT** Order of the matrix  $\mathbf{m}_{N+1}$ , and the entries of the matrices  $\mathbf{J}$ ,  $\mathbf{u}_N^{(N+1)}$ ,  $\mathbf{T}_N^{(N+1)}$  and  $\mathbf{v}_N^{(N+1)}$ .

**OUTPUT** Steady state probability vector  $\pi_{N+1}^{(N+1)} = \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \dots \quad \pi_N^{(N+1)} \right)$ .

##### ➤ Matrix reduction

**Step 1:** Partition  $\mathbf{m}_{N+1}$  as:  $\mathbf{m}_{N+1} = \left( \begin{array}{c|c} \mathbf{J} & \mathbf{u}_N^{(N+1)} \\ \hline \mathbf{v}_N^{(N+1)} & \mathbf{T}_N^{(N+1)} \end{array} \right)$ .

**Step 2:** Initialize  $n = 1$ ,  $1 \leq n < N$ .

**Step 3:** Compute  $\mathbf{m}_{(N+1)red}^{(n)} = \mathbf{T}_{N-n}^{(N-n+1)} + \mathbf{v}_{N-n}^{(N-n+1)} (1 - \mathbf{J})^{-1} \mathbf{u}_{N-n}^{(N-n+1)}$ .

**Step 4:** Increment  $n$  by 1. If  $n < N$ , go to step 3. O.w, go back substitution.

### ➤ Back substituting

**Step 1:** Initialize  $i = N - 1$ .

**Step 2:** Compute  $c_{N-1} = (p_{NN-1})(p_{N-1N})^{-1}$ .

**Step 3:** Decrement  $i$  by 1

**Step 4:** Compute  $c_i = \left( \sum_{r=i+1}^{N-1} p_{ri} c_r \right) (p_{ii+1})^{-1}$

**Step 5:** If  $0 \leq i \leq N - 2$ , go to step 4.

**Step 6:**  $\pi_N^{(N+1)} = \left( 1 + \sum_{r=0}^{N-1} c_r \right)^{-1}$ .

**Step 7:**  $\pi_r^{(N+1)} = c_r \pi_N^{(N+1)}$ ,  $r = 0, 1, 2, \dots, N - 1$

**Step 8:** Steady state probability vector is  $\pi \equiv \pi_{N+1}^{(N+1)} = \left( \pi_0^{(N+1)} \quad \pi_1^{(N+1)} \quad \dots \quad \pi_N^{(N+1)} \right)$ .

A Mathematica procedure for computing the stationary probability vector for  $4 \times 4$  TPM  $\mathbf{M}_4$  arising from NHMC-RW, is given as

**Consider the following matrix**

$$\mathbf{M}_4 = \begin{pmatrix} p_{00} & p_{01} & 0 & 0 \\ p_{10} & p_{11} & p_{12} & 0 \\ 0 & p_{21} & p_{22} & p_{23} \\ 0 & 0 & p_{32} & p_{33} \end{pmatrix}.$$

Then we get the following result

$$\mathbf{P} \mathbf{i} = \frac{1}{D(4)} \begin{pmatrix} p_{10} p_{21} p_{32} \\ p_{01} p_{21} p_{32} \\ p_{01} p_{12} p_{32} \\ p_{01} p_{12} p_{23} \end{pmatrix},$$

where,  $D(4) = p_{01} p_{12} p_{23} + (p_{01} p_{12} + (p_{01} + p_{10}) p_{21}) p_{32}$ .

## 5. Conclusions

Closed form expressions for the steady state probabilities for a regular TPM  $\mathbf{M}$  arising from the NHMC-RW  $\{X_k^{(N+1)}, k = 0, 1, \dots\}$  on the state space  $\{0, 1, \dots, N\}$ , between two partially segregating barriers, have been determined by the method of state reduction. Relationship between the stationary probability vectors in the two state spaces  $\{0, 1, \dots, N\}$  and  $\{0, 1, \dots, N - 1\}$  are obtained as well as some simple forms lead to explicit formulas for some special cases. Computational algorithm

for computing steady state (stationary) probabilities for the NHMC-RW are also given.

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## 7. Appendices.

In the appendices (a) and (b) we present proofs of lemma 2.1 and lemma 2.3, respectively. We have adopt the notations:  $\prod_{k+1}^k (.) = 1$  and  $\sum_{k+1}^k (.) = 0$ .

**Appendix (a).** Proof of the lemma 2.1, by simple iteration, we get

$$\begin{aligned}
 D(r+2) &= p_{r+1r} D(r+1) + \prod_{i=0}^r p_{ii+1} = p_{r+1r} \left( p_{rr-1} D(r) + \prod_{i=0}^{r-1} p_{ii+1} \right) + \prod_{i=0}^r p_{ii+1} \\
 &= p_{r+1r} p_{rr-1} \left( p_{r-1r-2} D(r-1) + \prod_{i=0}^{r-2} p_{ii+1} \right) + p_{r+1r} \prod_{i=0}^{r-1} p_{ii+1} + \prod_{i=0}^r p_{ii+1} \\
 &= p_{r+1r} p_{rr-1} \cdots p_{10} + p_{r+1r} p_{rr-1} \cdots p_{21} \prod_{i=0}^0 p_{ii+1} + \cdots + p_{r+1r} \prod_{i=0}^{r-1} p_{ii+1} + \prod_{i=0}^r p_{ii+1} \\
 &= \sum_{s=0}^{r+1} \left( \prod_{i=0}^{s-1} p_{ii+1} \right) \left( \prod_{j=s+1}^{r+1} p_{jj-1} \right).
 \end{aligned}$$

**Appendix (b).** Proof of the lemma 2.3 in more detail, using the method of state reduction. For  $N = 2$ , in this case we have  $s_3 = \{0, 1, 2\}$ . The states are divided into two subsets. The first subset contains state 0. The second subset contains the remaining states, and is denoted by  $s_2 = \{1, 2\}$ .

**First step:** Matrix reduction

The TPM  $M_3$  is partitioned into four sub-matrices called  $J, U_2^{(3)}, V_2^{(3)}$  and  $T_2^{(3)}$ , as

$$\mathbf{M}_3 = \begin{matrix} & \begin{matrix} 0 & \left( \begin{array}{c|cc} p_{00} & p_{01} & 0 \end{array} \right) \\ \hline 1 & \begin{pmatrix} p_{10} & p_{11} & p_{12} \end{pmatrix} \\ 2 & \begin{pmatrix} 0 & p_{21} & p_{22} \end{pmatrix} \end{matrix} \end{matrix} = \begin{pmatrix} \mathbf{J} & \mathbf{U}_2^{(3)} \\ \mathbf{V}_2^{(3)} & \mathbf{T}_2^{(3)} \end{pmatrix},$$

where,  $\mathbf{J} = p_{00}$ , is the single element in the upper left-hand corner of  $\mathbf{M}_3$ ,

$$\mathbf{U}_2^{(3)} = \begin{pmatrix} p_{01} & 0 \end{pmatrix}, \mathbf{V}_2^{(3)} = \frac{1}{2} \begin{pmatrix} p_{10} \\ 0 \end{pmatrix}, \text{ and } \mathbf{T}_2^{(3)} = \frac{1}{2} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

A **reduced matrix**,  $\mathbf{M}_{3red}^{(1)}$ , which represents a **regular** two-state NHMC-RW, is computed according to the formula  $\mathbf{M}_{3red}^{(1)} = \mathbf{T}_2^{(3)} + \mathbf{V}_2^{(3)}(1 - \mathbf{J})^{-1} \mathbf{U}_2^{(3)}$ . Since round off error can be reduced by eliminating subtractions, the factor  $(1 - p_{00})^{-1}$  is replaced by  $(p_{01})^{-1}$ .

The modified formula, called the matrix reduction formula, is

$$\mathbf{M}_{3red}^{(1)} = \mathbf{T}_2^{(3)} + \mathbf{V}_2^{(3)}(p_{01})^{-1} \mathbf{U}_2^{(3)}.$$

Thus, we have

$$\begin{aligned} \mathbf{M}_{3red}^{(1)} &= \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{10} \\ 0 \end{pmatrix} (p_{01})^{-1} \begin{pmatrix} p_{01} & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} (p_{11} + p_{10}) & p_{12} \\ p_{21} & p_{22} \end{pmatrix}. \end{aligned}$$

Using lemma 2.2, the matrix,  $\mathbf{M}_{3red}^{(1)}$  has the steady-state probability vector

$$\boldsymbol{\pi}_2^{(2)} = \begin{pmatrix} \pi_1^{(2)} & \pi_2^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{p_{21}}{p_{12} + p_{21}} & \frac{p_{12}}{p_{12} + p_{21}} \end{pmatrix}.$$

**Second step: Back substituting**

Solving  $\pi_1^{(2)}$  as a constant  $c_1$  times  $\pi_2^{(2)}$ :  $\pi_1^{(2)} = c_1 \pi_2^{(2)}$ , we get  $c_1 = p_{21} (p_{12})^{-1}$ .

Let  $\boldsymbol{\pi}_3^{(3)} = \begin{pmatrix} \pi_0^{(3)} & \pi_1^{(3)} & \pi_2^{(3)} \end{pmatrix}$  be the steady-state probability vector of  $\mathbf{M}_3$ :

$$\begin{pmatrix} \pi_0^{(3)} & \pi_1^{(3)} & \pi_2^{(3)} \end{pmatrix} = \begin{pmatrix} \pi_0^{(3)} & \pi_1^{(3)} & \pi_2^{(3)} \end{pmatrix} \begin{pmatrix} p_{00} & p_{01} & 0 \\ p_{10} & p_{11} & p_{12} \\ 0 & p_{21} & p_{22} \end{pmatrix}.$$

Thus, the first steady equation of this system is given as:

$$\pi_0^{(3)} = p_{00} \pi_0^{(3)} + p_{10} \pi_1^{(3)} \text{ or } (1 - p_{00}) \pi_0^{(3)} = p_{10} \pi_1^{(3)}.$$

Then,  $p_{01} \pi_0^{(3)} = p_{10} \pi_1^{(3)}$ .

Since (see Kemeny and Snell (1976)):

$$\pi_1^{(3)} = \frac{\pi_1^{(2)}}{\pi_1^{(2)} + \pi_2^{(2)}}, \text{ and } \pi_2^{(3)} = \frac{\pi_2^{(2)}}{\pi_1^{(2)} + \pi_2^{(2)}},$$

we get  $\frac{\pi_1^{(3)}}{\pi_2^{(3)}} = \frac{\pi_1^{(2)}}{\pi_2^{(2)}} = \frac{\pi_1}{\pi_2}$ , or  $\frac{\pi_1^{(3)}}{\pi_2^{(3)}} = \frac{c_1 \pi_2^{(2)}}{\pi_2^{(2)}}$ , or  $\pi_1^{(3)} = c_1 \pi_2^{(3)}$ .

Then,  $p_{01} \pi_0^{(3)} = p_{10} \pi_1^{(3)} = p_{10} \left( c_1 \pi_2^{(3)} \right)$ , or  $\pi_0^{(3)} = p_{10} c_1 (p_{01})^{-1} \pi_2^{(3)} = c_0 \pi_2^{(3)}$ , with

$$c_0 = (p_{10} c_1) (p_{01})^{-1}.$$

From normalizing equation,  $\pi_0^{(3)} + \pi_1^{(3)} + \pi_2^{(3)} = 1$ , we get  $\pi_2^{(3)} = (1 + c_0 + c_1)^{-1}$ ,

$\pi_1^{(3)} = c_1 \pi_2^{(3)} = c_1 (1 + c_0 + c_1)^{-1}$ , and  $\pi_0^{(3)} = c_0 \pi_2^{(3)} = c_0 (1 + c_0 + c_1)^{-1}$ , with

$c_0 = (p_{10} c_1) (p_{01})^{-1}$  and  $c_1 = p_{21} (p_{12})^{-1}$ . Thus,  $c_0 = (p_{10} p_{21}) (p_{01} p_{12})^{-1}$ .

Therefore,  $\pi_1^{(3)} = c_1 \pi_2^{(3)} = \frac{p_{01} p_{21}}{D(3)} = \frac{1}{D(3)} \left( \prod_{i=0}^0 p_{ii+1} \right) \left( \prod_{j=2}^2 p_{jj-1} \right)$ ,

$$\pi_0^{(3)} = c_0 \pi_2^{(3)} = \frac{p_{10} p_{21}}{D(3)} = \frac{1}{D(3)} \prod_{j=1}^2 p_{jj-1}, \text{ and } \pi_2^{(3)} = \frac{p_{01} p_{12}}{D(3)} = \frac{1}{D(3)} \prod_{i=0}^1 p_{ii+1},$$

where,  $D(2) = p_{01} + p_{10} = \sum_{s=0}^1 \left( \prod_{i=0}^{s-1} p_{ii+1} \right) \left( \prod_{j=s+1}^1 p_{jj-1} \right)$ , and

$$\begin{aligned} D(3) &= p_{21} (p_{10} + p_{01}) + p_{01} p_{12} = \sum_{s=0}^2 \left( \prod_{i=0}^{s-1} p_{ii+1} \right) \left( \prod_{j=s+1}^2 p_{jj-1} \right) \\ &= p_{21} D(2) + p_{01} p_{12}. \end{aligned}$$

This completes the proof of the lemma.

## FGM Bivariate Weibull Distribution

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**Abstract:** The bivariate Weibull distribution is an important lifetime distribution in survival analysis. In this paper, Farlie–Gumbel–Morgenstern (FGM) copula and Weibull marginal distribution are used for creating bivariate distribution which is called FGM Bivariate Weibull (FGMBW) distribution. FGMBW distribution is used for describing bivariate data that have weak correlation between variables in lifetime data. It is a good alternative to bivariate several lifetime distributions for modeling real-valued data in application. Some properties of the FGMBW distribution are obtained such as product moment, skewness, kurtosis, moment generation function, reliability function and hazard function. Three different estimation methods for parameters estimation are discussed for FGMBW distribution namely; maximum likelihood estimation, inference function for margins method and semi-parametric method. To evaluate the performance of the estimators, a Monte Carlo simulations study is conducted to compare the preferences between estimation methods. Also, a real data set is introduced, analyzed to investigate the model and useful results are obtained for illustrative purposes.

**Keywords:** Weibull Distribution, FGM copula, Maximum Likelihood Estimation, Inference Function for Margins and Semi-Parametric.

### 1. Introduction

The Weibull distribution has attained more attention in the literature and has inherent flexibility. The univariate Weibull distribution has the following cdf and pdf respectively

$$F(y; \alpha, \beta) = 1 - e^{-\left(\frac{y}{\beta}\right)^\alpha}; y > 0, \alpha, \beta > 0, \quad (1.1)$$

and

$$f(y; \alpha, \beta) = \frac{\alpha}{\beta} \left(\frac{y}{\beta}\right)^{\alpha-1} e^{-\left(\frac{y}{\beta}\right)^\alpha}; y > 0, \alpha, \beta > 0, \quad (1.2)$$

where  $\beta, \alpha$  are the scale and shape parameters respectively.

Recent researches have been made for the bivariate Weibull distribution. Galiani (2003) concluded that bivariate Weibull is specifically oriented towards applications in economics, finance and risk management. Flores (2009) used Weibull marginal to construct bivariate Weibull distributions. Kundu and Gupta (2013) introduced the Marshall–Olkin bivariate Weibull distribution.

A copula is a convenient approach for description of a multivariate distribution. Nelsen (2006) introduced Copulas as following; copula is function that join multivariate distribution functions with uniform  $[0, 1]$  margins. A copula is a convenient approach to describe a multivariate

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distribution with dependence structure. The n-dimensional copula ( $C$ ) exists for all  $y_1, \dots, y_n$ ,  $F(y_1, \dots, y_n) = C(F_1(y_1), \dots, F_n(y_n))$ , if  $F$  is continuous, then  $C$  is uniquely defined.

Sklar (1973) states that, considered the two random variables  $Y_1$  and  $Y_2$ , with distribution functions  $F_1(y_1)$  and  $F_2(y_2)$  the following cdf and pdf for copula are respectively

$$F(y_1, y_2) = C(F_1(y_1), F_2(y_2)), \quad (1.3)$$

and

$$f(y_1, y_2) = f_1(y_1)f_2(y_2)c(F_1(y_1), F_2(y_2)). \quad (1.4)$$

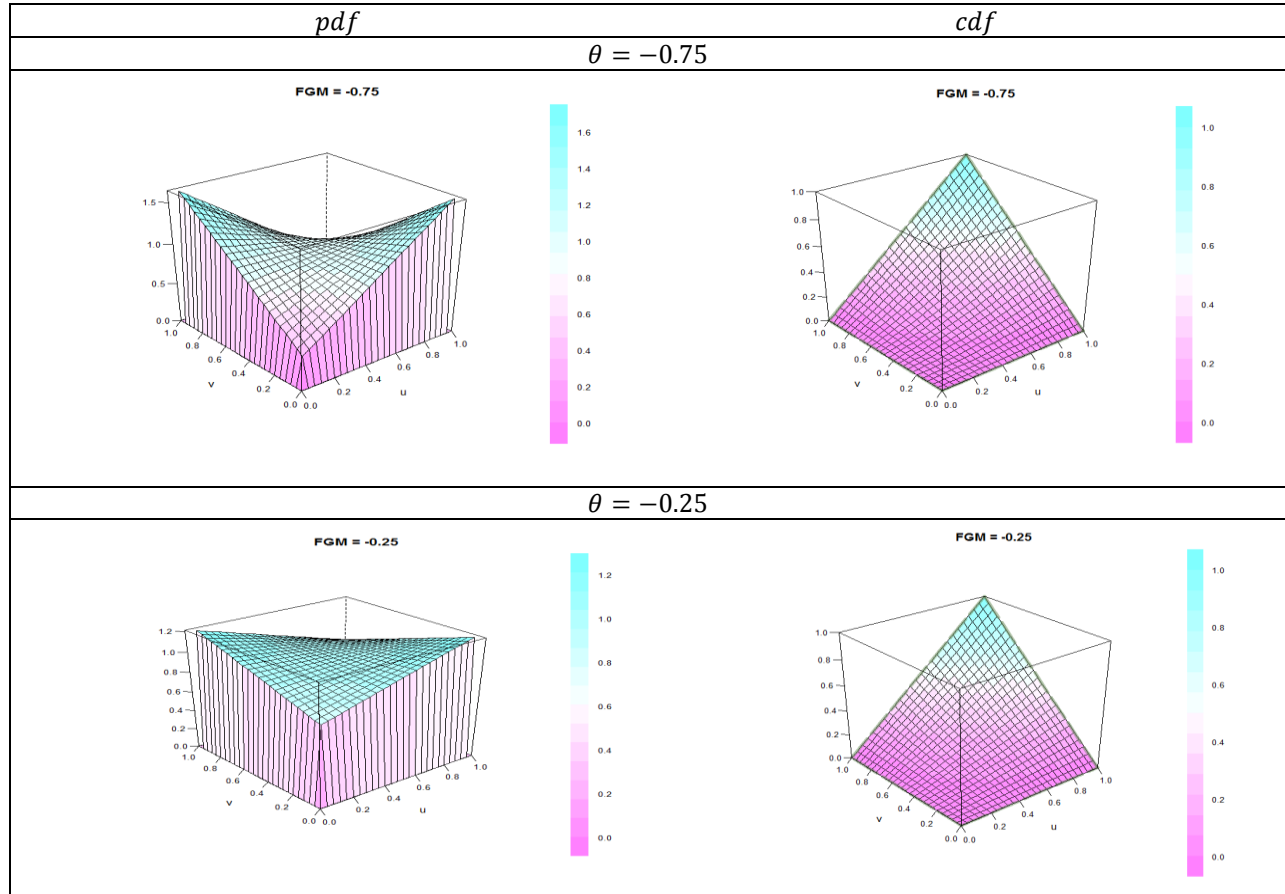
Farlie–Gumbel–Morgenstern (FGM) is one of the most popular parametric families of copulas, the family was discussed by Gumbel (1960). The joint cdf and joint pdf for FGM copula as following respectively

$$C(y_1, y_2) = F_1(y_1)F_2(y_2) \left( 1 + \theta(1 - F_1(y_1))(1 - F_2(y_2)) \right); -1 < \theta < 1 \quad (1.5)$$

and

$$c(y_1, y_2) = (1 + \theta(1 - 2F_1(y_1))(1 - 2F_2(y_2))). \quad (1.6)$$

Figure 1 3-dimention for the pdf and cdf of FGM copula with different value of parameter of copula  $\theta$ .



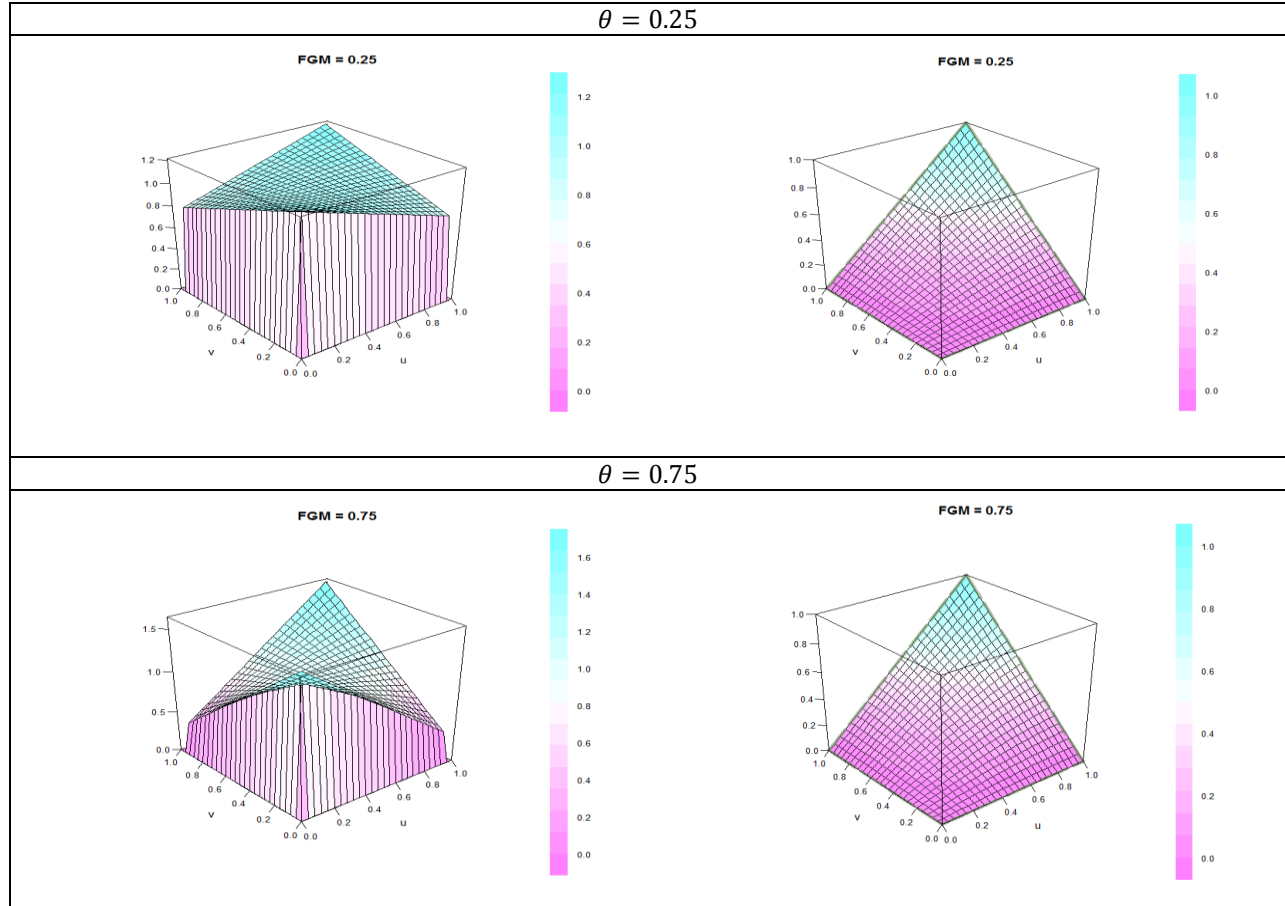


**Fig 1.** FGM Copula with Various Value of  $\theta$ .

Fredricks and Nelsen (2007) drives the formula for Spearman's and Kendall's correlation coefficient as follows

$$\rho_{sperman} = (12 \int \int uv(1 + \theta(1 - u)(1 - v)) du dv) - 3 = \frac{\theta}{3}, \quad (1.7)$$

$$\rho_{Kendall} = 1 - 4 \int \int \frac{\partial C}{\partial u} C(u, v) \frac{\partial C}{\partial v} C(u, v) du dv = \frac{2}{9} \theta, \quad (1.8)$$

Such that  $-\frac{1}{3} \leq \rho_{sperman} \leq \frac{1}{3}$ ,  $-\frac{2}{9} \leq \rho_{Kendall} \leq \frac{2}{9}$ .

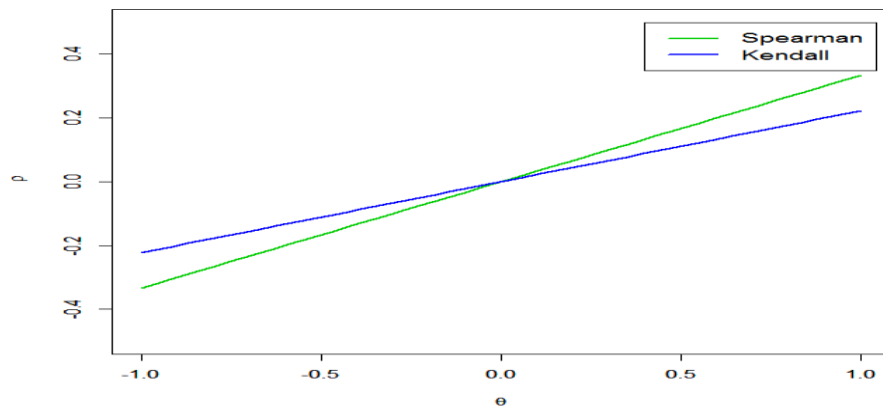


**Fig2. Correlation of FGM Copula with Various Value of Copula Parameter.**

In this article, we study the bivariate extension of the Weibull distribution based on FGM copula function (FGMBW) and discuss its statistical properties. FGMBW distribution is used for describing bivariate data that have weak correlation between variables in lifetime data. It is a good alternative to bivariate several lifetime distributions for modeling non-negative real-valued data in application.

The objective of this article is twofold: to study the properties of the FGMBW distribution, and to estimate the parameters of the model by different estimation methods. The attractive feature of the marginal function of FGMBW distribution is the same as the basic distribution (Weibull). Other features of the FGMBW distribution: it contains closed forms for its cdf, product moment, moment generation function, and hazard rate function. The final motivation of the article is to develop a guideline for introducing the best estimation method for the FGMBW distribution, which we think would be of deep interest to statisticians. A simulation study is conducted to compare the preferences between estimation methods. Also, a real data set is introduced and analyzed to investigate the model. The uniqueness of this study comes from the fact that we introduce a comprehensive description of mathematical and statistical properties of FGMBW distribution with the hope that they will attract wider applications in medicine, economics, life testing and other areas of research.

The rest of this paper is organized as follows: FGM bivariate Weibull distribution is obtain in section 2. Some statistical properties of FGMBW distribution in section 3. Parameter estimation methods for the FGMBW distribution based in copula in section 4. In section 5, asymptotic confidence intervals are discussed. In section 6, the potentiality of the new model is illustrated by simulation study. In section 7, Application of real data are discussed. Finally, Conclusion of some remarks for FGMBW model are addressed in section 8.

**2. FGM Bivariate Weibull Distribution**

According to Sklar theorem the joint pdf of bivariate Weibull distribution for any copula is as follows

$$f(y_1, y_2) = \frac{\alpha_1}{\beta_1} \left(\frac{y_1}{\beta_1}\right)^{\alpha_1-1} e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}} \frac{\alpha_2}{\beta_2} \left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}} c\left(\left(1 - e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right), \left(1 - e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\right) \quad (2.1)$$

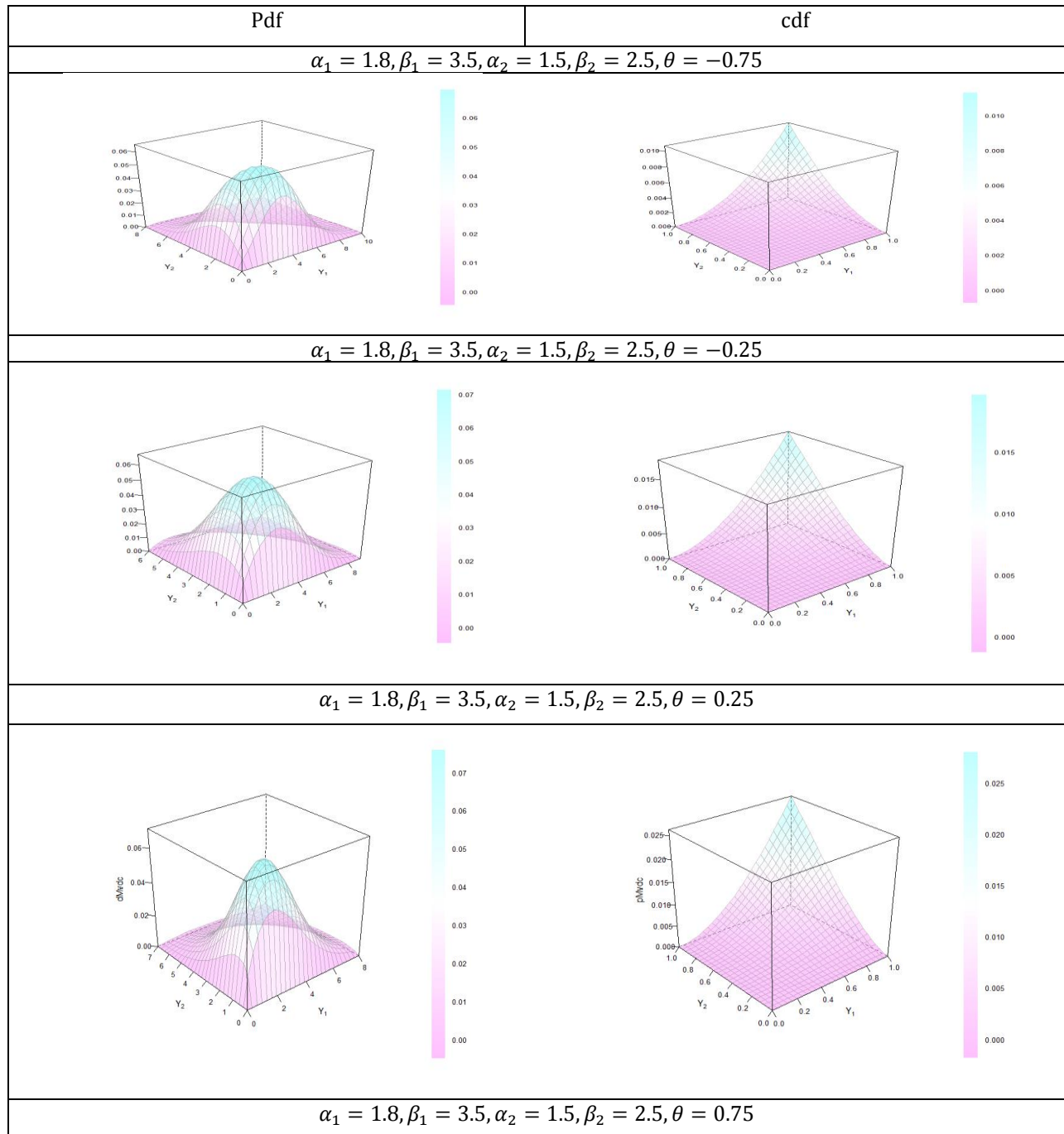
The cdf of a FGMBW distribution can be expressed as

$$F(y_1, y_2) = \left(1 - e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right) \left(1 - e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right) \left[1 + \theta \left(1 - \left(1 - e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right)\right) \left(1 - \left(1 - e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\right)\right] \quad (2.2)$$

The pdf of a FGMBW distribution is defined as

$$f(y_1, y_2) = \frac{\alpha_1}{\beta_1} \left( \frac{y_1}{\beta_1} \right)^{\alpha_1-1} e^{-\left( \frac{y_1}{\beta_1} \right)^{\alpha_1}} \frac{\alpha_2}{\beta_2} \left( \frac{y_2}{\beta_2} \right)^{\alpha_2-1} e^{-\left( \frac{y_2}{\beta_2} \right)^{\alpha_2}} \left[ 1 + \theta \left( 1 - 2 \left( 1 - e^{-\left( \frac{y_1}{\beta_1} \right)^{\alpha_1}} \right) \right) \left( 1 - 2 \left( 1 - e^{-\left( \frac{y_2}{\beta_2} \right)^{\alpha_2}} \right) \right) \right] \quad (2.3)$$

Figure 2 show the plot 3-dimention for the pdf and cdf of FGMBW distribution with different value of  $\alpha_1, \beta_1, \alpha_2, \beta_2$  and  $\theta$ .



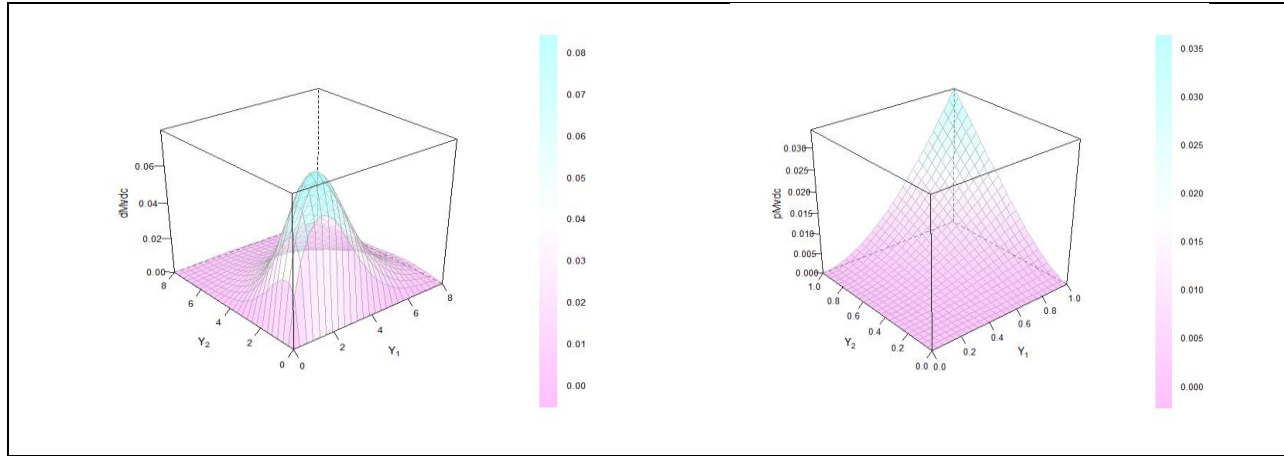


**Fig3.** The pdf and cdf of FGMBW Distribution with Various Value of the Parameters.

### 3. Properties of FGMBW Distribution

In this section, we give some important statistical properties of the FGMBW distribution such as Marginal Distributions, product moments, moment generating function, conditional distribution, generating random variables, reliability function. Establishing algebraic expressions to determine some statistical properties of the FGMBW distribution can be more efficient than computing them directly by numerical simulation.

#### 3.1. The Marginal Distributions

The marginal density functions for  $Y_1$  and  $Y_2$  respectively,

$$f(y_1; \alpha_1, \beta_1) = \frac{\alpha_1}{\beta_1} \left(\frac{y_1}{\beta_1}\right)^{\alpha_1-1} e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}; y_1 > 0, \alpha_1, \beta_1 > 0, \quad (3.1)$$

$$f(y_2; \alpha_2, \beta_2) = \frac{\alpha_2}{\beta_2} \left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}; y_2 > 0, \alpha_2, \beta_2 > 0, \quad (3.2)$$

Which are Weibull distributed, where the marginal distribution of  $Y_1$  and  $Y_2$  can be calculated directly by

$$f(y_i) = \int_{all y_j} f(y_1, y_2) dy_j; i, j = 1, 2, i \neq j.$$

#### 3.2. Conditional Distribution

The conditional probability distribution of  $Y_2$  given  $Y_1$  is given as follows

$$f(y_2|y_1) = \frac{\alpha_2}{\beta_2} \left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}} [1 + \theta(1 - 2u(y_1))(1 - 2u(y_2))], \quad (3.3)$$

and the conditional cdf is

$$F(y_2|y_1) = u(y_2)[1 - \theta + 2\theta u(y_1)] + \theta v(y_2) - 2\theta u(y_1)v(y_2), \quad (3.4)$$

where,  $u(y_i) = 1 - e^{-\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}}$   $i = 1, 2$  and  $v(y_i) = 1 - e^{-2\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}}$   $i = 1, 2$

Next, we can show that

$$E(y_2|y_1) = \beta_2 \Gamma\left(1 + \frac{1}{\alpha_2}\right) + \theta(1 - 2u(y_1)) \left[1 - \beta_2 \Gamma\left(1 + \frac{1}{\alpha_2}\right)\right]$$

### 3.3. Generating Random Variables

Nelsen (2006) discussed generating a sample from a specified joint distribution. By conditional distribution method, the joint distribution function is as follows

$$f(y_1, y_2) = f(y_1)f(y_2|y_1)$$

By using the following steps, we can generate a bivariate sample by using the conditional approach:

1. Generate  $U$  and  $V$  independently from a uniform(0, 1) distribution.
2. Set  $Y_1 = \beta_1[-\ln(1 - U)]^{\alpha_1}$ .
3. Set  $F(y_2|y_1) = V$  to find  $Y_2$  by numerical simulation.
4. Repeat Steps 1–3 ( $n$ ) times to obtain  $(y_{1i}, y_{2i}), i = 1, 2, \dots, n$ .

### 3.4. Moment Generating Function

Let  $(Y_1, Y_2)$  denote a random variable with the probability density function (2.3). Then, the moment generating function of  $(Y_1, Y_2)$  is given by,

$$M_{(y_1, y_2)}(t_1 t_2) = \sum_{n=0}^{\infty} \left( \frac{t_1^n \beta_1^n}{n!} \Gamma\left(1 + \frac{n}{\alpha_1}\right) \right) \sum_{m=0}^{\infty} \left( \frac{t_2^m \beta_2^m}{m!} \Gamma\left(1 + \frac{m}{\alpha_2}\right) \right) \left[ 1 + \theta - 2\theta \frac{1}{2^{(1+\frac{m}{\alpha_2})}} \right. \\ \left. - 2\theta \frac{1}{2^{(1+\frac{n}{\alpha_1})}} + 4\theta \frac{1}{2^{(1+\frac{n}{\alpha_1})}} \frac{1}{2^{(1+\frac{m}{\alpha_2})}} \right], \quad (3.5)$$

To prove the moment generating function start with

$$M_{(y_1, y_2)}(t_1 t_2) = E(e^{t_1 y_1} e^{t_2 y_2}) = \int_0^{\infty} \int_0^{\infty} e^{t_1 y_1} e^{t_2 y_2} f(y_1, y_2) dy_1 dy_2$$

### 3.5. Product Moments

If the random variable  $(Y_1, Y_2)$  is distributed as FGMBW, then its  $r^{th}$  and  $s^{th}$  moments around zero can be expressed as follows

$$\mu'_{rs} = \left( \beta_1^r \Gamma\left(\frac{r}{\alpha_1} + 1\right) \beta_2^s \Gamma\left(\frac{s}{\alpha_2} + 1\right) \right) \left[ 1 + \theta - \frac{\theta}{2^{(s/\alpha_2)}} - \frac{\theta}{2^{(r/\alpha_1)}} + \frac{\theta}{2^{(s/\alpha_2)} 2^{(r/\alpha_1)}} \right] \quad (3.6)$$

To prove that start with

$$\mu'_{rs} = E(Y_1^r Y_2^s) = \int_0^{\infty} \int_0^{\infty} y_1^r y_2^s f(y_1, y_2) dy_1 dy_2$$

We use Mardia's (1970) measures of multivariate and bivariate skewness (SK) and kurtosis (KU). Mardia defined bivariate SK and KU, respectively, as

$$SK = (1 - \rho^2)^{-3} [\gamma_{30}^2 + \gamma_{03}^2 + 3(1 + 2\rho^2)(\gamma_{12}^2 + \gamma_{21}^2) - 2\rho^3\gamma_{30}\gamma_{03} + 6\rho\{\gamma_{30}(\rho\gamma_{12} - \gamma_{21}) + \gamma_{03}(\rho\gamma_{21} - \gamma_{12}) - (2 + \rho^2)\gamma_{21}\gamma_{12}\}] \quad (3.7)$$

$$KU = \frac{\gamma_{40} + \gamma_{04} + 2\gamma_{22} + 4\rho(\rho\gamma_{22} - \gamma_{13} - \gamma_{31})}{(1 - \rho^2)^2} \quad (3.8)$$

Where  $\gamma_{rs} = \frac{\mu_{rs}}{\sigma_1^r \sigma_2^s}$ ,  $\rho = \text{corr}(Y_1, Y_2) = \frac{E((Y_1 - E(Y_1))(Y_2 - E(Y_2)))}{\sigma_1 \sigma_2}$  where  $\mu_{rs}$  is the central moment

of order  $(r, s)$  of  $(Y_1, Y_2)$ ,  $\sigma_1 = \sqrt{E(Y_1 - E(Y_1))^2}$  and  $\sigma_2 = \sqrt{E(Y_2 - E(Y_2))^2}$

$$\mu_{rs} = E((Y_1 - E(Y_1))^r (Y_2 - E(Y_2))^s)$$

**Table 1:** Covariance, Skewness, and Kurtosis of FGMBW Distribution

| $\alpha_1 = 1.8, \beta_1 = 3.5, \alpha_2 = 1.5, \beta_2 = 2.5$ |         |         |        |          |
|----------------------------------------------------------------|---------|---------|--------|----------|
| $\theta$                                                       | $cov$   | $\rho$  | $SK$   | $KU$     |
| -1                                                             | -0.8308 | -0.3030 | 1.8771 | -39.8120 |
| -0.8                                                           | -0.6646 | -0.2424 | 1.8317 | -27.0918 |
| -0.6                                                           | -0.4985 | -0.1818 | 1.7982 | -16.3708 |
| -0.4                                                           | -0.3323 | -0.1212 | 1.7747 | -6.9536  |
| -0.2                                                           | -0.1662 | -0.0606 | 1.7605 | 1.6790   |
| 0                                                              | 0.0000  | 0.0000  | 1.7556 | 9.9473   |
| 0.2                                                            | 0.1662  | 0.0606  | 1.7608 | 18.2257  |
| 0.4                                                            | 0.3323  | 0.1212  | 1.7777 | 26.8865  |
| 0.6                                                            | 0.4985  | 0.1818  | 1.8086 | 36.3420  |
| 0.8                                                            | 0.6646  | 0.2424  | 1.8572 | 47.0963  |
| 1                                                              | 0.8308  | 0.3030  | 1.9292 | 59.8180  |
| $\alpha_1 = 1.5, \beta_1 = 1.2, \alpha_2 = 2.1, \beta_2 = 2.8$ |         |         |        |          |
| $\theta$                                                       | $cov$   | $\rho$  | $SK$   | $KU$     |
| -1                                                             | -0.2795 | -0.3063 | 1.4634 | -20.5682 |
| -0.8                                                           | -0.2236 | -0.2450 | 1.4755 | -12.8286 |
| -0.6                                                           | -0.1677 | -0.1838 | 1.4774 | -6.3369  |
| -0.4                                                           | -0.1118 | -0.1225 | 1.4751 | -0.6539  |
| -0.2                                                           | -0.0559 | -0.0613 | 1.4722 | 4.5456   |
| 0                                                              | 0.0000  | 0.0000  | 1.4709 | 9.5227   |
| 0.2                                                            | 0.0559  | 0.0613  | 1.4725 | 14.5085  |
| 0.4                                                            | 0.1118  | 0.1225  | 1.4773 | 19.7317  |
| 0.6                                                            | 0.1677  | 0.1838  | 1.4852 | 25.4453  |
| 0.8                                                            | 0.2236  | 0.2450  | 1.4948 | 31.9583  |
| 1                                                              | 0.2795  | 0.3063  | 1.5027 | 39.6819  |

### 3.6. Reliability Function

Osmetti and Chiodini (2011) discussed that the reliability function is more convenient to express a joint survival function as a copula of its marginal survival functions, where  $Y_1$  and  $Y_2$  be random variable with survival functions  $\bar{F}(y_1)$  and  $\bar{F}(y_2)$  as following

The reliability function of the marginal distributions is defined as

$$R(y_j; \alpha_j, \beta_j) = 1 - F(y_j; \alpha_j, \beta_j) = e^{-\left(\frac{y_j}{\beta_j}\right)^{\alpha_j}}; y > 0, \alpha, \beta > 0, j = 1, 2$$

The expression of the joint survival function for copula is as following

$$R(y_1, y_2) = C(R(y_1), R(y_2))$$

Then the reliability function of FGMBW distribution is

$$R(y_1, y_2) = e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}} e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}} \left[ 1 + \theta \left( 1 - e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}} \right) \left( 1 - e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}} \right) \right] \quad (3.9)$$

Basu (1971) defined the bivariate failure rate function for the first time as

$$h(y_1, y_2) = \frac{f(y_1, y_2)}{R(y_1, y_2)}.$$

Then the hazard rate function of FGMBW distribution is

$$h(y_1, y_2) = \frac{\frac{\alpha_1}{\beta_1} \left(\frac{y_1}{\beta_1}\right)^{\alpha_1-1} \frac{\alpha_2}{\beta_2} \left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1} \left[ 1 + \theta \left( -1 + 2e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}} \right) \left( -1 + 2e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}} \right) \right]}{\left[ 1 + \theta \left( 1 - e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}} \right) \left( 1 - e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}} \right) \right]} \quad (3.10)$$

#### 4. Estimation based on Copulas

In the section, we introduce different estimation methods that used to estimate the parameters of FGMBW distribution, such as: maximum likelihood estimation (MLE), inference functions for margins (IFM) and semi-parametric method (SP). To more information about these methods see Chen (2007), Tsukahara (2005) and Weiß (2011).

##### 4.1. Maximum Likelihood Estimation (MLE)

Elaal et al. (2017), discussed the maximum likelihood estimator to estimate all model parameters jointly, it is a one-step parametric method. Therefore, the log-likelihood is given as

$$\ln L = \sum_{j=1}^n \left[ \ln \left( f_1(y_{1j}) f_2(y_{2j}) c(F_1(y_{1j}, \delta_1), F_2(y_{2j}, \delta_2); \theta) \right) \right]$$

The parameter estimates are obtained by maximizing the log-likelihood function with respect to each parameter separately. Considering Equation (2.3) and (1.6), let

$$a(y_j; \alpha_j, \beta_j) = 1 - 2 \left( 1 - e^{-\left(\frac{y_j}{\beta_j}\right)^{\alpha_j}} \right), j = 1, 2.$$

The likelihood function of a FGMBW distribution is defined as

$$L = \left( \frac{\alpha_2 \alpha_1}{\beta_2 \beta_1} \right)^n \prod_{i=1}^n \left( \left( \frac{y_{1i}}{\beta_1} \right)^{\alpha_1-1} \left( \frac{y_{2i}}{\beta_2} \right)^{\alpha_2-1} \right) e^{-\sum_{i=1}^n \left( \frac{y_{1i}}{\beta_1} \right)^{\alpha_1} - \sum_{i=1}^n \left( \frac{y_{2i}}{\beta_2} \right)^{\alpha_2}} \prod_{i=1}^n \left( 1 + \theta(a(y_{1i}, \alpha_1, \beta_1))(a(y_{2i}, \alpha_2, \beta_2)) \right)$$

and the log-likelihood function can be written as

$$\begin{aligned} \ln L = & n (\ln \alpha_1 - \ln \beta_1) + n (\ln \alpha_2 - \ln \beta_2) + (\alpha_1 - 1) \sum_{i=1}^n \ln \left( \frac{y_{1i}}{\beta_1} \right) - \sum_{i=1}^n \left( \frac{y_{1i}}{\beta_1} \right)^{\alpha_1} \\ & + (\alpha_2 - 1) \sum_{i=1}^n \ln \left( \frac{y_{2i}}{\beta_2} \right) - \sum_{i=1}^n \left( \frac{y_{2i}}{\beta_2} \right)^{\alpha_2} \\ & + \sum_{i=1}^n \ln \left( 1 + \theta(a(y_{1i}, \alpha_1, \beta_1))(a(y_{2i}, \alpha_2, \beta_2)) \right) \end{aligned} \quad (4.1)$$

The estimates of all parameters are obtained by differentiating the log-likelihood function in (4.1) with respect to each parameter separately, as following

$$\begin{aligned} \frac{\partial L}{\partial \alpha_1} &= \frac{n}{\alpha_1} + \sum_{i=1}^n \ln \left( \frac{y_{1i}}{\beta_1} \right) - \sum_{i=1}^n \left( \frac{y_{1i}}{\beta_1} \right)^{\alpha_1} \ln \frac{y_{1i}}{\beta_1} \\ &\quad + \sum_{i=1}^n \frac{-2 \theta(a(y_{2i}, \alpha_2, \beta_2)) e^{-\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}} \left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1} \ln \frac{y_{1i}}{\beta_1}}{\left(1 + \theta(a(y_{1i}, \alpha_1, \beta_1))(a(y_{2i}, \alpha_2, \beta_2))\right)} \\ \frac{\partial L}{\partial \alpha_2} &= \frac{n}{\alpha_2} + \sum_{i=1}^n \ln \left( \frac{y_{2i}}{\beta_2} \right) - \sum_{i=1}^n \left( \frac{y_{2i}}{\beta_2} \right)^{\alpha_2} \ln \frac{y_{2i}}{\beta_2} \\ &\quad + \sum_{i=1}^n \frac{-2 \theta(a(y_{1i}, \alpha_1, \beta_1)) e^{-\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2}} \left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2} \ln \frac{y_{2i}}{\beta_2}}{\left(1 + \theta(a(y_{1i}, \alpha_1, \beta_1))(a(y_{2i}, \alpha_2, \beta_2))\right)} \\ \frac{\partial L}{\partial \beta_1} &= \frac{-n}{\beta_1} - \frac{(\alpha_1 - 1)}{\beta_1} + \frac{\alpha_1}{\beta_1} \sum_{i=1}^n \left( \frac{y_{1i}}{\beta_1} \right)^{\alpha_1} + \sum_{i=1}^n \frac{2 \alpha_1 \theta y_{1i} \left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1-1} e^{-\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}} (a(y_{2i}, \alpha_2, \beta_2))}{\beta_1^2 \left(1 + \theta(a(y_{1i}, \alpha_1, \beta_1))(a(y_{2i}, \alpha_2, \beta_2))\right)} \\ \frac{\partial L}{\partial \beta_2} &= \frac{-n}{\beta_2} - \frac{n(\alpha_2 - 1)}{\beta_2} + \frac{\alpha_2}{\beta_2} \sum_{i=1}^n \left( \frac{y_{2i}}{\beta_2} \right)^{\alpha_2} \\ &\quad + \sum_{i=1}^n \frac{2 \alpha_2 \theta y_{2i} \left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2}} (a(y_{1i}, \alpha_1, \beta_1))}{\beta_2^2 \left(1 + \theta(a(y_{1i}, \alpha_1, \beta_1))(a(y_{2i}, \alpha_2, \beta_2))\right)} \end{aligned}$$

and

$$\frac{\partial L}{\partial \theta} = \sum_{i=1}^n \frac{(a(y_{1i}, \alpha_1, \beta_1))(a(y_{2i}, \alpha_2, \beta_2))}{\left(1 + \theta(a(y_{1i}, \alpha_1, \beta_1))(a(y_{2i}, \alpha_2, \beta_2))\right)}.$$

The MLE  $\hat{\delta} = (\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\theta})$  can be obtained by solving simultaneously the likelihood equations

$$\left. \frac{\partial L}{\partial \theta} \right|_{\theta=\hat{\theta}} = 0, \quad \left. \frac{\partial L}{\partial \beta_j} \right|_{\beta_j=\hat{\beta}_j} = 0, \quad \left. \frac{\partial L}{\partial \alpha_j} \right|_{\alpha_j=\hat{\alpha}_j} = 0, \quad j = 1, 2$$

But the equations has to be performed numerically using a nonlinear optimization algorithm.

#### 4.2. Estimation by Inference Functions for Margins (IFM)

Joe (2005) introduced this parametric method with two-step of estimation. In the first step, each marginal distribution is estimated separately.

$$\ln L_1 = \sum_{j=1}^n \ln f_1(y_{1j}, \delta_1); \ln L_2 = \sum_{i=1}^n \ln f_2(y_{2j}, \delta_2) \quad (4.2)$$

Then, in the second step the copula parameter is estimated by maximizing the log-likelihood function of the copula density using the ML estimates of the marginal  $\hat{F}_1(y_{1j}, \delta)$  and  $\hat{F}_2(y_{2j}, \delta)$ . Considering the Equation (2.3), the log likelihood function of a Weibull distribution is defined as

$$\ln L_j = n (\ln \alpha_j - \ln \beta_j) + (\alpha_j - 1) \sum_{i=1}^n \ln \left( \frac{y_{ji}}{\beta_j} \right) - \sum_{i=1}^n \left( \frac{y_{ji}}{\beta_j} \right)^{\alpha_j}; j = 1, 2 \quad (4.3)$$

The MLEs( $\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2$ ) can be obtained by solving simultaneously the likelihood equations

$$\left. \frac{\partial \ln L}{\partial \beta_j} \right|_{\beta_j = \hat{\beta}_j} = 0, \quad \left. \frac{\partial \ln L}{\partial \alpha_j} \right|_{\alpha_j = \hat{\alpha}_j} = 0, \quad j = 1, 2$$

then

$$\hat{F}_j(y_j) = 1 - e^{-\left(\frac{y_j}{\hat{\beta}_j}\right)^{\hat{\alpha}_j}}; j = 1, 2$$

and considering the previous step, the IFM estimate of a FGMBW distribution is defined as

$$\ln L_{IFM} = \sum_{i=1}^n \ln \left( 1 + \theta \left( 1 - 2\hat{F}_1(y_1) \left( 1 - 2\hat{F}_2(y_2) \right) \right) \right) \quad (4.4)$$

The estimates of all parameters are obtained by differentiating the log-likelihood function in (4.4) with respect to each parameter separately. Basing on this, differentiating the log-likelihood function with respect to  $\theta$  is given as

$$\frac{\partial \ln L_{IFM}}{\partial \theta} = \sum_{i=1}^n \frac{\left( a(y_{1i}, \hat{\alpha}_1, \hat{\beta}_1) \right) \left( a(y_{2i}, \hat{\alpha}_2, \hat{\beta}_2) \right)}{\left( 1 + \theta \left( a(y_{1i}, \hat{\alpha}_1, \hat{\beta}_1) \right) \left( a(y_{2i}, \hat{\alpha}_2, \hat{\beta}_2) \right) \right)}$$

The estimates of parameters are handled numerically simultaneously the likelihood equations

$$\left. \frac{\partial \ln L_{IFM}}{\partial \theta} \right|_{\theta = \hat{\theta}} = 0$$

There is no closed-form expression for the MLE  $\hat{\theta}$  and its computation has to be performed numerically using a nonlinear optimization algorithm.

#### 4.3. Estimation by Semi-parametric Method [SP]

Kim et al. (2007) introduced Estimation that carried out in two stages as in IFM, but the difference is that the marginal distributions are estimated non-parametrically by their sample empirical distributions. In this method, the observations are transformed into pseudo-observations using the empirical distribution function of each marginal distribution. The empirical distribution function is defined as

$$\tilde{F}_i(y_i) = \frac{\sum_{j=1}^n I(Y_{i,j} \leq y_i)}{n+1} ; i = 1, 2 \quad (4.5)$$

Then,  $\theta$  is estimated by the maximizer of the pseudo loglikelihood,

$$\sum_{j=1}^n \ln c(\tilde{F}_1(Y_1), \tilde{F}_2(Y_2); \theta) \quad (4.6)$$

Considering the Equation (4.6), the log likelihood function of a FGMBW distribution is defined as

$$\ln L_{SP} = \sum_{i=1}^n \ln \left( 1 + \theta \left( 1 - 2\tilde{F}_1(y_1) \left( 1 - 2\tilde{F}_2(y_2) \right) \right) \right) \quad (4.7)$$

There is no closed-form expression for the MLE  $\hat{\theta}$  by use of equation (4.7) and its computation has to be performed numerically using a statistical software.

## 5. Confidence Interval

In this section, we propose different confidence intervals. One is based on the asymptotic distribution of  $\alpha_k, \beta_k$  when  $k = 1, 2$  and  $\theta$  by two different bootstrap confidence intervals

### 5.1. Asymptotic Confidence Intervals

In this section, we propose the asymptotic confidence intervals using methods of estimations. Keeping this in mind, we may propose the asymptotic confidence intervals using ML, IFM, and SP methods can be used to construct the confidence intervals for the parameters. First, we obtain  $I(\hat{\delta})$  which is the observed inverse Fishers information matrix and it is defined as:

$$I(\hat{\delta}) = \begin{bmatrix} -L''_{\alpha_1\alpha_1} & -L''_{\alpha_1\beta_1} & -L''_{\alpha_1\alpha_2} & -L''_{\alpha_1\beta_2} & -L''_{\alpha_1\theta} \\ -L''_{\beta_1\alpha_1} & -L''_{\beta_1\beta_1} & -L''_{\beta_1\alpha_2} & -L''_{\beta_1\beta_2} & -L''_{\beta_1\theta} \\ -L''_{\alpha_2\alpha_1} & -L''_{\alpha_2\beta_1} & -L''_{\alpha_2\alpha_2} & -L''_{\alpha_2\beta_2} & -L''_{\alpha_2\theta} \\ -L''_{\beta_2\alpha_1} & -L''_{\beta_2\beta_1} & -L''_{\beta_2\alpha_2} & -L''_{\beta_2\beta_2} & -L''_{\beta_2\theta} \\ -L''_{\theta\alpha_1} & -L''_{\theta\beta_1} & -L''_{\theta\alpha_2} & -L''_{\theta\beta_2} & -L''_{\theta\theta} \end{bmatrix}^{-1} = \begin{bmatrix} I_{\hat{\alpha}_1\hat{\alpha}_1} & I_{\hat{\alpha}_1\hat{\beta}_1} & I_{\hat{\alpha}_1\hat{\alpha}_2} & I_{\hat{\alpha}_1\hat{\beta}_2} & I_{\hat{\alpha}_1\hat{\theta}} \\ I_{\hat{\beta}_1\hat{\alpha}_1} & I_{\hat{\beta}_1\hat{\beta}_1} & I_{\hat{\beta}_1\hat{\alpha}_2} & I_{\hat{\beta}_1\hat{\beta}_2} & I_{\hat{\beta}_1\hat{\theta}} \\ I_{\hat{\alpha}_2\hat{\alpha}_1} & I_{\hat{\alpha}_2\hat{\beta}_1} & I_{\hat{\alpha}_2\hat{\alpha}_2} & I_{\hat{\alpha}_2\hat{\beta}_2} & I_{\hat{\alpha}_2\hat{\theta}} \\ I_{\hat{\beta}_2\hat{\alpha}_1} & I_{\hat{\beta}_2\hat{\beta}_1} & I_{\hat{\beta}_2\hat{\alpha}_2} & I_{\hat{\beta}_2\hat{\beta}_2} & I_{\hat{\beta}_2\hat{\theta}} \\ I_{\hat{\theta}\hat{\alpha}_1} & I_{\hat{\theta}\hat{\beta}_1} & I_{\hat{\theta}\hat{\alpha}_2} & I_{\hat{\theta}\hat{\beta}_2} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix} \quad (5.1)$$

An approximate 95% two side confidence intervals for  $\delta$  where  $\hat{\delta} = (\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\theta})$  are respectively

$$\hat{\alpha}_i \pm Z_{0.025} \sqrt{I_{\hat{\alpha}_i\hat{\alpha}_i}}, \quad \hat{\beta}_i \pm Z_{0.025} \sqrt{I_{\hat{\beta}_i\hat{\beta}_i}} ; i = 1, 2, \quad \text{and } \hat{\theta} \pm Z_{0.025} \sqrt{I_{\hat{\theta}\hat{\theta}}}$$

### 5.2. Bootstrap Confidence Interval

Here, we construct two parametric bootstrap confidence interval for  $\alpha_k, \beta_k$  when  $k = 1, 2$  and  $\theta$  as follow:

#### 5.2.1. Percentile Bootstrap Confidence Interval (PB)

I. Compute the MLE of  $\delta = (\alpha_k, \beta_k, \theta)$  and  $k = 1, 2$  for FGMBW distribution.

II. Generated a bootstrap samples using  $\alpha_k, \beta_k$  and  $\theta$  to obtain the bootstrap estimate of  $\alpha_k$  say  $\hat{\alpha}_k^b$ ,  $\beta_k$  say  $\hat{\beta}_k^b$  and  $\theta$  say  $\hat{\theta}^b$  using the bootstrap sample.

- III. Repeat step (II) B times to have  $(\alpha_k^{b(1)}, \alpha_k^{b(2)}, \dots, \alpha_k^{b(B)})$ ,  $(\beta_k^{b(1)}, \beta_k^{b(2)}, \dots, \beta_k^{b(B)})$  and  $(\theta^{b(1)}, \theta^{b(2)}, \dots, \theta^{b(B)})$ .
- IV. Arrange  $(\alpha_k^{b(1)}, \alpha_k^{b(2)}, \dots, \alpha_k^{b(B)})$ ,  $(\beta_k^{b(1)}, \beta_k^{b(2)}, \dots, \beta_k^{b(B)})$  and  $(\theta^{b(1)}, \theta^{b(2)}, \dots, \theta^{b(B)})$  in ascending order as  $(\alpha_k^{b[1]}, \alpha_k^{b[2]}, \dots, \alpha_k^{b[B]})$ ,  $(\beta_k^{b[1]}, \beta_k^{b[2]}, \dots, \beta_k^{b[B]})$  and  $(\theta^{b[1]}, \theta^{b[2]}, \dots, \theta^{b[B]})$ .
- V. A two side 100 (1 -  $\gamma$ )% percentile bootstrap confidence interval for the unknown parameters  $\alpha_k, \beta_k$  and  $\theta$  is given by  $\{\hat{\alpha}_k^{b[B\gamma/2]}, \hat{\alpha}_k^{b[B(1-\gamma/2)]}\}$ ,  $\{\hat{\beta}_k^{b[B\gamma/2]}, \hat{\beta}_k^{b[B(1-\gamma/2)]}\}$  and  $\{\hat{\theta}^{b[B\gamma/2]}, \hat{\theta}^{b[B(1-\gamma/2)]}\}$ .

### 5.2.2. Bootstrap-t Confidence Interval (TB)

- I. Same as the steps (I-II) in Boot-p
- II. Compute the t-statistic of  $\delta = (\alpha_k, \beta_k, \theta)$  and  $k = 1, 2$  as  $T = \frac{(\hat{\delta}_k^b - \delta_k)}{\sqrt{V(\hat{\delta}_k^b)}}$  where  $V(\hat{\delta}_k^b)$  is asymptotic variances of  $\hat{\delta}_k^b$  and it can be obtained using the Fisher information matrix.
- III. Repeat step II-III B times and obtain  $T^{(1)}, T^{(2)}, \dots, T^{(B)}$ .
- IV. Arrange  $T^{(1)}, T^{(2)}, \dots, T^{(B)}$  in ascending order as  $T^{[1]}, T^{[2]}, \dots, T^{[B]}$ .
- V. A two side 100 (1 -  $\gamma$ )% percentile bootstrap-t confidence interval for the unknown parameters  $\alpha_k, \beta_k$  and  $\theta$  is given by  $\{\hat{\alpha}_k + T_k^{[B\gamma/2]} \sqrt{V(\hat{\alpha}_k)}, \hat{\alpha}_k + T_k^{[B(1-\gamma/2)]} \sqrt{V(\hat{\alpha}_k)}\}$ ,  $\{\hat{\beta}_k + T_k^{[B\gamma/2]} \sqrt{V(\hat{\beta}_k)}, \hat{\beta}_k + T_k^{[B(1-\gamma/2)]} \sqrt{V(\hat{\beta}_k)}\}$  and  $\{\hat{\theta} + T^{[B\gamma/2]} \sqrt{V(\hat{\theta})}, \hat{\theta} + T^{[B(1-\gamma/2)]} \sqrt{V(\hat{\theta})}\}$ .

## 6. Simulation Study

In this section; a simulation is done for comparison between estimation methods based on copula such as: MLE, IFM and SP. For estimating FGMBW distribution parameters by R language.

**Simulation Algorithm:** A simulation experiments were carried out based on the following data-generated form Weibull Distributions, where  $Y_1, Y_2$  are distributed as Weibull with  $\beta_i$  shape parameters and  $\alpha_i$  scale parameter,  $i=1,2$  the values of the parameters  $\alpha_1, \beta_1, \alpha_2, \beta_2$  and  $\theta$  is chosen as the following cases for the random variables generating:

Case1:  $(\alpha_1 = 1.8, \beta_1 = 3.5, \alpha_2 = 1.5, \beta_2 = 2.5, \theta = -0.75)$ ,

Case2:  $(\alpha_1 = 1.8, \beta_1 = 3.5, \alpha_2 = 1.5, \beta_2 = 2.5, \theta = -0.25)$ ,

Case 3:  $(\alpha_1 = 1.8, \beta_1 = 3.5, \alpha_2 = 1.5, \beta_2 = 2.5, \theta = 0.25)$  and

Case4:  $(\alpha_1 = 1.8, \beta_1 = 3.5, \alpha_2 = 1.5, \beta_2 = 2.5, \theta = 0.75)$ . For different sample size  $n = 30, 50, 100$ , and 150. All computations are obtained based on the R language. The simulation methods are compared using the criteria of parameters estimation, the comparison is performed by

calculate in the Bias, the MSE, the length of confidence interval (L.CI) and confidence interval under bootstrap for each method as following

$$Bias = (\hat{\delta} - \delta) . \quad (6.1)$$

Where  $\hat{\delta}$  is the estimated value of  $\delta$ .

$$MSE = Mean(\hat{\delta} - \delta)^2. \quad (6.2)$$

and

$$L. CI = Upper. CI - Lower. CI \quad (6.3)$$

We restricted the number of repeated-samples to 1000.

**Table 2:** Estimation of the Parameters of FGMBW Distribution: Case 1

| n   |     |                  | Mean    | Bias    | MSE    | L.CI   | PT     | PP     |
|-----|-----|------------------|---------|---------|--------|--------|--------|--------|
| 30  | MLE | $\hat{\alpha}_1$ | 1.8637  | 0.0637  | 0.0719 | 1.0216 | 0.0331 | 0.0326 |
|     |     | $\hat{\beta}_1$  | 3.5156  | 0.0156  | 0.1403 | 1.4677 | 0.0457 | 0.0456 |
|     |     | $\hat{\alpha}_2$ | 1.5620  | 0.0620  | 0.0544 | 0.8817 | 0.0281 | 0.0276 |
|     |     | $\hat{\beta}_2$  | 2.5010  | 0.0010  | 0.0999 | 1.2394 | 0.0403 | 0.0400 |
|     |     | $\hat{\theta}$   | -0.9667 | -0.2167 | 1.3285 | 5.9241 | 0.1822 | 0.1837 |
|     | IFM | $\hat{\alpha}_1$ | 1.8708  | 0.0708  | 0.0702 | 1.0017 | 0.0316 | 0.0323 |
|     |     | $\hat{\beta}_1$  | 3.5102  | 0.0102  | 0.1171 | 1.3412 | 0.0453 | 0.0458 |
|     |     | $\hat{\alpha}_2$ | 1.5701  | 0.0701  | 0.0532 | 0.8618 | 0.0272 | 0.0281 |
|     |     | $\hat{\beta}_2$  | 2.5026  | 0.0026  | 0.0875 | 1.1601 | 0.0360 | 0.0359 |
|     |     | $\hat{\theta}$   | -0.8110 | -0.0610 | 0.4436 | 2.6011 | 0.0827 | 0.0812 |
|     | SP  | $\hat{\theta}$   | -0.8565 | -0.1065 | 0.5057 | 2.7576 | 0.0861 | 0.0862 |
| 50  | MLE | $\hat{\alpha}_1$ | 1.8378  | 0.0378  | 0.0437 | 0.8067 | 0.0253 | 0.0258 |
|     |     | $\hat{\beta}_1$  | 3.5043  | 0.0043  | 0.0884 | 1.1657 | 0.0352 | 0.0366 |
|     |     | $\hat{\alpha}_2$ | 1.5380  | 0.0380  | 0.0314 | 0.6792 | 0.0221 | 0.0215 |
|     |     | $\hat{\beta}_2$  | 2.5000  | 0.0000  | 0.0689 | 1.0292 | 0.0316 | 0.0321 |
|     |     | $\hat{\theta}$   | -0.8352 | -0.0852 | 0.4845 | 2.7095 | 0.0866 | 0.0886 |
|     | IFM | $\hat{\alpha}_1$ | 1.8421  | 0.0421  | 0.0432 | 0.7984 | 0.0250 | 0.0252 |
|     |     | $\hat{\beta}_1$  | 3.5038  | 0.0038  | 0.0805 | 1.1129 | 0.0354 | 0.0350 |
|     |     | $\hat{\alpha}_2$ | 1.5423  | 0.0423  | 0.0314 | 0.6745 | 0.0213 | 0.0213 |
|     |     | $\hat{\beta}_2$  | 2.4978  | -0.0022 | 0.0621 | 0.9769 | 0.0305 | 0.0309 |
|     |     | $\hat{\theta}$   | -0.7797 | -0.0297 | 0.2230 | 1.8484 | 0.0596 | 0.0596 |
|     | SP  | $\hat{\theta}$   | -0.8071 | -0.0571 | 0.2494 | 1.9458 | 0.0597 | 0.0599 |
| 100 | MLE | $\hat{\alpha}_1$ | 1.8240  | 0.0240  | 0.0219 | 0.5731 | 0.0184 | 0.0186 |
|     |     | $\hat{\beta}_1$  | 3.5029  | 0.0029  | 0.0425 | 0.8088 | 0.0252 | 0.0248 |
|     |     | $\hat{\alpha}_2$ | 1.5219  | 0.0219  | 0.0143 | 0.4602 | 0.0144 | 0.0146 |
|     |     | $\hat{\beta}_2$  | 2.4975  | -0.0025 | 0.0301 | 0.6808 | 0.0219 | 0.0224 |
|     |     | $\hat{\theta}$   | -0.7662 | -0.0162 | 0.0932 | 1.1954 | 0.0379 | 0.0386 |
|     | IFM | $\hat{\alpha}_1$ | 1.8252  | 0.0252  | 0.0220 | 0.5732 | 0.0179 | 0.0177 |
|     |     | $\hat{\beta}_1$  | 3.5032  | 0.0032  | 0.0427 | 0.8102 | 0.0247 | 0.0246 |
|     |     | $\hat{\alpha}_2$ | 1.5232  | 0.0232  | 0.0142 | 0.4592 | 0.0142 | 0.0145 |
|     |     | $\hat{\beta}_2$  | 2.4972  | -0.0028 | 0.0300 | 0.6787 | 0.0217 | 0.0214 |
|     |     | $\hat{\theta}$   | -0.7582 | -0.0082 | 0.0878 | 1.1618 | 0.0357 | 0.0365 |
|     | SP  | $\hat{\theta}$   | -0.7722 | -0.0222 | 0.0927 | 1.1911 | 0.0377 | 0.0368 |
| 150 | MLE | $\hat{\alpha}_1$ | 1.8172  | 0.0172  | 0.0139 | 0.4582 | 0.0147 | 0.0147 |
|     |     | $\hat{\beta}_1$  | 3.4995  | -0.0005 | 0.0281 | 0.6572 | 0.0200 | 0.0204 |
|     |     | $\hat{\alpha}_2$ | 1.5129  | 0.0129  | 0.0090 | 0.3695 | 0.0115 | 0.0113 |
|     |     | $\hat{\beta}_2$  | 2.5002  | 0.0002  | 0.0207 | 0.5638 | 0.0182 | 0.0187 |
|     |     | $\hat{\theta}$   | -0.7529 | -0.0029 | 0.0525 | 0.8982 | 0.0278 | 0.0274 |
|     | IFM | $\hat{\alpha}_1$ | 1.8177  | 0.0177  | 0.0141 | 0.4597 | 0.0144 | 0.0144 |
|     |     | $\hat{\beta}_1$  | 3.4991  | -0.0009 | 0.0281 | 0.6571 | 0.0206 | 0.0212 |
|     |     | $\hat{\alpha}_2$ | 1.5139  | 0.0139  | 0.0090 | 0.3690 | 0.0120 | 0.0121 |
|     |     | $\hat{\beta}_2$  | 2.5008  | 0.0008  | 0.0212 | 0.5713 | 0.0176 | 0.0184 |
|     |     | $\hat{\theta}$   | -0.7483 | 0.0017  | 0.0502 | 0.8788 | 0.0285 | 0.0289 |
|     | SP  | $\hat{\theta}$   | -0.7573 | -0.0073 | 0.0518 | 0.8923 | 0.0280 | 0.0289 |

**Table 3:** Estimation of the Parameters of FGMBW Distribution: Case 2

| $n$ |     |                  | Mean    | Bias    | MSE    | L.CI   | PT     | PP     |
|-----|-----|------------------|---------|---------|--------|--------|--------|--------|
| 30  | MLE | $\hat{\alpha}_1$ | 1.8689  | 0.0689  | 0.0715 | 1.0135 | 0.0330 | 0.0321 |
|     |     | $\hat{\beta}_1$  | 3.5107  | 0.0107  | 0.1249 | 1.3853 | 0.0428 | 0.0417 |
|     |     | $\hat{\alpha}_2$ | 1.5653  | 0.0653  | 0.0528 | 0.8644 | 0.0279 | 0.0283 |
|     |     | $\hat{\beta}_2$  | 2.5049  | 0.0049  | 0.0969 | 1.2206 | 0.0390 | 0.0388 |
|     |     | $\hat{\theta}$   | -0.2658 | -0.0158 | 0.7961 | 3.4989 | 0.1079 | 0.1120 |
|     | IFM | $\hat{\alpha}_1$ | 1.8708  | 0.0708  | 0.0702 | 1.0017 | 0.0316 | 0.0323 |
|     |     | $\hat{\beta}_1$  | 3.5102  | 0.0102  | 0.1171 | 1.3412 | 0.0453 | 0.0458 |
|     |     | $\hat{\alpha}_2$ | 1.5694  | 0.0694  | 0.0534 | 0.8645 | 0.0273 | 0.0286 |
|     |     | $\hat{\beta}_2$  | 2.5050  | 0.0050  | 0.0890 | 1.1699 | 0.0362 | 0.0359 |
|     |     | $\hat{\theta}$   | -0.2365 | 0.0135  | 0.3439 | 2.2993 | 0.0726 | 0.0722 |
|     | SP  | $\hat{\theta}$   | -0.2488 | 0.0012  | 0.4100 | 2.5112 | 0.0789 | 0.0797 |
| 50  | MLE | $\hat{\alpha}_1$ | 1.8420  | 0.0420  | 0.0435 | 0.8015 | 0.0253 | 0.0252 |
|     |     | $\hat{\beta}_1$  | 3.5045  | 0.0045  | 0.0821 | 1.1237 | 0.0342 | 0.0348 |
|     |     | $\hat{\alpha}_2$ | 1.5413  | 0.0413  | 0.0311 | 0.6725 | 0.0219 | 0.0219 |
|     |     | $\hat{\beta}_2$  | 2.4984  | -0.0016 | 0.0641 | 0.9933 | 0.0307 | 0.0301 |
|     |     | $\hat{\theta}$   | -0.2505 | -0.0005 | 0.2220 | 1.8478 | 0.0593 | 0.0604 |
|     | IFM | $\hat{\alpha}_1$ | 1.8421  | 0.0421  | 0.0432 | 0.7984 | 0.0250 | 0.0252 |
|     |     | $\hat{\beta}_1$  | 3.5038  | 0.0038  | 0.0805 | 1.1129 | 0.0354 | 0.0350 |
|     |     | $\hat{\alpha}_2$ | 1.5421  | 0.0421  | 0.0311 | 0.6714 | 0.0213 | 0.0211 |
|     |     | $\hat{\beta}_2$  | 2.4985  | -0.0015 | 0.0635 | 0.9881 | 0.0309 | 0.0318 |
|     |     | $\hat{\theta}$   | -0.2448 | 0.0052  | 0.2089 | 1.7924 | 0.0577 | 0.0572 |
|     | SP  | $\hat{\theta}$   | -0.2538 | -0.0038 | 0.2326 | 1.8913 | 0.0580 | 0.0585 |
| 100 | MLE | $\hat{\alpha}_1$ | 1.8252  | 0.0252  | 0.0221 | 0.5743 | 0.0183 | 0.0184 |
|     |     | $\hat{\beta}_1$  | 3.5035  | 0.0035  | 0.0429 | 0.8121 | 0.0253 | 0.0252 |
|     |     | $\hat{\alpha}_2$ | 1.5230  | 0.0230  | 0.0142 | 0.4579 | 0.0144 | 0.0147 |
|     |     | $\hat{\beta}_2$  | 2.4976  | -0.0024 | 0.0308 | 0.6883 | 0.0224 | 0.0227 |
|     |     | $\hat{\theta}$   | -0.2472 | 0.0028  | 0.0933 | 1.1977 | 0.0383 | 0.0399 |
|     | IFM | $\hat{\alpha}_1$ | 1.8252  | 0.0252  | 0.0220 | 0.5732 | 0.0179 | 0.0177 |
|     |     | $\hat{\beta}_1$  | 3.5032  | 0.0032  | 0.0427 | 0.8102 | 0.0247 | 0.0246 |
|     |     | $\hat{\alpha}_2$ | 1.5233  | 0.0233  | 0.0141 | 0.4567 | 0.0142 | 0.0146 |
|     |     | $\hat{\beta}_2$  | 2.4978  | -0.0022 | 0.0306 | 0.6863 | 0.0219 | 0.0218 |
|     |     | $\hat{\theta}$   | -0.2458 | 0.0042  | 0.0921 | 1.1904 | 0.0372 | 0.0383 |
|     | SP  | $\hat{\theta}$   | -0.2506 | -0.0006 | 0.0987 | 1.2319 | 0.0393 | 0.0395 |
| 150 | MLE | $\hat{\alpha}_1$ | 1.8177  | 0.0177  | 0.0140 | 0.4596 | 0.0148 | 0.0149 |
|     |     | $\hat{\beta}_1$  | 3.4992  | -0.0008 | 0.0282 | 0.6582 | 0.0200 | 0.0206 |
|     |     | $\hat{\alpha}_2$ | 1.5135  | 0.0135  | 0.0090 | 0.3685 | 0.0115 | 0.0112 |
|     |     | $\hat{\beta}_2$  | 2.5008  | 0.0008  | 0.0213 | 0.5724 | 0.0183 | 0.0183 |
|     |     | $\hat{\theta}$   | -0.2427 | 0.0073  | 0.0547 | 0.9165 | 0.0286 | 0.0291 |
|     | IFM | $\hat{\alpha}_1$ | 1.8177  | 0.0177  | 0.0141 | 0.4597 | 0.0144 | 0.0144 |
|     |     | $\hat{\beta}_1$  | 3.4991  | -0.0009 | 0.0281 | 0.6571 | 0.0206 | 0.0212 |
|     |     | $\hat{\alpha}_2$ | 1.5136  | 0.0136  | 0.0090 | 0.3679 | 0.0119 | 0.0122 |
|     |     | $\hat{\beta}_2$  | 2.5008  | 0.0008  | 0.0214 | 0.5731 | 0.0178 | 0.0185 |
|     |     | $\hat{\theta}$   | -0.2421 | 0.0079  | 0.0543 | 0.9133 | 0.0295 | 0.0292 |
|     | SP  | $\hat{\theta}$   | -0.2464 | 0.0036  | 0.0568 | 0.9348 | 0.0300 | 0.0305 |

**Table 4:** Estimation of the Parameters of FGMBW Distribution: Case 3

| $n$ |     |                  | Mean   | Bias    | MSE    | L.CI   | PT     | PP     |
|-----|-----|------------------|--------|---------|--------|--------|--------|--------|
| 30  | MLE | $\hat{\alpha}_1$ | 1.8690 | 0.0690  | 0.0708 | 1.0079 | 0.0329 | 0.0327 |
|     |     | $\hat{\beta}_1$  | 3.5106 | 0.0106  | 0.1212 | 1.3650 | 0.0421 | 0.0410 |
|     |     | $\hat{\alpha}_2$ | 1.5656 | 0.0656  | 0.0532 | 0.8675 | 0.0278 | 0.0286 |
|     |     | $\hat{\beta}_2$  | 2.5061 | 0.0061  | 0.0935 | 1.1987 | 0.0383 | 0.0376 |
|     |     | $\hat{\theta}$   | 0.3188 | 0.0688  | 0.6125 | 3.0574 | 0.0983 | 0.1024 |
|     | IFM | $\hat{\alpha}_1$ | 1.8708 | 0.0708  | 0.0702 | 1.0017 | 0.0316 | 0.0323 |
|     |     | $\hat{\beta}_1$  | 3.5102 | 0.0102  | 0.1171 | 1.3412 | 0.0453 | 0.0458 |
|     |     | $\hat{\alpha}_2$ | 1.5683 | 0.0683  | 0.0536 | 0.8675 | 0.0273 | 0.0281 |
|     |     | $\hat{\beta}_2$  | 2.5070 | 0.0070  | 0.0905 | 1.1792 | 0.0365 | 0.0368 |
|     |     | $\hat{\theta}$   | 0.2886 | 0.0386  | 0.3395 | 2.2800 | 0.0717 | 0.0711 |
|     | SP  | $\hat{\theta}$   | 0.3064 | 0.0564  | 0.3973 | 2.4622 | 0.0775 | 0.0793 |
| 50  | MLE | $\hat{\alpha}_1$ | 1.8422 | 0.0422  | 0.0435 | 0.8008 | 0.0252 | 0.0248 |
|     |     | $\hat{\beta}_1$  | 3.5046 | 0.0046  | 0.0823 | 1.1250 | 0.0344 | 0.0356 |
|     |     | $\hat{\alpha}_2$ | 1.5418 | 0.0418  | 0.0311 | 0.6721 | 0.0219 | 0.0221 |
|     |     | $\hat{\beta}_2$  | 2.5001 | 0.0001  | 0.0649 | 0.9989 | 0.0311 | 0.0313 |
|     |     | $\hat{\theta}$   | 0.2618 | 0.0118  | 0.2252 | 1.8606 | 0.0603 | 0.0614 |
|     | IFM | $\hat{\alpha}_1$ | 1.8421 | 0.0421  | 0.0432 | 0.7984 | 0.0250 | 0.0252 |
|     |     | $\hat{\beta}_1$  | 3.5038 | 0.0038  | 0.0805 | 1.1129 | 0.0354 | 0.0350 |
|     |     | $\hat{\alpha}_2$ | 1.5419 | 0.0419  | 0.0312 | 0.6729 | 0.0214 | 0.0216 |
|     |     | $\hat{\beta}_2$  | 2.4990 | -0.0010 | 0.0649 | 0.9989 | 0.0313 | 0.0327 |
|     |     | $\hat{\theta}$   | 0.2566 | 0.0066  | 0.2134 | 1.8116 | 0.0585 | 0.0577 |
|     | SP  | $\hat{\theta}$   | 0.2664 | 0.0164  | 0.2400 | 1.9202 | 0.0589 | 0.0612 |
| 100 | MLE | $\hat{\alpha}_1$ | 1.8254 | 0.0254  | 0.0220 | 0.5737 | 0.0183 | 0.0184 |
|     |     | $\hat{\beta}_1$  | 3.5036 | 0.0036  | 0.0428 | 0.8115 | 0.0253 | 0.0253 |
|     |     | $\hat{\alpha}_2$ | 1.5234 | 0.0234  | 0.0142 | 0.4580 | 0.0146 | 0.0147 |
|     |     | $\hat{\beta}_2$  | 2.4986 | -0.0014 | 0.0314 | 0.6948 | 0.0227 | 0.0232 |
|     |     | $\hat{\theta}$   | 0.2513 | 0.0013  | 0.0931 | 1.1969 | 0.0390 | 0.0391 |
|     | IFM | $\hat{\alpha}_1$ | 1.8252 | 0.0252  | 0.0220 | 0.5732 | 0.0179 | 0.0177 |
|     |     | $\hat{\beta}_1$  | 3.5032 | 0.0032  | 0.0427 | 0.8102 | 0.0247 | 0.0246 |
|     |     | $\hat{\alpha}_2$ | 1.5233 | 0.0233  | 0.0141 | 0.4565 | 0.0143 | 0.0144 |
|     |     | $\hat{\beta}_2$  | 2.4983 | -0.0017 | 0.0314 | 0.6949 | 0.0221 | 0.0221 |
|     |     | $\hat{\theta}$   | 0.2498 | -0.0002 | 0.0920 | 1.1895 | 0.0377 | 0.0387 |
|     | SP  | $\hat{\theta}$   | 0.2541 | 0.0041  | 0.0982 | 1.2290 | 0.0392 | 0.0403 |
| 150 | MLE | $\hat{\alpha}_1$ | 1.8178 | 0.0178  | 0.0141 | 0.4601 | 0.0148 | 0.0148 |
|     |     | $\hat{\beta}_1$  | 3.4993 | -0.0007 | 0.0281 | 0.6572 | 0.0199 | 0.0205 |
|     |     | $\hat{\alpha}_2$ | 1.5135 | 0.0135  | 0.0089 | 0.3656 | 0.0115 | 0.0114 |
|     |     | $\hat{\beta}_2$  | 2.5012 | 0.0012  | 0.0214 | 0.5741 | 0.0182 | 0.0181 |
|     |     | $\hat{\theta}$   | 0.2537 | 0.0037  | 0.0555 | 0.9236 | 0.0286 | 0.0300 |
|     | IFM | $\hat{\alpha}_1$ | 1.8177 | 0.0177  | 0.0141 | 0.4597 | 0.0144 | 0.0144 |
|     |     | $\hat{\beta}_1$  | 3.4991 | -0.0009 | 0.0281 | 0.6571 | 0.0206 | 0.0212 |
|     |     | $\hat{\alpha}_2$ | 1.5133 | 0.0133  | 0.0089 | 0.3662 | 0.0118 | 0.0121 |
|     |     | $\hat{\beta}_2$  | 2.5008 | 0.0008  | 0.0215 | 0.5757 | 0.0180 | 0.0185 |
|     |     | $\hat{\theta}$   | 0.2529 | 0.0029  | 0.0551 | 0.9203 | 0.0292 | 0.0297 |
|     | SP  | $\hat{\theta}$   | 0.2555 | 0.0055  | 0.0573 | 0.9389 | 0.0304 | 0.0310 |

**Table 5:** Estimation of the Parameters of FGMBW Distribution: Case 4

| $n$ |     |                  | Mean   | Bias    | MSE    | L.CI   | PT     | PP     |
|-----|-----|------------------|--------|---------|--------|--------|--------|--------|
| 30  | MLE | $\hat{\alpha}_1$ | 1.8624 | 0.0624  | 0.0708 | 1.0148 | 0.0329 | 0.0320 |
|     |     | $\hat{\beta}_1$  | 3.5162 | 0.0162  | 0.1393 | 1.4622 | 0.0453 | 0.0454 |
|     |     | $\hat{\alpha}_2$ | 1.5591 | 0.0591  | 0.0536 | 0.8783 | 0.0281 | 0.0280 |
|     |     | $\hat{\beta}_2$  | 2.5108 | 0.0108  | 0.1062 | 1.2774 | 0.0404 | 0.0403 |
|     |     | $\hat{\theta}$   | 1.0056 | 0.2556  | 2.4411 | 6.0451 | 0.1982 | 0.1930 |
|     | IFM | $\hat{\alpha}_1$ | 1.8708 | 0.0708  | 0.0702 | 1.0017 | 0.0316 | 0.0323 |
|     |     | $\hat{\beta}_1$  | 3.5102 | 0.0102  | 0.1171 | 1.3412 | 0.0453 | 0.0458 |
|     |     | $\hat{\alpha}_2$ | 1.5673 | 0.0673  | 0.0542 | 0.8739 | 0.0275 | 0.0277 |
|     |     | $\hat{\beta}_2$  | 2.5088 | 0.0088  | 0.0915 | 1.1857 | 0.0368 | 0.0371 |
|     |     | $\hat{\theta}$   | 0.8440 | 0.0940  | 0.4596 | 2.6332 | 0.0829 | 0.0814 |
|     | SP  | $\hat{\theta}$   | 0.8984 | 0.1484  | 0.5178 | 2.7615 | 0.0871 | 0.0881 |
| 50  | MLE | $\hat{\alpha}_1$ | 1.8396 | 0.0396  | 0.0434 | 0.8023 | 0.0252 | 0.0248 |
|     |     | $\hat{\beta}_1$  | 3.5073 | 0.0073  | 0.0895 | 1.1728 | 0.0362 | 0.0373 |
|     |     | $\hat{\alpha}_2$ | 1.5404 | 0.0404  | 0.0318 | 0.6808 | 0.0221 | 0.0227 |
|     |     | $\hat{\beta}_2$  | 2.5047 | 0.0047  | 0.0685 | 1.0260 | 0.0322 | 0.0319 |
|     |     | $\hat{\theta}$   | 0.8232 | 0.0732  | 0.3642 | 2.3494 | 0.0760 | 0.0791 |
|     | IFM | $\hat{\alpha}_1$ | 1.8421 | 0.0421  | 0.0432 | 0.7984 | 0.0250 | 0.0252 |
|     |     | $\hat{\beta}_1$  | 3.5038 | 0.0038  | 0.0805 | 1.1129 | 0.0354 | 0.0350 |
|     |     | $\hat{\alpha}_2$ | 1.5419 | 0.0419  | 0.0318 | 0.6795 | 0.0217 | 0.0222 |
|     |     | $\hat{\beta}_2$  | 2.4992 | -0.0008 | 0.0659 | 1.0072 | 0.0317 | 0.0330 |
|     |     | $\hat{\theta}$   | 0.7824 | 0.0324  | 0.2460 | 1.9413 | 0.0633 | 0.0606 |
|     | SP  | $\hat{\theta}$   | 0.8139 | 0.0639  | 0.2702 | 2.0232 | 0.0619 | 0.0632 |
| 100 | MLE | $\hat{\alpha}_1$ | 1.8248 | 0.0248  | 0.0220 | 0.5733 | 0.0183 | 0.0185 |
|     |     | $\hat{\beta}_1$  | 3.5043 | 0.0043  | 0.0421 | 0.8042 | 0.0251 | 0.0248 |
|     |     | $\hat{\alpha}_2$ | 1.5236 | 0.0236  | 0.0143 | 0.4598 | 0.0148 | 0.0149 |
|     |     | $\hat{\beta}_2$  | 2.5002 | 0.0002  | 0.0317 | 0.6987 | 0.0229 | 0.0235 |
|     |     | $\hat{\theta}$   | 0.7586 | 0.0086  | 0.0911 | 1.1830 | 0.0393 | 0.0388 |
|     | IFM | $\hat{\alpha}_1$ | 1.8252 | 0.0252  | 0.0220 | 0.5732 | 0.0179 | 0.0177 |
|     |     | $\hat{\beta}_1$  | 3.5032 | 0.0032  | 0.0427 | 0.8102 | 0.0247 | 0.0246 |
|     |     | $\hat{\alpha}_2$ | 1.5234 | 0.0234  | 0.0143 | 0.4595 | 0.0145 | 0.0147 |
|     |     | $\hat{\beta}_2$  | 2.4987 | -0.0013 | 0.0320 | 0.7020 | 0.0222 | 0.0220 |
|     |     | $\hat{\theta}$   | 0.7515 | 0.0015  | 0.0867 | 1.1551 | 0.0366 | 0.0371 |
|     | SP  | $\hat{\theta}$   | 0.7663 | 0.0163  | 0.0935 | 1.1978 | 0.0382 | 0.0379 |
| 150 | MLE | $\hat{\alpha}_1$ | 1.8178 | 0.0178  | 0.0141 | 0.4604 | 0.0148 | 0.0147 |
|     |     | $\hat{\beta}_1$  | 3.5000 | 0.0000  | 0.0277 | 0.6526 | 0.0197 | 0.0202 |
|     |     | $\hat{\alpha}_2$ | 1.5134 | 0.0134  | 0.0087 | 0.3624 | 0.0115 | 0.0114 |
|     |     | $\hat{\beta}_2$  | 2.5018 | 0.0018  | 0.0213 | 0.5722 | 0.0180 | 0.0178 |
|     |     | $\hat{\theta}$   | 0.7538 | 0.0038  | 0.0521 | 0.8953 | 0.0276 | 0.0283 |
|     | IFM | $\hat{\alpha}_1$ | 1.8177 | 0.0177  | 0.0141 | 0.4597 | 0.0144 | 0.0144 |
|     |     | $\hat{\beta}_1$  | 3.4991 | -0.0009 | 0.0281 | 0.6571 | 0.0206 | 0.0212 |
|     |     | $\hat{\alpha}_2$ | 1.5132 | 0.0132  | 0.0088 | 0.3651 | 0.0118 | 0.0118 |
|     |     | $\hat{\beta}_2$  | 2.5006 | 0.0006  | 0.0217 | 0.5781 | 0.0181 | 0.0180 |
|     |     | $\hat{\theta}$   | 0.7500 | 0.0000  | 0.0509 | 0.8852 | 0.0278 | 0.0281 |
|     |     | $\hat{\theta}$   | 0.7598 | 0.0098  | 0.0530 | 0.9024 | 0.0295 | 0.0292 |

On the basis of the results summarized in table, some conclusions can be drawn which are stated as follows: It is observed that as sample size increases for fixed vector values of  $\delta$  the MSE of the estimates decreases in all the considered methods and for large size all of them are nearly equivalent but IFM performs better than another method when increase  $\theta$  (parameter of copula). While  $\theta$  decreases, the MLE is better than other methods Based on MSE term, but the SP method is discussed where, the marginal distributions are estimated non-parametrically by their sample empirical distributions and it estimated the parameter of copula, this compare between parametric estimation and non-parametrically estimation. MSE of the Copula Parameter Estimate for Different Cases with Variation of Sample Size. It is noted that result, IFM method is the best method because it is a two steps of estimation, first, the marginal distribution parameters estimated and second the copula parameter is estimated, taking into consideration of previous parameter estimates of marginal distribution. That get more efficiency. Therefore, it has a lower MSE for copula parameter. The value of MSE of SP method is increasing approaches to the value of MSE in case of IFM method. Finally, in large samples size, the value of MSE approaches to the three methods, where there are no significant differences in MSE values for modal methods and MLE method.

## 7. Application of Real Data

We study the parameter estimation of the appropriate distribution of data, where the correlation between the two variables (bivariate data) is low. And through this access to a fit model specialized in the study of weak relations and the extent of their impact and effectiveness.

The economic data set, which is reproduced in Table (1) consists of 31 yearly time series observations [1980:2010] on response variable: Exports of goods and services ( $y_1$ ) and GDP growth ( $y_2$ ). The main reasons for selecting the economic data for the present study may due to the fact that, economic is an important sector for many developed and developing countries. Thus, the government is interested in increasing GPA growth and Exports of goods and services. To show the usefulness of the proposed bivariate estimators obtained from section 2 to section 4 with real situations, we considered here the real economic data to estimate parameters of distribution for the GPA growth and Exports of goods and services. The Data is relevant to the distribution based on FGM copula, since the correlation between data is weak  $\left(\frac{-1}{3}:\frac{1}{3}\right)$ .

**Table 6:** Data of Economics.

| years | $x_1$ | $x_2$ | years | $x_1$ | $x_2$ |
|-------|-------|-------|-------|-------|-------|
| 1980  | 30.51 | 10.01 | 1995  | 22.55 | 4.64  |
| 1981  | 33.37 | 3.76  | 1996  | 20.75 | 4.99  |
| 1982  | 27.03 | 9.91  | 1997  | 18.84 | 5.49  |
| 1983  | 25.48 | 7.40  | 1998  | 16.21 | 4.04  |
| 1984  | 22.35 | 6.09  | 1999  | 15.05 | 6.11  |
| 1985  | 19.91 | 6.60  | 2000  | 16.20 | 5.37  |
| 1986  | 15.73 | 2.65  | 2001  | 17.48 | 3.54  |
| 1987  | 12.56 | 2.52  | 2002  | 18.32 | 2.37  |
| 1988  | 17.32 | 7.93  | 2003  | 21.80 | 3.19  |
| 1989  | 17.89 | 4.97  | 2004  | 28.23 | 4.09  |
| 1990  | 20.05 | 5.70  | 2005  | 30.34 | 4.48  |

|      |       |      |      |       |      |
|------|-------|------|------|-------|------|
| 1991 | 27.82 | 1.08 | 2006 | 29.95 | 6.85 |
| 1992 | 28.40 | 4.43 | 2007 | 30.25 | 7.09 |
| 1993 | 25.84 | 2.90 | 2008 | 33.04 | 7.16 |
| 1994 | 22.57 | 3.97 | 2009 | 24.96 | 4.67 |
|      |       |      | 2010 | 21.35 | 5.15 |

$x_1$ : Exports of goods and services (EGS) (% of GDP) 1980:2010

$x_2$ : GDP growth (% per year) 1980:2010

Source: The employed of economics data are collected by World Bank National Accounts data and OECD National Accounts data.

**Table 7:** The Correlation Coefficient and Test of Correlation for Real Data

|           | Corr   | p-value |
|-----------|--------|---------|
| Pearson's | 0.2709 | 0.1404  |
| Kendall's | 0.1398 | 0.2792  |

**The Goodness of Fit Test for Copulas:** The simplest goodness-of-fit test for copulas consists in comparing the distance between a nonparametric estimate  $\hat{C}_n$  of  $C$  and a parametric estimate  $C_{\theta_n}$  derived from an estimator  $\theta_n$  of  $\theta$  which is consistent when the null hypothesis  $H_0$  holds. Let  $(F_{1n}, F_{2n})$  be the empirical distribution functions of  $(F_1, F_2)$ , respectively. A natural substitute for the unobservable  $U_i = (F_{1i}, F_{2i})$  where  $i = 1, \dots, n$ , it is given by

$$\hat{U}_i = (\hat{U}_{1i}, \hat{U}_{2i}) = \left( \frac{n}{n+1} F_{1n}(X_{1i}), \frac{n}{n+1} F_{2n}(X_{2i}) \right)$$

The scaling factor  $\frac{n}{n+1}$  used in defining  $\hat{U}_i$  avoids numerical issues that sometimes occur, when a parametric copula density is evaluated at pseudo-observations. A natural estimator  $\hat{C}_n$  of  $C$ , called the empirical copula, is then defined, for all  $(u_1, u_2); u \in [0,1]$ , by

$$\hat{C}_n(u_1, u_2) = \frac{1}{n} \sum_{i=1}^n I(\hat{U}_{1i} \leq u_1, \hat{U}_{2i} \leq u_2)$$

Genest et al. (2013) introduced Multiplier bootstrap-based goodness-of-fit test, this consideration leads naturally to Anderson–Darling-type statistics such as

$$R_n = n \int_0^1 \left[ \frac{\hat{C}_n(u_1, u_2) - C_{\theta_n}(u_1, u_2)}{[C_{\theta_n}(u_1, u_2)(1 - C_{\theta_n}(u_1, u_2)) + \delta_m]^m} \right]^2 d\hat{C}_n(u_1, u_2)$$

Involving a consistent, rank-based estimator  $\theta_n$  of  $\theta$ , and tuning parameters  $m \geq 0$  and  $\delta_m \geq 0$ .

We use the concludes of Genest to fit of Farlie-Gumbel-Morgenstern (FGM) by R package then

**Table 8:** Goodness of Fit test of FGM Copula

|                                | statistic | $\hat{\theta}$ | p-value  |
|--------------------------------|-----------|----------------|----------|
| Anderson–Darling-type( $R_n$ ) | 0.526309  | 0.627073       | 0.179441 |

This by using a parametric bootstrap  $N=10000$  time and the empirical copula estimate Goodness of fit test One-sample Kolmogorov-Smirnov test.

**Table 9:** Goodness of Fit test of Weibull Distribution

|         | $x_1$  |         |          |          | $x_2$  |         |          |          |
|---------|--------|---------|----------|----------|--------|---------|----------|----------|
|         | D      | P-Value | AIC      | BIC      | D      | P-Value | AIC      | BIC      |
| Weibull | 0.1250 | 0.6717  | 199.8782 | 202.7462 | 0.0625 | 0.9991  | 135.3818 | 138.2497 |
| Gamma   | 0.0978 | 0.8999  | 199.3264 | 202.1943 | 0.0713 | 0.9942  | 135.8115 | 138.6794 |
| Gen.Exp | 0.0920 | 0.9337  | 200.5567 | 203.4247 | 0.0912 | 0.9379  | 137.3278 | 140.1957 |

FGM bivariate Gamma (FGMBG) is discussed by Balakrishnan and Johnson (1985) and FGM Bivariate Generalized Exponential (FGMBGE) is discussed by Elaal et al. (2017).

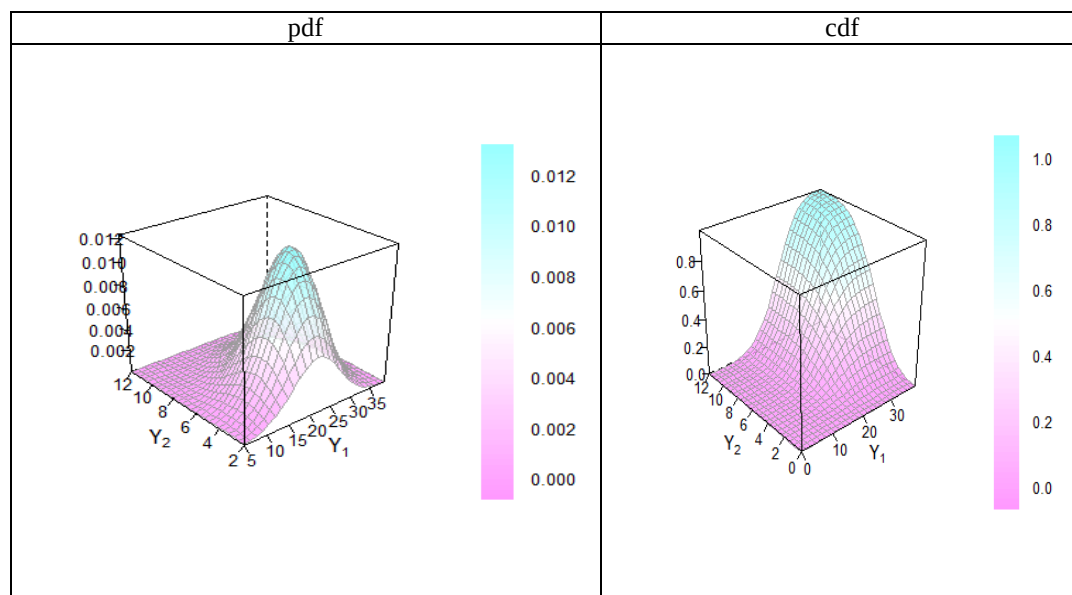
**Table10:** The Estimates of Parameters of FGM Distributions.

|             | $\hat{\alpha}_1$ | $\hat{\beta}_1$ | $\hat{\alpha}_2$ | $\hat{\beta}_2$ | $\hat{\theta}$ | AIC     | BIC     |
|-------------|------------------|-----------------|------------------|-----------------|----------------|---------|---------|
| FGM-Weibull | 4.5225           | 25.3230         | 2.6953           | 5.8395          | 0.6712         | 335.617 | 342.789 |
| FGM-Gamma   | 15.71089         | 1.46326         | 5.66429          | 0.91133         | 0.60489        | 335.703 | 342.869 |
| FGM-GE      | 50.9233          | 0.19577         | 7.40937          | 0.50802         | 0.6338         | 338.418 | 345.589 |

**Table (11):** The Estimates and the Corresponding Stander Deviation of Parameters of FGMBW Distribution for Medical Data

|     |       | $\hat{\alpha}_1$ | $\hat{\beta}_1$ | $\hat{\alpha}_2$ | $\hat{\beta}_2$ | $\hat{\theta}$ |
|-----|-------|------------------|-----------------|------------------|-----------------|----------------|
| MLE | Coef  | 4.5225           | 25.3230         | 2.6953           | 5.8395          | 0.6712         |
|     | (std) | (0.6379)         | (1.0822)        | (0.3667)         | (0.4145)        | (0.5200)       |
| IFM | Coef  | 4.5031           | 25.2076         | 2.6745           | 5.7750          | 0.6562         |
|     | (std) | (0.6341)         | (1.0631)        | (0.3660)         | (0.4092)        | (0.5067)       |
| SP  | Coef  | -                | -               | -                | -               | 0.6270         |
|     | (std) | -                | -               | -                | -               | (0.5590)       |

The graph of pdf and cdf of FGMBW Distribution for data as following


**Fig. 4:** The pdf and cdf of FGMBW Distribution for data

## 8. Conclusion

In this paper, we have proposed a FGMBW distribution based on FGM copula function. Moreover, we have the reliability functions for FGMBW distribution; therefore, it can be used quite effectively in life testing data. Additionally, the new FGMBW model can be used as an alternative to any bivariate Weibull distribution; it might work better, where the marginal function of FGMBW distribution has the same basic distribution and has closed forms for product moment. A comparison between different estimation methods of the FGMBW distribution are concluded. The result show that the best method of estimation is IFM method. Hence, we can argue that IFM estimators and MLE are the best performing estimators for FGMBW distribution. The real data application show by MLE method to give the best model, where the FGMBG, the FGMBGE and the FGMBW are discussed in the real data. The results confirmed the best model is FGMBW to this data.

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# A New Generating Family of Distributions Based on Kumaraswamy Half Logistic Distribution

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## Abstract

In this article we will introduce a new generating family of distributions called the Kumaraswamy type I half logistic (KwTIHL-G). The KwTIHL-G family is provides more flexibility to model specific real data. Three new sub-models are investigated. The statistical properties as explicit expressions for the moments, probability weighted moments, quantile function, and order statistics are studied. Estimation parameters of the family are obtained based on maximum likelihood procedure. A real data set is employed to show the usefulness of the KwTIHL-G family.

**Keywords:** Half logistic distribution; Moments; Order statistics; Maximum likelihood method.

## 1. Introduction

In recent years, many researchers are interested to expand family of distributions in order to obtain better fit for data. These numerous distributions extend known distributions as well as provide great flexibility to model specific real data. So, several ways for generating new distributions from classic ones were developed. Some of the well-known generators are the beta-family of distributions studied by (Eugene et al. 2002). (Zografos and Balakrishnan, 2009) suggested a generated family using gamma distribution which is defined as follows

$$F(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log[1-G(x)]} t^{\delta-1} e^{-t} dt, \quad (1)$$

Kumaraswamy generalized family provided by (Cordeiro and de Castro, 2011). Ristic and Balakrishnan, 2011) proposed an alternative gamma generator for any continuous distribution. Further, some generated families were studied by several authors, for example, the kummer beta by (Pescim et al., 2012), exponentiated generalized by (Cordeiro et al., 2013), Weibull-G by (Bourguignon et al., 2014), exponentiated half-logistic by (Cordeiro et al., 2014), the type I half-logistic by (Cordeiro et al., 2015), the new Kumaraswamy Kumaraswamy family of generalized distributions with application has been presented by Mahmoud et al. (2015), Garhy generated family of distributions introduced by Elgarhy et al. (2016) and the Kumaraswamy Weibull by (Hassan and Elgarhy, 2016a), odd Burr generalized by Alizadeh et al. (2017), generalized odd log-logistic by Cordeiro et al. (2017), a new generalized odd log-logistic by Haghbin et al. (2017), odd Lindley-G by Gomes-Silva, et al. (2017), odd Frechet-G by Haq and Elgarhy (2018), and among others.

In this article, we introduce a recently generated family of distributions using the half logistic distribution as a generator. This article can be organized as follows. In the next section, the Kumaraswamy type I half logistic- generated (KwTIHL-G) family is defined. In Section 3, three new sub-models of the generated family are considered. Section 4 concerns with some general

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properties of the family. In Section 5, estimation of the parameters of the family is implemented through maximum likelihood method. An illustrative purpose on the basis of real data is investigated, in Section 6. Finally, concluding remarks are handled in Section 7.

## 2. Kumaraswamy Type I Half Logistic Family

The Kumaraswamy half logistic distribution is a member of the family of logistic distributions which has the following cumulative distribution function (cdf)

$$G(x) = 1 - \left\{ 1 - \left[ \frac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}} \right]^a \right\}^b, \quad x, \lambda, a, b > 0 \quad (2)$$

The associated probability density function (pdf) is as follows

$$g(x) = ab \left[ \frac{2\lambda e^{-\lambda x}}{(1 + e^{-\lambda x})^2} \right] \left[ \frac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}} \right]^{a-1} \left\{ 1 - \left[ \frac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}} \right]^a \right\}^{b-1}. \quad (3)$$

On the basis of cdf (1), we use the half logistic generator instead of gamma generator to obtain the Kumaraswamy type I half logistic family which is denoted by KwTIHL-G. Hence the cdf of KwTIHL-G family can be expressed as follows

$$F(x) = \int_0^{-\log[1-G(x)]} ab \left[ \frac{2\lambda e^{-\lambda t}}{(1 + e^{-\lambda t})^2} \right] \left[ \frac{1 - e^{-\lambda t}}{1 + e^{-\lambda t}} \right]^{a-1} \left\{ 1 - \left[ \frac{1 - e^{-\lambda t}}{1 + e^{-\lambda t}} \right]^a \right\}^{b-1} dt$$

$$F(x) = 1 - \left\{ 1 - \left[ \frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^a \right\}^b, \quad x, \lambda, a, b > 0, \quad (4)$$

where,  $\lambda$  is a scale parameter,  $a, b$  are two shape parameters and  $G(x)$  is a baseline cdf. The distribution function (4) provides broadly KwTIHL-G distributions. Therefore, the pdf of the KwTIHL-G family is as follows

$$f(x) = \frac{2ab\lambda g(x)(1 - G(x))^{\lambda-1}}{\left[ 1 + (1 - G(x))^\lambda \right]^2} \left[ \frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^{a-1} \left\{ 1 - \left[ \frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^a \right\}^{b-1} \quad (5)$$

Hereafter, we denote by  $X \sim \text{KwTIHL} - G$  a random variable  $X$  has pdf (5).

The survival function, hazard rate function and reserved hazard rate function are, respectively, given by

$$\bar{F}(x) = 1 - F(x) = \left\{ 1 - \left[ \frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^a \right\}^b,$$

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\frac{2ab\lambda g(x)(1 - G(x))^{\lambda-1}}{\left[ 1 + (1 - G(x))^\lambda \right]^2} \left[ \frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^{a-1}}{1 - \left[ \frac{1 - (1 - G(x))^\lambda}{1 + (1 - G(x))^\lambda} \right]^a},$$

and

$$\tau(x) = \frac{f(x)}{F(x)} = \frac{\frac{2ab\lambda g(x)(1-G(x))^{\lambda-1}}{[1+(1-G(x))^{\lambda}]^2} \left[ \frac{1-(1-G(x))^{\lambda}}{1+(1-G(x))^{\lambda}} \right]^{a-1} \left\{ 1 - \left[ \frac{1-(1-G(x))^{\lambda}}{1+(1-G(x))^{\lambda}} \right]^a \right\}^{b-1}}{1 - \left\{ 1 - \left[ \frac{1-(1-G(x))^{\lambda}}{1+(1-G(x))^{\lambda}} \right]^a \right\}^b}.$$

**Note that:**

If  $a=b=1$  we get the type I half-logistic family of distributions (Cordeiro et al. 2015).

If  $b=1$  we get the exponentiated half-Logistic family of distributions (Cordeiro et al. 2014).

### 3. Sub-models

In this section, we define and describe three sub-models of the KwTIHL generated family namely, KwTIHL-uniform, KwTIHL -Frechet and KwTIHL -exponential.

#### 3.1 KwTIHL-uniform distribution

The pdf of type I half logistic- uniform KwTIHLU is derived from (5), by taking  $g(x, \theta) = \frac{1}{\theta}$ ,  $0 < x < \theta$  and  $G(x, \theta) = \frac{x}{\theta}$  as the following

$$f(x) = \frac{2ab\lambda \theta^{\lambda}(\theta-x)^{\lambda-1} [\theta^{\lambda} - (\theta-x)^{\lambda}]^{a-1}}{[\theta^{\lambda} + (\theta-x)^{\lambda}]^{a+1}} \left\{ 1 - \left[ \frac{\theta^{\lambda} - (\theta-x)^{\lambda}}{\theta^{\lambda} + (\theta-x)^{\lambda}} \right]^a \right\}^{b-1}, \quad 0 < x < \theta$$

The corresponding cdf takes the following form

$$F(x) = 1 - \left\{ 1 - \left[ \frac{\theta^{\lambda} - (\theta-x)^{\lambda}}{\theta^{\lambda} + (\theta-x)^{\lambda}} \right]^a \right\}^b.$$

Moreover, the survival and the hazard rate functions are given, respectively, as follows

$$\bar{F}(x) = \left\{ 1 - \left[ \frac{\theta^{\lambda} - (\theta-x)^{\lambda}}{\theta^{\lambda} + (\theta-x)^{\lambda}} \right]^a \right\}^b,$$

and

$$h(x) = \frac{2ab\lambda \theta^{\lambda}(\theta-x)^{\lambda-1} [\theta^{\lambda} - (\theta-x)^{\lambda}]^{a-1}}{[\theta^{\lambda} + (\theta-x)^{\lambda}]^{a+1} \left\{ 1 - \left[ \frac{\theta^{\lambda} - (\theta-x)^{\lambda}}{\theta^{\lambda} + (\theta-x)^{\lambda}} \right]^a \right\}}.$$

The plots of pdf and hazard rate function for the KwTIHLU are showed in Figures 1 and 2 respectively

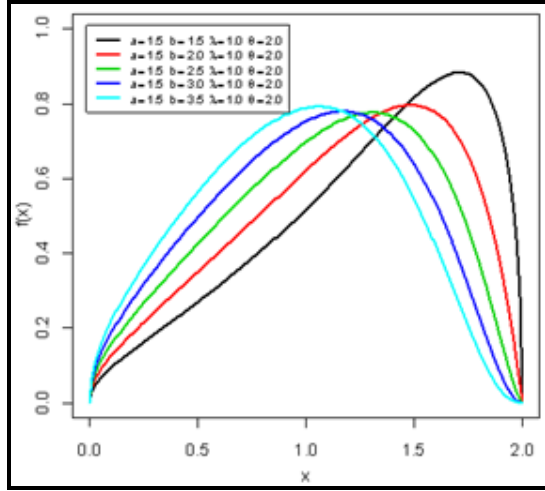


Figure 1. pdf of KwTIHLU distribution

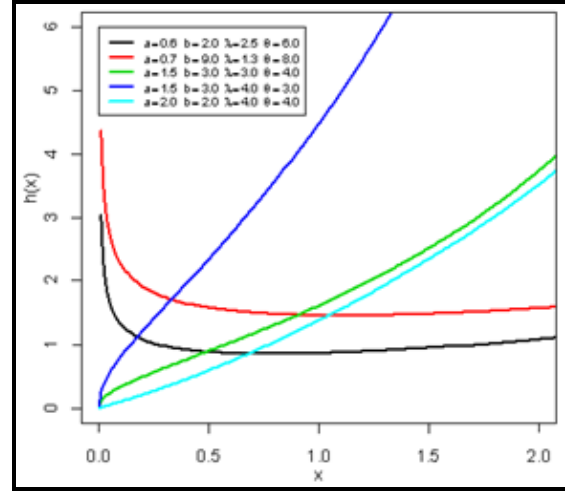


Figure 2. Hazard rate function of KwTIHLU distribution

### 3.2 KwTIHL -Frechet distribution

Let us consider the Frechet distribution with distribution functions given by  $G(x) = e^{-\left(\frac{\mu}{x}\right)^\delta}$ . Then the KwTIHLF distribution has the following cdf, pdf, survival, and hazard rate functions

$$F(x) = 1 - \left\{ 1 - \frac{\left[ 1 - \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^a}{1 + \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda} \right\}^b, \quad x, a, b, \mu, \lambda, \delta > 0$$

$$f(x) = \frac{2ab\lambda\delta\mu^\delta e^{-\left(\frac{\mu}{x}\right)^\delta} \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^{\lambda-1} \left[ 1 - \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^{a-1}}{x^{\delta+1} \left[ 1 + \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^{a+1}} \left\{ 1 - \frac{\left[ 1 - \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^a}{1 + \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda} \right\}^{b-1},$$

$$\bar{F}(x) = \left\{ 1 - \frac{\left[ 1 - \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^a}{1 + \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda} \right\}^b,$$

and

$$h(x) = \frac{2ab\lambda\delta\mu^\delta e^{-\left(\frac{\mu}{x}\right)^\delta} \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^{\lambda-1} \left[ \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^{a-1}}{x^{\delta+1} \left[ 1 + \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^{a+1} \left\{ 1 - \frac{\left[ 1 - \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda \right]^a}{1 + \left[ 1 - e^{-\left(\frac{\mu}{x}\right)^\delta} \right]^\lambda} \right\}^a}.$$

The plots of pdf and hazard rate function for the KwTIHLF are displayed in Figures 3 and 4 respectively

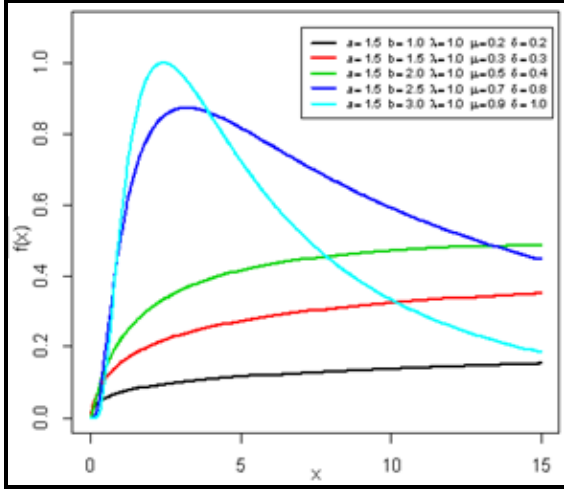


Figure 3. pdf of KwTIHLF distribution

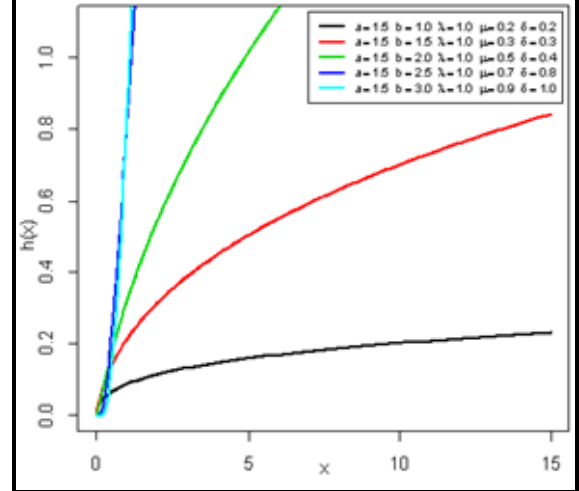


Figure 4. Hazard rate function of KwTIHLF distribution

### 3.3 KwTIHL -Exponential Distribution

The cdf and pdf of KwTIIHL-Exponential (KwTIHLE) distribution are derived from (5) and (6) taking  $G(x, \alpha) = 1 - e^{-\alpha x}$  as the following

$$F(x) = 1 - \left\{ 1 - \left[ \frac{1 - e^{-\alpha \lambda x}}{1 + e^{-\alpha \lambda x}} \right]^a \right\}^b, \quad x, a, b, \alpha, \lambda > 0,$$

and,

$$f(x) = \frac{2ab \lambda \alpha e^{-\alpha \lambda x} [1 - e^{-\alpha \lambda x}]^{a-1}}{[1 + e^{-\alpha \lambda x}]^{a+1}} \left\{ 1 - \left[ \frac{1 - e^{-\alpha \lambda x}}{1 + e^{-\alpha \lambda x}} \right]^a \right\}^{b-1}$$

Further, the survival and hazard rate functions are as follows

$$\bar{F}(x) = \left\{ 1 - \left[ \frac{1 - e^{-\alpha \lambda x}}{1 + e^{-\alpha \lambda x}} \right]^a \right\}^b,$$

and

$$h(x) = \frac{2ab \lambda \alpha e^{-\alpha \lambda x} [1 - e^{-\alpha \lambda x}]^{a-1}}{[1 + e^{-\alpha \lambda x}]^{a+1} \left\{ 1 - \left[ \frac{1 - e^{-\alpha \lambda x}}{1 + e^{-\alpha \lambda x}} \right]^a \right\}}.$$

The plots of pdf and hazard rate function for the KwTIHLE are presented in Figures 5 and 6 respectively

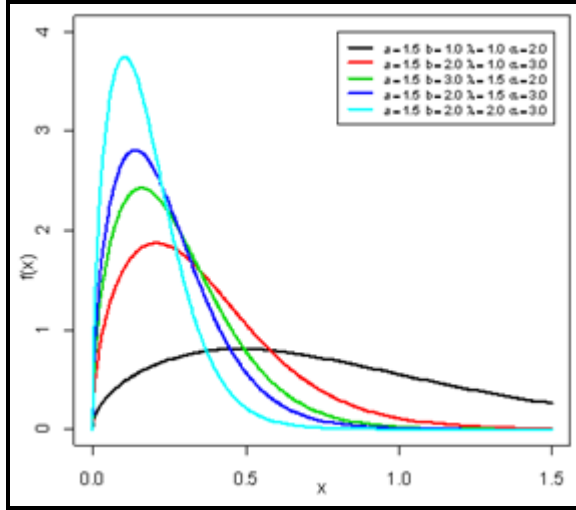


Figure 5. pdf of KwTIHLE distribution

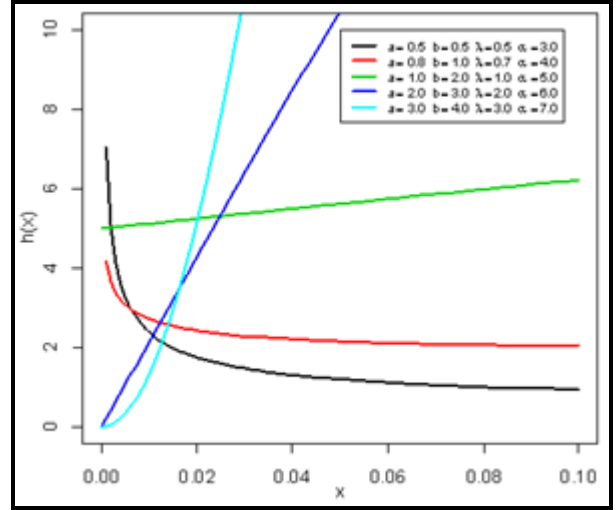


Figure 6. Hazard rate function of KwTIHLE distribution

#### 4. Some Statistical Properties

This section provides some statistical properties of KwTIHL-G family of distributions.

##### 4.1 Quantile function

Let  $X$  denotes a random variable has the cdf (4), the quantile function; say  $Q(u)$  of  $X$  is given by

$$Q(u) = G^{-1} \left[ 1 - \frac{\left[ 1 - \left( 1 - (1 - u)^{\frac{1}{b}} \right)^{\frac{1}{a}} \right]^{\frac{1}{\lambda}}}{1 + \left( 1 - (1 - u)^{\frac{1}{b}} \right)^{\frac{1}{a}}} \right], \quad (6)$$

where,  $u$  is a uniform distribution on the interval  $(0,1)$  and  $G^{-1}(\cdot)$  is the inverse function of  $G(\cdot)$ . In particular, the first quartile, the median, and the third quartile are obtained by putting  $u = 0.25, 0.50$  and  $0.75$  respectively, in (6).

##### 4.2 Useful representation

In this subsection some representations of the cdf and pdf for the Kumaraswamy type I half logistic family of distributions will be presented. The mathematical relation given below will be useful in this subsection.

It is well-known that, if  $\beta > 0$  and  $|z| < 1$  the generalized binomial theorem is written as follows

$$(1 - z)^{\beta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} z^i. \quad (7)$$

Then, by applying the binomial theorem (7) in (5), the distribution function of KwTIHL-G distribution where  $b$  is real becomes

$$f(x) = 2ab\lambda g(x)(1 - G(x))^{\lambda-1} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \left[1 - (1 - G(x))^{\lambda}\right]^{a(i+1)-1} \times \left[1 + (1 - G(x))^{\lambda}\right]^{-(a(i+1)+1)}. \quad (8)$$

Using the generalized binomial theorem, we can write

$$\left[1 + (1 - G(x))^{\lambda}\right]^{-[a(i+1)+1]} = \sum_{j=0}^{\infty} (-1)^j \binom{a(i+1)+j}{j} [1 - G(x)]^{\lambda j}. \quad (9)$$

Inserting (9) and (7) in (8), the KwTIHL-G density function can be written as follows

$$f(x) = 2ab\lambda g(x) \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i} \binom{a(i+1)+j}{j} \binom{a(i+1)-1}{k} (1 - G(x))^{\lambda(k+j+1)-1}.$$

Then, using binomial expansion again in the last equation, where  $\lambda$  is real non integer, leads to:

$$f(x) = \sum_{i,j,k,l=0}^{\infty} \eta^* g(x)(G(x))^l, \quad (10)$$

where,

$$\eta^* = 2ab\lambda (-1)^{i+j+k+l} \binom{b-1}{i} \binom{a(i+1)+j}{j} \binom{a(i+1)-1}{k} \binom{\lambda(k+j+1)-1}{l}$$

Further, an expansion for  $[F(x)]^h$  is derived, for  $h$  is integer, again, the binomial expansion is worked out.

$$[F(x)]^h = \sum_{p=0}^h \sum_{q,u,v,w=0}^{\infty} W^*(G(x))^w, \quad (11)$$

where,

$$W^* = (-1)^{p+q+u+v+w} \binom{h}{p} \binom{bp}{q} \binom{aq}{u} \binom{aq+v-1}{v} \binom{\lambda(u+v)}{w}.$$

### 4.3 The probability weighted moments

Class of moments, called the probability-weighted moments (PWMs), has been proposed by Greenwood et al. (1979). This class is used to derive estimators of the parameters and quantiles of distributions expressible in inverse form. For a random variable  $X$  the PWMs, denoted by  $\tau_{r,s}$ , can be calculated through the following relation

$$\tau_{r,s} = E(X^r F(x)^s) = \int_{-\infty}^{\infty} x^r f(x) F(x)^s dx \quad (12)$$

The PWMs of KwTIHL-G is obtained by substituting (10) and (11) into (12), and replacing  $h$  with  $s$ , as follows

$$\tau_{r,s} = \sum_{p=0}^s \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \eta^* w^* \int_{-\infty}^{\infty} x^r g(x) G(x)^{l+w} dx.$$

Then,

$$\tau_{r,s} = \sum_{p=0}^s \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \eta^* w^* \tau_{r,l+w},$$

where,

$$\tau_{r,l+w} = \int_{-\infty}^{\infty} x^r g(x) G(x)^{l+w} dx.$$

#### 4.4 Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the  $r^{\text{th}}$  moment for the KwTIHL-G family. If  $X$  has the pdf (10), then  $r^{\text{th}}$  moment is obtained as follows

$$\mu_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \sum_{i,j,k,l=0}^{\infty} \eta^* \int_{-\infty}^{\infty} x^r g(x) G(x)^l dx$$

Then,

$$\mu_r = \sum_{i,j,k,l=0}^{\infty} \eta^* \tau_{r,l}.$$

For a random variable  $X$  it is known that, the moment generating function is defined as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r = \sum_{r,i,j,k,l=0}^{\infty} \frac{t^r}{r!} \eta^* \tau_{r,l}.$$

#### 4.5 Order statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function  $F(x)$ . Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  the corresponding ordered random sample from a population of size  $n$ . According to (David, 1981), the pdf of the  $k^{\text{th}}$  order statistic, is defined as

$$f_{X_{(k)}}(x) = \frac{f(x)}{B(k, n-k+1)} \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} F(x)^{m+k-1}, \quad (13)$$

$B(.,.)$  stands for beta function. The pdf of the  $k^{\text{th}}$  order statistic for KwTIHL-G family is derived by substituting (10) and (11) in (13), replacing  $h$  with  $m+k-1$

$$f_{X_{(k)}}(x) = \frac{g(x)}{B(k, n-k+1)} \sum_{m=0}^{n-k} \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \sum_{p=0}^{m+k-1} C^* G(x)^{l+w}, \quad (14)$$

where,  $C^* = (-1)^m \binom{n-k}{m} \eta^* w^*$ ,  $g(.)$  and  $G(.)$  are the pdf and cdf of the KwTIHL-G family, respectively.

Further, the  $r^{\text{th}}$  moment of  $k^{\text{th}}$  order statistics for KwTIHL-G family is defined by:

$$E(X^r_{(k)}) = \int_{-\infty}^{\infty} x^r_{(k)} f(x_{(k)}) dx_{(k)}. \quad (15)$$

By substituting (14) in (15), leads to

$$E(X^r_{(k)}) = \frac{1}{B(k, n-k+1)} \sum_{m=0}^{n-k} \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \sum_{P=0}^{m+k-1} C^* \int_{-\infty}^{\infty} x^r_{(k)} g(x_{(k)}) G(x_{(k)})^{l+w} dx_{(k)}.$$

Then,

$$E(X^r_{(k)}) = \frac{1}{B(k, n-k+1)} \sum_{m=0}^{n-k} \sum_{i,j,k,l=0}^{\infty} \sum_{q,u,\gamma,w=0}^{\infty} \sum_{P=0}^{m+k-1} C^* \tau_{r,l+w}.$$

## 5. Maximum Likelihood Method

This section deals with the maximum likelihood estimators of the unknown parameters for the KwTIHL-G family of distributions on the basis of complete samples. Let  $X_1, X_2, \dots, X_n$  be the observed values from the KwTIHL-G family with set of parameters  $\Phi = (a, b, \lambda, \xi)^T$ . The log-likelihood function for parameter vector  $\Phi = (a, b, \lambda, \xi)^T$  is obtained as follows

$$\begin{aligned} \ln(L, \phi) &= n \ln(2) + n \ln(a) + n \ln(b) + n \ln(\lambda) + \sum_{i=1}^n \ln[g(x_i, \xi)] + (\lambda - 1) \sum_{i=1}^n \ln[1 - G(x_i, \xi)] \\ &\quad + (a - 1) \sum_{i=1}^n \ln[1 - (1 - G(x_i, \xi))^\lambda] - (a + 1) \sum_{i=1}^n \ln[1 + (1 - G(x_i, \xi))^\lambda] \\ &\quad + (b - 1) \sum_{i=1}^n \ln \left[ 1 - \left[ \frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a \right]. \end{aligned}$$

$$\begin{aligned} U_a &= \frac{n}{a} + \sum_{i=1}^n \ln[1 - (1 - G(x_i, \xi))^\lambda] - \sum_{i=1}^n \ln[1 + (1 - G(x_i, \xi))^\lambda] \\ &\quad + (b - 1) \sum_{i=1}^n \frac{\left[ \frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a \ln \left[ \frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]}{1 - \left[ \frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a}, \end{aligned}$$

$$U_b = \frac{n}{b} + \sum_{i=1}^n \ln \left[ 1 - \left[ \frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^a \right],$$

$$\begin{aligned}
U_\lambda = & \frac{n}{\lambda} + \sum_{i=1}^n \ln[1 - G(x_i, \xi)] + (a-1) \sum_{i=1}^n \frac{(1 - G(x_i, \xi))^\lambda \ln[1 - G(x_i, \xi)]}{1 - (1 - G(x_i, \xi))^\lambda}, \\
& -(a+1) \sum_{i=1}^n \frac{(1 - G(x_i, \xi))^\lambda \ln[1 - G(x_i, \xi)]}{1 + (1 - G(x_i, \xi))^\lambda} \\
& + a(b-1) \sum_{i=1}^n \frac{\left[ \frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^{a-1} \frac{2(1 - G(x_i, \xi))^\lambda \ln[1 - G(x_i, \xi)]}{[1 + (1 - G(x_i, \xi))^\lambda]^2}}{1 - \left[ \frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^\lambda}
\end{aligned}$$

and,

$$\begin{aligned}
U_{\xi_k} = & \sum_{i=1}^n \frac{g'_k(x_i, \xi)}{g(x_i, \xi)} - (\lambda-1) \sum_{i=1}^n \frac{G'_k(x_i, \xi)}{1 - G(x_i, \xi)} + \lambda(\alpha-1) \sum_{i=1}^n \frac{G'_k(x_i, \xi) [1 - G(x_i, \xi)]^{\lambda-1}}{1 - [1 - G(x_i, \xi)]^\lambda} \\
& + \lambda(\alpha+1) \sum_{i=1}^n \frac{G'_k(x_i, \xi) [G(x_i, \xi)]^{\lambda-1}}{1 + [1 - G(x_i, \xi)]^\lambda} \\
& - a(b-1) \sum_{i=1}^n \frac{\left[ \frac{1 - (1 - G(x_i, \xi))^\lambda}{1 + (1 - G(x_i, \xi))^\lambda} \right]^{a-1} \frac{2\lambda(1 - G(x_i, \xi))^{\lambda-1} G'_k(x_i, \xi)}{[1 + (1 - G(x_i, \xi))^\lambda]^2}}{[1 + (1 - G(x_i, \xi))^\lambda]^\lambda}.
\end{aligned}$$

where,  $g'_k(x_i, \xi) = \partial g(x_i, \xi) / \partial \xi_k$  and  $G'_k(x_i, \xi) = \partial G(x_i, \xi) / \partial \xi_k$ .

Setting  $U_a, U_b, U_\lambda$ , and  $U_{\xi_k}$  equal to zero and solving these equations simultaneously yield the maximum likelihood estimate (MLE)  $\hat{\Phi} = (\hat{a}, \hat{b}, \hat{\lambda}, \hat{\xi})$  of  $\Phi = (a, b, \lambda, \xi)^T$  these equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods.

## 6. Application

In this section, one real data set is used to illustrate the potentiality of the KwTIHL family. Application of the KwTIHL distributions based on four distributions; namely, Kumaraswamy type I half logistic exponential (KwTIHLE), type I half logistic Weibull (TIHLW), Exponentiated Weibull Weibull (EWW) (Hassan and Elgarhy al., 2016b), and Weibull Weibull (WW) (Abouelmagd et al., 2017).

**Data set:** is obtained from (Hinkley, 1977). It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul.

The estimate of the unknown parameters of each distribution is obtained by the maximum-likelihood method. In order to compare the four distribution models, various criteria were used. Criteria like;  $-2 \ln L$ , Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct

Akaike information criterion (*CAIC*), Hannan information criterion(*HQIC*), the Kolmogorov-Smirnov ( $K - S$ ), Anderson Darling ( $A^*$ ) and Cramer-von Mises ( $W^*$ ) statistics are considered for the data set.

**Table (1)** show the "best" distribution corresponds to the smallest values of  $-2 \ln L$ , *AIC*, *BIC*, *CAIC*, *HQIC*,  $K - S$ ,  $A^*$  and  $W^*$  criteria.

**Table (1):** MLEs of the model parameters and its standard error (S.E) (in parentheses) for data set

| Model                                                   | MLEs and S. E      |                     |                   |                      |                  |
|---------------------------------------------------------|--------------------|---------------------|-------------------|----------------------|------------------|
| <b>KwTIHLE</b> ( $\alpha, b, \alpha, \lambda$ )         | 1.82<br>( 1.198 )  | 25.172<br>( 0.322 ) | 0.31<br>( 0.247 ) | 0.579<br>( 0.55607 ) | -                |
| <b>TIHLW</b> ( $\alpha, \beta, \lambda$ )               | 0.626<br>(0.124)   | 1.532<br>(0.293)    | 0.889<br>(0.1791) | -                    | -                |
| <b>EWW</b> ( $\alpha, \alpha, \beta, \lambda, \gamma$ ) | 78.61<br>(0.14836) | 79.35<br>(0.561)    | 20.486<br>(0.131) | 0.624<br>(0.024)     | 0.014<br>(0.148) |
| <b>WW</b> ( $\alpha, \beta, \lambda, \gamma$ )          | 39.853<br>(0.414)  | 3.154<br>(0.518)    | 0.196<br>(0.102)  | 0.5<br>(0.072)       |                  |

The following table gives the values of measurements for the data set.

**Table (2):** measurements for all models based on the data set

| Distribution   | $-2 \ln L$ | <i>AIC</i> | <i>BIC</i> | <i>CAIC</i> | <i>HQIC</i> | $K - S$ | $A^*$   | $W^*$   |
|----------------|------------|------------|------------|-------------|-------------|---------|---------|---------|
| <b>KwTIHLE</b> | 77.244     | 85.244     | 83.153     | 86.844      | 87.037      | 0.06879 | 0.17008 | 0.02196 |
| <b>TIHLW</b>   | 106.639    | 112.639    | 112.211    | 113.562     | 115.358     | 0.069   | 0.22044 | 0.02953 |
| <b>EWW</b>     | 129.022    | 139.022    | 141.522    | 136.407     | 141.263     | 0.113   | 0.34353 | 0.05525 |
| <b>WW</b>      | 138.194    | 146.194    | 145.623    | 147.794     | 149.819     | 0.07549 | 0.20247 | 0.02622 |

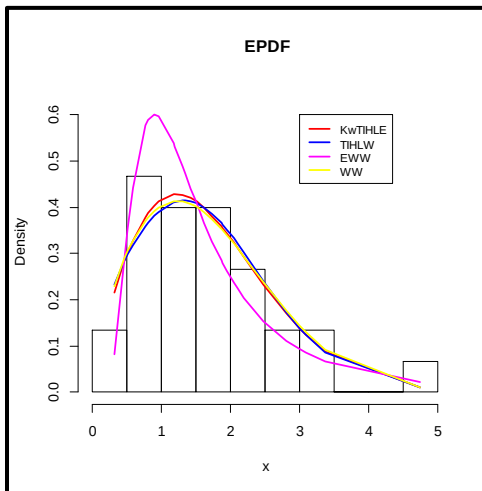


Figure 7. Estimated densities for the data set

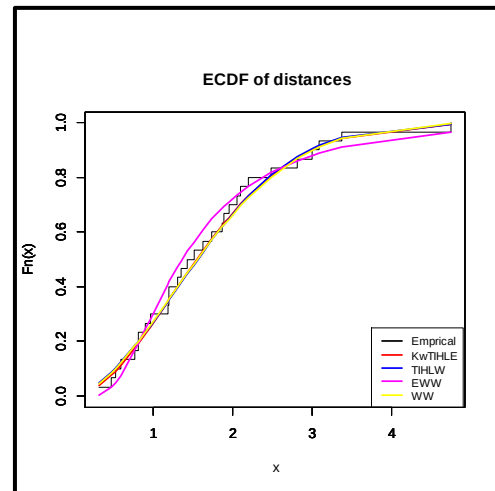


Figure 8. Estimated cumulative densities for the data set

The values in Table 2, indicate that the KwTIHL exponential distribution is a strong competitor to other distributions used here for fitting data set. A density and cdf plots compares the fitted densities of the models with the empirical histogram of the observed data (Figure 7 and Figure 8). The fitted density for the KwTIHL exponential model is the closest to the empirical histogram of the other fitted models.

## 7. Conclusion

In this article, we introduced the new KwTIHL-G family of distributions. More specifically, KwTIHL-G family covers several new distributions. We wish a broadly statistical application in some area for this new generalization. Properties of the KwTIHL-G are discussed, such as, expressions for the density function, moments, quantile function and order statistics. The maximum likelihood method is employed for estimating the model parameters. Three sub-models are provided. Further, the derived properties of the generated family are valid to these selected models. We fit some KwTIHL-G distributions to one real data set to demonstrate the potentiality of this family.

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## New 3-Components Finite Mixture Distribution

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### Abstract

Two new distributions based on mixing different types of distributions are introduced in this paper. They are one-parameter and two-parameter distributions. The two-parameter distribution was derived from the one-parameter distribution by using a power transformation. Its failure rate can be (constant, decreasing, bathtub, decreasing - increasing and increasing) depending on the value of its two parameters. Some properties of these two distributions are discussed. Maximum likelihood estimation is investigated for the two-parameter model. Finally, investigating the mean square error of the maximum likelihood estimators, scaled root mean square error, the coverage probability, and the average width of the confidence intervals of the unknown parameters using simulation study is carried out.

**Keywords:** maximum likelihood estimates; hazard function; linear failure rate distribution; exponential distribution; gamma distribution; finite mixture distribution; power transformation.

## 1 Introduction

Sometimes the known statistical distributions do not suit the available data; accordingly generating a new distribution is needed. mixing distributions (Everitt and Hand,1981) is one of the means that can be utilized to construct new

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distributions from other existing ones. The finite exponential mixture is formed as follow

$$f(x) = \sum_{i=1}^c p_i \frac{1}{\mu_i} e^{-\frac{x}{\mu_i}}, x > 0 \quad (1)$$

where  $\mu_i > 0$ ,  $p_i > 0$   $i = 1, \dots, c$  and  $\sum_{i=1}^c p_i = 1$

Recently, many new distributions were generated e.g. Mixture of two Inverse Weibull Distribution (Sultan *et al.*, 2007), Mixture of two Fréchet Distribution (Mubark, 2010), and mixture gamma distribution Erisoglu *et al.*(2013).

The use of reliability function (survival function) and hazard function to analyze lifetime data is an integral scope of statistical analysis. Although hazard function for the exponential distribution is always constant, it can be either constant or increasing for linear failure rate distribution (LFRD).

Sarhan *et al.* (2014) introduced a new lifetime distribution and its power transformation based on finite mixtures. Their distribution have different shapes of the hazard function including constant; decreasing; bathtub; decreasing increasing and increasing.

The aim of this paper is to introduce two new distributions. The first one is a one-parameter distribution which is referred to as NLM( $\beta$ )(New lifetime Mixture) and the second one is a power transformation of the first distribution. We refer to the second distribution as NLMP( $\alpha, \beta$ )(New Lifetime Mixture Power). The hazard function of NLMP( $\alpha, \beta$ ) can be constant, decreasing, bathtub, decreasing - increasing, and increasing. This variety of shapes of its hazard function enables NLMP( $\alpha, \beta$ ) to fit different data sets. The maximum likelihood estimate of the two parameters of NLMP( $\alpha, \beta$ ) were obtained and a simulation study was carried out

to examine the accuracy of the maximum likelihood estimators of the NLMP( $\alpha, \beta$ ) parameters as well as the coverage probability and average width of the confidence intervals for both parameters.

## 2 Main Results

Suppose the random variable T is a mixture of three components : G( $2, \beta$ ), Exp( $2\beta$ ), LFRD( $\beta, \beta^2$ ) with mixing weights  $a_1 = \frac{\beta}{1+\beta}$  and  $a_2, a_3 = \frac{1}{2(1+\beta)}$ . where

G( $2, \beta$ ) is a Gamma distribution, Exp( $2\beta$ ) is an exponential distribution and LFRD( $\beta, \beta^2$ ) is a linear failure rate distribution. Then NLM( $\beta$ ) probability density function (pdf) is

$$f(t) = \frac{1}{(1+\beta)} \beta e^{-t\beta} (e^{-t\beta} + t\beta^2 + \frac{1}{2} e^{(-\frac{1}{2}t^2\beta^2)} (1+t\beta)), t > 0 \quad (2)$$

where  $\beta$  is shape parameter. From (1) we can get some characteristics of NLM( $\beta$ )

The cumulative distribution function of NLM( $\beta$ ) is

$$F(t) = 1 - \frac{e^{-2t\beta} + e^{(-\frac{1}{2}t\beta(2+t\beta))} + 2\beta e^{(-t\beta)} (1+t\beta)}{2(1+\beta)} \quad (3)$$

while its hazard rate function h(t) is given by

$$h(t) = \frac{2\beta e^{-t\beta} (e^{(-t\beta)} + t\beta^2 + \frac{1}{2} e^{(-\frac{1}{2}t^2\beta^2)} (1+t\beta))}{e^{-2t\beta} + e^{(-\frac{1}{2}t\beta(2+t\beta))} + 2\beta e^{(-t\beta)} (1+t\beta)} \quad (4)$$

The density  $f(t)$  and hazard function h(t) are plotted in Figure 1 for four choices of  $\beta$  when  $\beta=(0.32, 0.84, 2.7$  and  $4.18)$ .

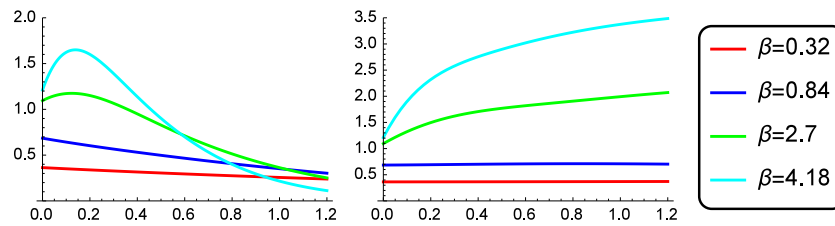


Figure 1: PDF and  $h(t)$  for  $NLM(\beta)$  for some parameter values

## 2.1 Power transformation of the $NLM(\beta)$ distribution

To increase the distribution flexibility, we apply the power transformation  $x = t^{\frac{1}{\alpha}}, \alpha > 0$  to the  $NLM(\beta)$  which pdf of  $X$  will be

$$f(x) = \frac{\alpha x^{\alpha-1} \beta e^{-\beta x^\alpha}}{(1+\beta)} (e^{-x^\alpha \beta} + \beta^2 x^\alpha + \frac{1}{2} e^{-\frac{1}{2} x^{2\alpha} \beta^2} (1 + \beta x^\alpha)), x > 0 \quad (5)$$

where  $\beta$  is shape parameter and  $\alpha$  is scale parameter. The random variable  $X$ , denoted by  $NLM(\alpha, \beta)$  is a mixture of three components:  $PG(\alpha, 2, \beta)$ ,  $W(\alpha, 2\beta)$  and  $PLFRD(\alpha, \beta, \beta^2)$  with mixture weights  $a_1 = \frac{\beta}{(1+\beta)}$ ,  $a_2 = \frac{1}{2(1+\beta)}$  and  $a_3 = \frac{1}{2(1+\beta)}$  respectively, where

$PG(\alpha, 2, \beta)$  is power gamma distribution,  $W(\alpha, 2\beta)$  is weibull distribution and  $PLFRD(\alpha, \beta, \beta^2)$  is power linear failure rate distribution. Figure 2 shows the density of  $X$  at different values for  $(\alpha, \beta)$ . From (5) we can get some of the characteristics of  $NLM(\alpha, \beta)$ . The cumulative distribution function of  $NLM(\alpha, \beta)$  is

$$F(x) = 1 - \frac{e^{-2x^\alpha \beta} + e^{-\frac{1}{2} x^\alpha \beta (2+x^\alpha \beta)} + 2e^{-x^\alpha \beta}}{2(1+\beta)} \quad (6)$$

while its hazard rate function  $h(x)$  is given by

$$h(x) = \frac{2\alpha x^{\alpha-1} \beta e^{-\beta x^\alpha} (e^{-x^\alpha \beta} + \beta^2 x^\alpha + \frac{1}{2} e^{-\frac{1}{2} (x^\alpha)^2 \beta^2} (1 + \beta x^\alpha))}{e^{-2x^\alpha \beta} + e^{-\frac{1}{2} x^\alpha \beta (2+x^\alpha \beta)} + 2e^{-x^\alpha \beta} \beta} \quad (7)$$

The hazard function of X is plotted at Figure 3 for some selected values for  $(\alpha, \beta)$ .

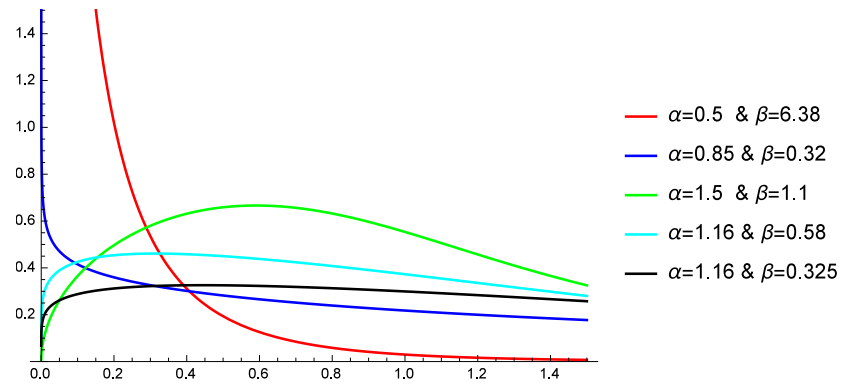


Figure 2: pdf for NLMP( $\alpha, \beta$ ) for some parameter values

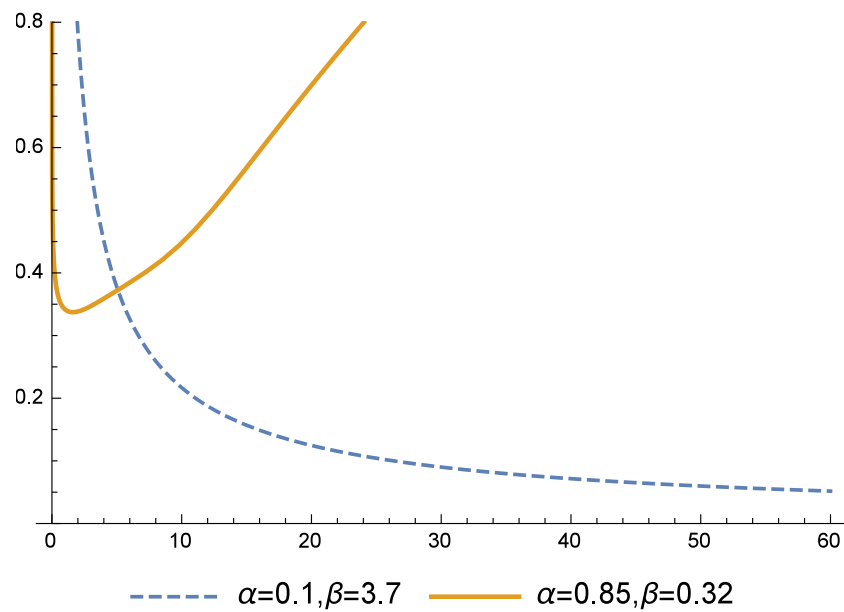


Figure 3: Hazard function  $h(x)$  for NLMP( $\alpha, \beta$ ) for some parameter values of  $(\alpha, \beta)$

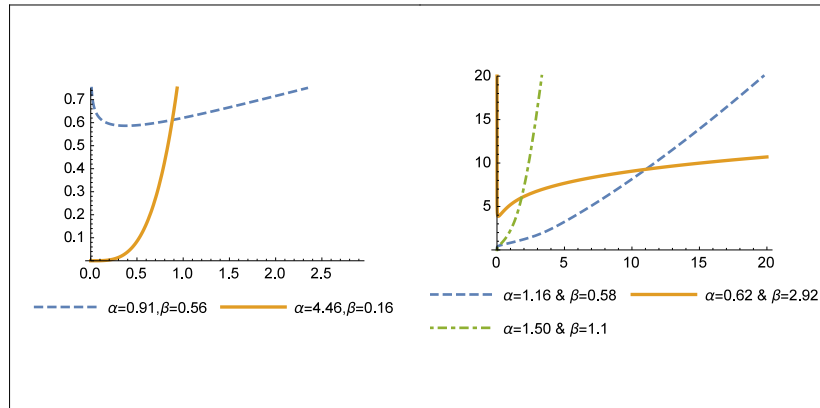


Figure 4: Hazard function  $h(x)$  for  $NLMP(\alpha, \beta)$  for some parameter values of  $(\alpha, \beta)$

As shown in Figures 3 and 4 the hazard shape is bathtub for parameters value  $(\alpha=0.85, \beta=0.32)$ ,  $(\alpha=0.91, \beta=0.56)$  and decreasing for values  $(\alpha=0.1, \beta=3.7)$  and increasing for values  $(\alpha=4.45, \beta=0.16)$ ,  $(\alpha=1.5, \beta=1.1)$ ,  $(\alpha=0.62, \beta=2.9)$  and  $(\alpha=1.16, \beta=0.58)$ .

## 2.2 Maximum Likelihood Estimation(MLE)

MLE is widely utilized to estimate the parameters of different statistical models. Suppose that  $n$  refers to independent and identical items, with lifetimes following  $NLMP(\alpha, \beta)$ , and put on a life test simultaneously. Let  $x_1, x_2, \dots, x_n$  be the failure times of the items and let  $x = (x_1, \dots, x_n)$ . The likelihood function for  $(\alpha, \beta)$  is

$$l(x, \alpha, \beta) = \frac{\alpha^n \beta^n}{(1 + \beta)^n} \prod_{i=1}^n x_i^{\alpha-1} e^{-\beta \sum_{i=1}^n x_i^\alpha} * \prod_{i=1}^n (e^{-x_i^\alpha \beta} + \beta^2 x_i^\alpha + \frac{1}{2} e^{-\frac{1}{2} x_i^{2\alpha} \beta^2} (1 + \beta x_i^\alpha)) \quad (8)$$

The log-likelihood function is

$$\begin{aligned}
 ll(x, \alpha, \beta) = & n * \log \alpha + n * \log \beta - n * \log(1 + \beta) - \beta \sum_{i=1}^n x_i^\alpha + (\alpha - 1) \sum_{i=1}^n \log x_i \\
 & + \sum_{i=1}^n \log A_i(\alpha, \beta)
 \end{aligned} \tag{9}$$

where

$$A_i(\alpha, \beta) = e^{(-x_i^\alpha \beta)} + \beta^2 x_i^\alpha + \frac{1}{2} e^{-\frac{1}{2} x_i^{2\alpha} \beta^2} (1 + \beta x_i^\alpha) \tag{10}$$

After taking partial derivatives of the log-likelihood in (8) with respect to  $\alpha$  and  $\beta$ , and equating the derivatives to zero, the following equations are obtained:

$$L_\alpha = \frac{n}{\alpha} - \beta \sum_{i=1}^n x_i^\alpha \log x_i + \sum_{i=1}^n \log x_i + \sum_{i=1}^n \frac{(A_{i,\alpha}(\alpha, \beta))}{(A_i(\alpha, \beta))} \tag{11}$$

$$L_\beta = \frac{n}{\beta} - \frac{n}{1 + \beta} - \sum_{i=1}^n x_i^\alpha + \sum_{i=1}^n \frac{(A_{i,\beta}(\alpha, \beta))}{(A_i(\alpha, \beta))} \tag{12}$$

where:-

$$\begin{aligned}
 A_{i,\alpha}(\alpha, \beta) = & \frac{1}{2} e^{(-\frac{1}{2} x_i^{2\alpha} \beta^2)} x_i^\alpha \beta \log x_i - e^{(-x_i^\alpha \beta)} x_i^\alpha \beta \log x_i + x_i^\alpha \beta^2 \log x_i \\
 & - \frac{1}{2} e^{(-\frac{1}{2} x_i^{2\alpha} \beta^2)} x_i^{2\alpha} \beta^2 (1 + x_i^\alpha \beta) \log x_i
 \end{aligned} \tag{13}$$

$$A_{i,\beta}(\alpha, \beta) = -e^{-x_i^\alpha \beta} x_i^\alpha + \frac{1}{2} e^{(-\frac{1}{2} x_i^{2\alpha} \beta^2)} x_i^\alpha + 2x_i^\alpha \beta - \frac{1}{2} e^{(-\frac{1}{2} x_i^{2\alpha} \beta^2)} x_i^{2\alpha} \beta (1 + x_i^\alpha \beta) \tag{14}$$

Equations (11) and (12) do not lend themselves analytical solutions and they must be solved numerically for this purpose, the popular numerical software package *Mathematica*[9] was used to obtain the results shown in Tables 1 and 2.

The asymptotic variance-covariance matrix for the estimators  $\hat{\alpha}$  and  $\hat{\beta}$  can be obtained approximately by substituting the estimated values of the parameters in the matrix of second partial derivateites of the log-likelihood function whose elements

are given by:

$$L_{\alpha,\alpha} = -\frac{n}{\alpha^2} - \beta \sum_{i=1}^n x_i^\alpha (\log x_i)^2 + \sum_{i=1}^n \log x_i + \sum_{i=1}^n \frac{A_i(\alpha, \beta) * A_{i,\alpha^2}(\alpha, \beta) - (A_{i,\alpha}(\alpha, \beta))^2}{(A_i(\alpha, \beta))^2} \quad (15)$$

$$L_{\beta,\beta} = -\frac{n}{\beta^2} + \frac{n}{(1+\beta)^2} - \sum_{i=1}^n x_i^\alpha + \sum_{i=1}^n \frac{A_{i\alpha,\beta} * A_{(i,\beta^2)}(\alpha, \beta) - (A_{(i,\beta)}(\alpha, \beta))^2}{(A_i(\alpha, \beta))^2} \quad (16)$$

$$L_{\alpha,\beta} = -\sum_{i=1}^n x_i^\alpha \log x_i + \sum_{i=1}^n \frac{A_i(\alpha, \beta) * A_{i,\alpha,\beta}(\alpha, \beta) - (A_{i,\alpha}(\alpha, \beta)) * (A_{i,\beta}(\alpha, \beta))}{(A_i(\alpha, \beta))^2} \quad (17)$$

where:-

$$A_{i,\alpha^2}(\alpha, \beta) = \frac{1}{2}e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^\alpha\beta\log x_i^2 - e^{-x_i^\alpha\beta}x_i^\alpha\beta\log x_i^2 + x_i^\alpha\beta^2\log x_i^2 + e^{-x_i^\alpha\beta}x_i^{2\alpha}\beta^2\log x_i^2 - e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^{3\alpha}\beta^3\log x_i^2 - e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^{2\alpha}\beta^2(1+x_i^\alpha\beta)\log x_i^2 + \frac{1}{2}e^{(-\frac{1}{2}x_i^{2\alpha}\beta^2)}x_i^{4\alpha}\beta^4(1+x_i^\alpha\beta)\log x_i^2 \quad (18)$$

$$A_{i,\beta^2}(\alpha, \beta) = 2x_i^\alpha + e^{-x_i^\alpha\beta}x_i^{2\alpha} - e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^{3\alpha}\beta - \frac{1}{2}e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^{2\alpha}(1+x_i^\alpha\beta) + \frac{1}{2}e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^{4\alpha}\beta^2(1+x_i^\alpha\beta) \quad (19)$$

$$A_{i,\alpha,\beta}(\alpha, \beta) = \frac{1}{2}e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^\alpha\log x_i + 2x_i^\alpha\beta\log x_i - e^{-x_i^\alpha\beta}x_i^\alpha\log x_i + e^{-x_i^\alpha\beta}x_i^{2\alpha}\beta\log x_i - e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^{3\alpha}\beta^2\log x_i - e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^{2\alpha}\beta(1+x_i^\alpha\beta)\log x_i + \frac{1}{2}e^{-\frac{1}{2}x_i^{2\alpha}\beta^2}x_i^{4\alpha}\beta^3(1+x_i^\alpha\beta)\log x_i \quad (20)$$

Then the information matrix is  $I = -E \begin{bmatrix} L_{\alpha,\alpha} & L_{\alpha,\beta} \\ L_{\alpha,\beta} & L_{\beta,\beta} \end{bmatrix}$

Cohen (1965) suggested that approximate variance covariance matrix may be obtained by replacing expected values by their (MLEs). For more details see Meeker and Escobar (1998).

### 3 Simulation study

We used a simulation study to investigate the accuracy of point and interval estimates of the two parameters of NLMP( $\alpha, \beta$ ) for replications  $m = 10,000$ , the sample sizes  $n$  are 25, 50, 100, and 150, and the parameter values  $(\alpha, \beta) = (0.5, 6.38), (0.85, 0.32), (1.5, 1.1), (1.16, 0.58), (1.16, 0.325), (1, 0.9)$  and  $(1, 3.91)$ . The following steps were followed to obtain the results :

1. Specify initial values for parameters  $\alpha$  and  $\beta$
2. Specify the sample size  $n$
3. Use  $n$  in (2) to generate  $m$  random samples with size  $n$  from NLMP( $\alpha, \beta$ )
4. Calculate the log-likelihood function and obtain  $(\hat{\alpha}, \hat{\beta})$
5. Solve the equations and get the maximum of log-likelihood to obtain  $(\hat{\alpha}, \hat{\beta})$
6. Calculate mean, bias and relative bias for  $(\hat{\alpha}, \hat{\beta})$
7. Calculate the mean squared error (MSE), root mean square error and scaled root mean square error for each estimator
8. Calculate coverage probability and average width for each parameter
9. Calculate a 95% CI for each parameter

Table 1: Mean estimates( $\hat{\alpha}, \hat{\beta}$ ), Bias, Relative Bias and Coverage Probability(CP)for selected values of( $\alpha$ and $\beta$ )

| $\alpha$ | $\beta$ | n   | Mean<br>$\alpha$ | Mean<br>$\beta$ | Bias<br>$\alpha$ | Bias<br>$\beta$ | R.Bias<br>$\alpha$ | R.Bias<br>$\beta$ | CP<br>$\alpha$ | CP<br>$\beta$ |
|----------|---------|-----|------------------|-----------------|------------------|-----------------|--------------------|-------------------|----------------|---------------|
| 0.5      | 6.38    | 25  | 0.5246           | 7.0122          | 0.0246           | 0.6322          | 0.0493             | 0.0991            | 1              | 0.8341        |
|          |         | 50  | 0.5122           | 6.6799          | 0.0122           | 0.2999          | 0.0245             | 0.0470            | 1              | 0.8117        |
|          |         | 100 | 0.5063           | 6.5200          | 0.0063           | 0.1450          | 0.0127             | 0.0227            | 1              | 0.7967        |
|          |         | 150 | 0.5044           | 6.4731          | 0.0044           | 0.0932          | 0.0089             | 0.0146            | 1              | 0.7845        |
| 0.85     | 0.32    | 25  | 1.2253           | 0.3936          | 0.3753           | 0.0736          | 0.4416             | 0.2301            | 0.4            | 0.9722        |
|          |         | 50  | 0.8713           | 0.3299          | 0.0213           | 0.0099          | 0.0251             | 0.0311            | 1              | 0.8904        |
|          |         | 100 | 0.8610           | 0.3245          | 0.0110           | 0.0045          | 0.0129             | 0.0140            | 1              | 0.8767        |
|          |         | 150 | 0.8578           | 0.3226          | 0.0078           | 0.0026          | 0.0092             | 0.0082            | 1              | 0.8739        |
| 1.5      | 1.1     | 25  | 1.5933           | 1.1362          | 0.0934           | 0.0362          | 0.0622             | 0.0329            | 1              | 0.7856        |
|          |         | 50  | 1.5447           | 1.1166          | 0.0447           | 0.0166          | 0.0298             | 0.0151            | 1              | 0.7845        |
|          |         | 100 | 1.5246           | 1.1066          | 0.0247           | 0.0066          | 0.0164             | 0.0060            | 1              | 0.7755        |
|          |         | 150 | 1.5176           | 1.1039          | 0.0177           | 0.0039          | 0.0118             | 0.0036            | 1              | 0.7794        |
| 1.16     | 0.58    | 25  | 1.2197           | 0.6175          | 0.0598           | 0.0375          | 0.0515             | 0.0647            | 1              | 0.8546        |
|          |         | 50  | 1.1897           | 0.6000          | 0.0298           | 0.0200          | 0.0257             | 0.0345            | 1              | 0.8363        |
|          |         | 100 | 1.1759           | 0.5884          | 0.0159           | 0.0084          | 0.0137             | 0.0144            | 1              | 0.8219        |
|          |         | 150 | 1.1699           | 0.5873          | 0.0100           | 0.0073          | 0.0086             | 0.0125            | 1              | 0.8285        |
| 1.16     | 0.33    | 25  | 1.2198           | 0.3459          | 0.0598           | 0.0209          | 0.0516             | 0.0644            | 1              | 0.9209        |
|          |         | 50  | 1.1914           | 0.3346          | 0.0315           | 0.0096          | 0.0271             | 0.0294            | 1              | 0.8952        |
|          |         | 100 | 1.1724           | 0.3310          | 0.0124           | 0.0060          | 0.0107             | 0.0183            | 1              | 0.8859        |
|          |         | 150 | 1.1706           | 0.3284          | 0.0106           | 0.0034          | 0.0092             | 0.0104            | 1              | 0.8758        |
| 1        | 0.9     | 25  | 1.0554           | 0.9391          | 0.0554           | 0.0391          | 0.0554             | 0.0434            | 1              | 0.8009        |
|          |         | 50  | 1.0249           | 0.9217          | 0.0250           | 0.0217          | 0.0250             | 0.0241            | 1              | 0.8011        |
|          |         | 100 | 1.0148           | 0.9102          | 0.0148           | 0.0102          | 0.0148             | 0.0113            | 1              | 0.7883        |
|          |         | 150 | 1.0096           | 0.9060          | 0.0096           | 0.0060          | 0.0096             | 0.0067            | 1              | 0.7878        |
| 1.00     | 3.90    | 25  | 1.0534           | 4.1820          | 0.0535           | 0.2721          | 0.0535             | 0.0696            | 1              | 0.8560        |
|          |         | 50  | 1.0258           | 4.0345          | 0.0258           | 0.1246          | 0.0258             | 0.0319            | 1              | 0.8356        |
|          |         | 100 | 1.0123           | 3.9710          | 0.0124           | 0.0615          | 0.0124             | 0.0157            | 1              | 0.8318        |
|          |         | 150 | 1.0090           | 3.9463          | 0.0090           | 0.0363          | 0.0090             | 0.0093            | 1              | 0.8290        |

Table 1 presents mean estimated  $(\alpha, \beta)$ , bias, relative bias and coverage probability. As it was expected, this table shows that the bias decreases when the sample size increases also the relative bias decreases when the sample size increases. The coverage probability for  $(\alpha)$  almost equal 1 and for  $(\beta)$  is close to 85%.

Table 2: Mean Square Error (MSE), Root of Mean Square Error (R MSE), Scaled Root Mean Square Error (S.R MSE) and Average Width (AW))for selected values of ( $\alpha$ and $\beta$ )

| $\alpha$ | $\beta$ | n   | MSE      | MSE     | R MSE    | R MSE   | S.R          | S.R         | AW       | AW      |
|----------|---------|-----|----------|---------|----------|---------|--------------|-------------|----------|---------|
| g        | a       | n   | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | MSE $\alpha$ | MSE $\beta$ | $\alpha$ | $\beta$ |
| 0.50     | 6.38    | 25  | 0.0071   | 3.5144  | 0.0845   | 1.8746  | 0.1690       | 0.2938      | 0.19721  | 2.02380 |
|          |         | 50  | 0.0031   | 1.2565  | 0.0559   | 1.1209  | 0.1117       | 0.1757      | 0.13848  | 1.35690 |
|          |         | 100 | 0.0014   | 0.5205  | 0.0380   | 0.7215  | 0.0760       | 0.1131      | 0.09771  | 0.93864 |
|          |         | 150 | 0.0009   | 0.3316  | 0.0306   | 0.5759  | 0.0612       | 0.0903      | 0.07974  | 0.76090 |
| 0.85     | 0.30    | 25  | 0.1819   | 0.0334  | 0.4265   | 0.1828  | 0.5018       | 0.5712      | 0.76611  | 0.31838 |
|          |         | 50  | 0.0099   | 0.0098  | 0.0994   | 0.0988  | 0.1169       | 0.3088      | 0.34841  | 0.19585 |
|          |         | 100 | 0.0047   | 0.0044  | 0.0684   | 0.0664  | 0.0804       | 0.2076      | 0.24300  | 0.13593 |
|          |         | 150 | 0.0030   | 0.0028  | 0.0545   | 0.0532  | 0.0641       | 0.1663      | 0.19738  | 0.11031 |
| 1.5      | 1.1     | 25  | 0.0931   | 0.1377  | 0.3051   | 0.3712  | 0.2033       | 0.3374      | 0.86337  | 0.56160 |
|          |         | 50  | 0.0396   | 0.0648  | 0.1991   | 0.2546  | 0.1327       | 0.2315      | 0.59429  | 0.39087 |
|          |         | 100 | 0.0190   | 0.0325  | 0.1378   | 0.1803  | 0.0919       | 0.1639      | 0.41552  | 0.27439 |
|          |         | 150 | 0.0121   | 0.0214  | 0.1099   | 0.1462  | 0.0733       | 0.1329      | 0.33796  | 0.22358 |
| 1.16     | 0.58    | 25  | 0.0464   | 0.0598  | 0.2153   | 0.2446  | 0.1857       | 0.4217      | 0.67962  | 0.41318 |
|          |         | 50  | 0.0207   | 0.0289  | 0.1440   | 0.1690  | 0.1241       | 0.2930      | 0.46941  | 0.28521 |
|          |         | 100 | 0.0098   | 0.0138  | 0.0991   | 0.1176  | 0.0855       | 0.2029      | 0.32820  | 0.19880 |
|          |         | 150 | 0.0063   | 0.0089  | 0.0796   | 0.0944  | 0.0686       | 0.1627      | 0.26673  | 0.16198 |
| 1.16     | 0.33    | 25  | 0.0439   | 0.0228  | 0.2095   | 0.1510  | 0.1806       | 0.4645      | 0.81515  | 0.29453 |
|          |         | 50  | 0.0190   | 0.0099  | 0.1379   | 0.0997  | 0.1189       | 0.3067      | 0.54059  | 0.19933 |
|          |         | 100 | 0.0084   | 0.0046  | 0.0917   | 0.0680  | 0.0791       | 0.2092      | 0.36930  | 0.13878 |
|          |         | 150 | 0.0055   | 0.0029  | 0.0742   | 0.0539  | 0.0640       | 0.1660      | 0.30048  | 0.11243 |
| 1        | 0.90    | 25  | 0.0390   | 0.1114  | 0.1974   | 0.3338  | 0.1974       | 0.3709      | 0.54977  | 0.51634 |
|          |         | 50  | 0.0168   | 0.0535  | 0.1298   | 0.2313  | 0.1298       | 0.2570      | 0.38097  | 0.36025 |
|          |         | 100 | 0.0081   | 0.0267  | 0.0901   | 0.1633  | 0.0901       | 0.1814      | 0.26781  | 0.25275 |
|          |         | 150 | 0.0052   | 0.0173  | 0.0724   | 0.1315  | 0.0724       | 0.1462      | 0.21792  | 0.20573 |
| 1        | 3.90    | 25  | 0.0320   | 0.9125  | 0.1788   | 0.9553  | 0.1788       | 0.2443      | 0.46961  | 1.39696 |
|          |         | 50  | 0.0137   | 0.3517  | 0.1170   | 0.5930  | 0.1170       | 0.1517      | 0.32690  | 0.94676 |
|          |         | 100 | 0.0064   | 0.1517  | 0.0798   | 0.3895  | 0.0798       | 0.0996      | 0.22946  | 0.65776 |
|          |         | 150 | 0.0042   | 0.0977  | 0.0648   | 0.3126  | 0.0648       | 0.0799      | 0.18713  | 0.53368 |

Table 2 presents the mean square error (MSE), root of mean square error, relative mean square error and average width which all is decrease when the sample size increase and for average width is decrease while n is increasing.

## 4 Real Data

Here, we analyze four data sets to illustrate the applicability of the new distribution proposed. We apply goodness-of-fit tests to verify which distribution better fits these data sets. We consider the well-known Kolmogorov-Smirnov (KS) statistic, the Cramé-von Mises (CM), and Anderson-Darling (AD) statistics (Chen and Balakrishnan(1995)), Furthermore, we consider the Akaike information criterion  $AIC = -2\hat{L} + 2q$  where  $\hat{L}$  is the loglikelihood function at the MLE of the parameters and  $q$  is the number of model parameters.

Data Set 1: the data set was reported by Jorgensen (1982). It consisted of 40 observations of the active repair times (in hours) for airborne communication transceiver. The data are:

0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50

We showed that  $NLMP(\alpha, \beta)$  distribution is fitted to this real data set.

Table 3: Parameter estimates,Statistics (p- value)for K-S, C-M and A-D for the first data set

|           | $\hat{\alpha}$ | $\hat{\beta}$ | K-S    | C-M   | A-D    |
|-----------|----------------|---------------|--------|-------|--------|
| Statistic | 1.0232         | 0.2           | 0.1308 | 0.116 | 0.8503 |
| p-value   |                |               | 0.461  | 0.509 | 0.4448 |

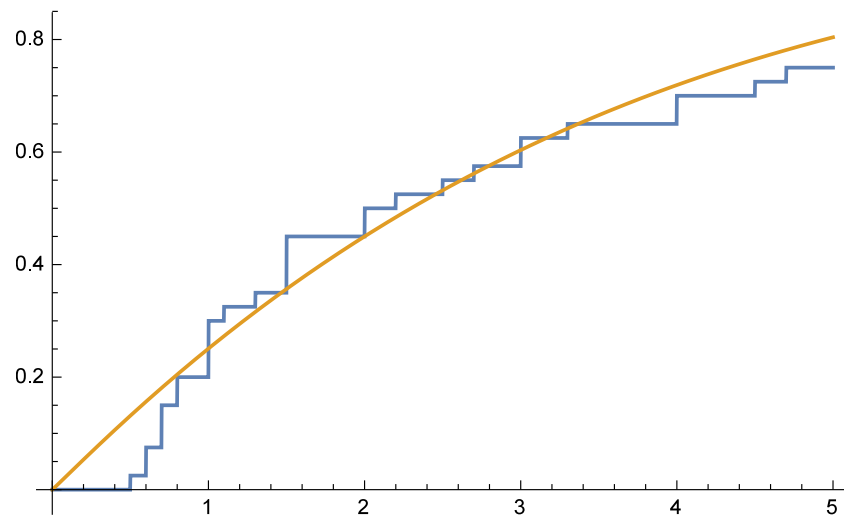


Figure 5: Empirical distribution for data set 1

Data Set 2: the data is provided in Murthy et al. (2004) about time between failures for repairable item. The data are listed as following:

.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

We showed that  $NLMP(\alpha, \beta)$  distribution is fitted to this real data set.

Table 4: Parameter estimates, Statistics (p-value) for K-S, C-M and A-D for the second data set

|           | $\hat{\alpha}$ | $\hat{\beta}$ | K-S     | C-M    | A-D    |
|-----------|----------------|---------------|---------|--------|--------|
| Statistic | 1.5605         | 0.4181        | 0.0777  | 0.0239 | 0.1526 |
| p-value   |                |               | 0.98697 | 0.9954 | 0.9984 |

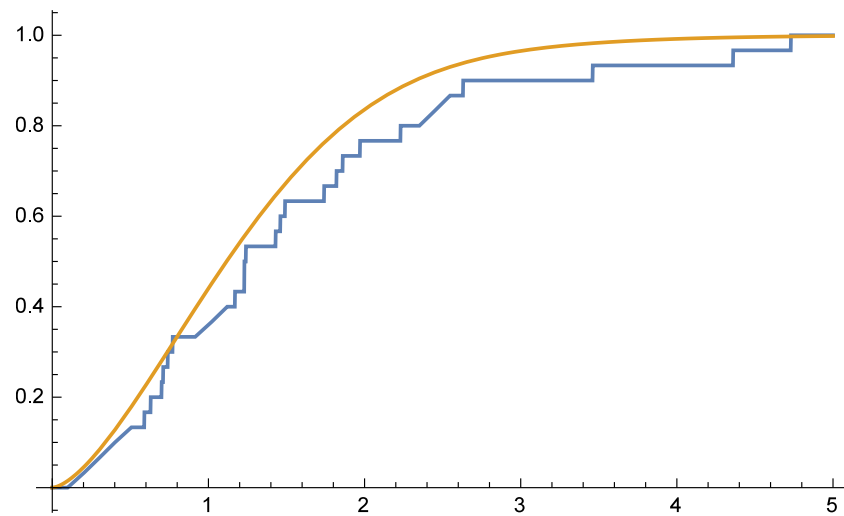


Figure 6: Empirical distribution for data set 2

Data Set 3: the data is provided in Meeker and Escobar (1998) of 57 times (in thousands of operating hours) of unscheduled maintenance actions for the number 4 diesel engine of the U.S.S. Grampus, up to 16 thousand hours of operation

0.86, 1.258, 1.317, 1.442, 1.897, 2.011, 2.122, 2.4, 3.203, 3.298, .902, 3.91, 4, 4.247, 4.411, 4.456, 4.517, 4.899, 1.91, 5.676, 5.755, 6.137, 6.221, 6.311, 6.613, 6.975, 7.335, 8.158, 8.498, 8.69, 9.042, 9.33, 9.394, 9.426, 9.872, 10.191, 11.511, 11.575, 12.1, 12.126, 12.368, 12.681, 12.795, 13.399, 13.668, 13.78, 13.877, 14.007, 14.028, 14.035, 14.173, 14.173, 14.449, 14.587, 14.61, 15.07, 16

We showed that  $NLMP(\alpha, \beta)$  distribution is fitted to this real data set.

Table 5: Parameter estimates, Statistics (p- value) for K-S, C-M and A-D for the third data set

|           | $\hat{\alpha}$ | $\hat{\beta}$ | K-S    | C-M    | A-D    |
|-----------|----------------|---------------|--------|--------|--------|
| Statistic | 1.689          | 0.014         | 0.129  | 0.194  | 1.444  |
| p-value   |                |               | 0.2669 | 0.2792 | 0.1903 |

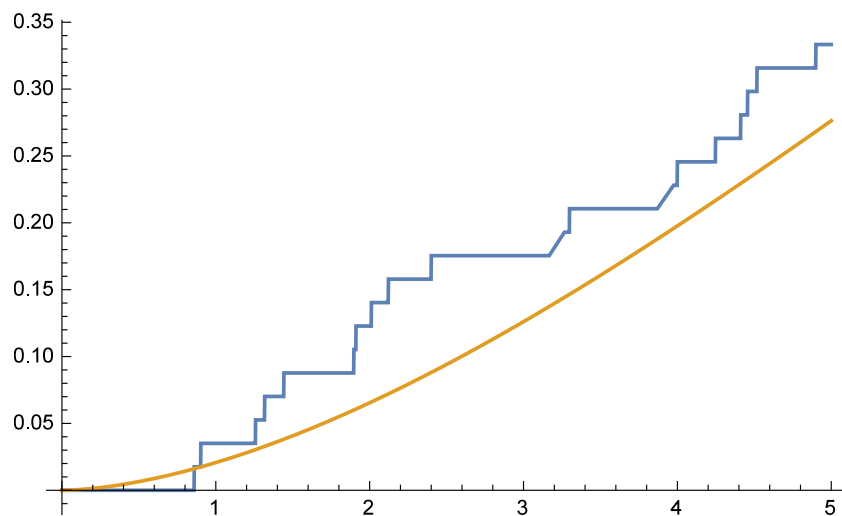


Figure 7: Empirical distribution for data set 3

Data Set 4: the data is provided by Ugarte et al. (2008) carbon retained by leaves measured in kilogram/hectare for thirty eight different plots of mountainous regions of Navarra (Spain), depending on the forest classification: areas with ninety percent or more beech trees (*Fagus Sylvatica*) are labeled monospecific, while areas with many species of trees are labeled multispecific

1501.82, 6989.43, 2424.02, 4150.29, 8693.35, 2643.77, 13148.37, 6149.39, 23587.21, 7248.37, 4788.22, 6009.51, 5349.65, 5741.32, 7065.81, 7261.37, 2358.42, 10357.884 , 2499.05, 3022.90 ,4234.86, 4482.03, 6363.71, 3329.91, 8740.47, 3664.95, 4515.97, 8497.71, 4569.89,

8069.63, 7366.79 ,1525.41, 3363.02, 2420.57, 3576.74, 3708.05, 5819.12, 5479.38

We showed that  $NLMP(\alpha, \beta)$  distribution is fitted to this real data set.

Table 6: Parameter estimates,Statistics (p- value)for K-S, C-M and A-D for the forth data set

|           | $\hat{\alpha}$ | $\hat{\beta}$ | K-S    | C-M    | A-D     |
|-----------|----------------|---------------|--------|--------|---------|
| Statistic | 1.538          | 0.0000008     | 0.1256 | 0.1069 | 0.9186  |
| p-value   |                |               | 0.544  | 0.551  | 0.40163 |

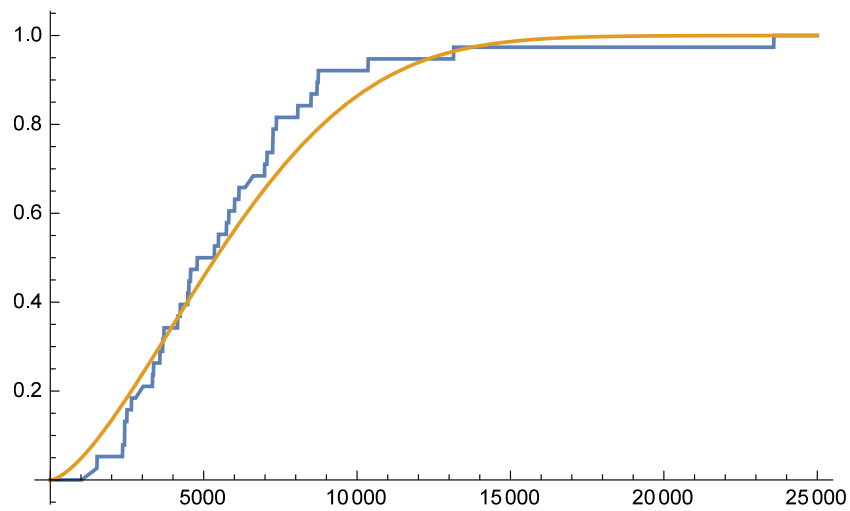


Figure 8: Empirical distribution for data set 4

## 5 Conclusion

In this paper new one-parameter and two-parameter distributions were introduced, the first was called the  $NLM(\beta)$  and second was called  $NLMP(\alpha, \beta)$ , The  $NLMP(\alpha, \beta)$  was obtained by using a power transformation of the  $NLM(\beta)$  distributed variable. The  $NLM(\beta)$  obtained by mixing three distributions (Gamma, exponential and Linear Failure Rate Distribution) with different weights ( $a_1 = \frac{\beta}{1+\beta}$ ,  $a_2 = \frac{1}{2(1+\beta)}$  and  $a_3 = \frac{1}{2(1+\beta)}$ ). Moreover, the  $NLMP(\alpha, \beta)$  obtained by mixing three distributions (power gamma, Weibull and power linear failure rate distribution) with the same weights used in  $NLM(\beta)$ . The  $NLMP(\alpha, \beta)$  provides more flexibility than the  $NLM(\beta)$  in terms of the shape of the density and hazard rate functions. The  $NLMP(\alpha, \beta)$  hazard shape has different types of shapes including Bathtub, decreasing and increasing hazard function. The maximum likelihood estimates of the parameters and their variance covariance matrix were derived using *Mathematica* [9] for numerically solving equations have no analytical solutions. Finally, we examined the accuracy of the maximum likelihood estimators of the  $NLMP(\alpha, \beta)$  parameters as well as the coverage probability and average width of the confidence intervals for the parameters using simulation. The results of the simulation study show that the bias for any estimator is decreased when the sample size increases also the relative bias is decreasing when the sample size increases and coverage probability. we found that the coverage probability for  $(\alpha)$  almost equal 1 and for  $(\beta)$  is close to 85%. Mean square error (MSE), root of mean square error, relative mean square error and average width decrease when the sample size increases. Also using  $(\alpha)=1$  in simulation study is a special case for the  $NLMP(\alpha, \beta)$  which lead to the one parameter distribution  $NLM(\beta)$ . After that we applied on some real data and the result is as follows. For data set 1, The  $NLMP(\alpha, \beta)$  is fitted to this real data since the p-value is greater than  $(\alpha = 0.05)$

for all tests as shown in Table 3.

For data set 2, it is clear that  $NLMP(\alpha, \beta)$  is fitted to this real data since the p-value is greater than ( $\alpha = 0.05$ ) for all tests as shown in Table 4.

For data set 3, the  $NLMP(\alpha, \beta)$  is fitted to this real data since the p-value is greater than ( $\alpha = 0.05$ ) for all tests as shown in Table 5.

For data set 4, it is clear that  $NLMP(\alpha, \beta)$  is fitted to this real data since the p-value is greater than ( $\alpha = 0.05$ ) for all tests as shown in Table 6.

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# A New Generalization of Modified Weibull Distribution

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## Abstract

This paper presented The (P-A-L) Modified Weibull Distribution. We can compute several properties of this distribution. The maximum likelihood estimators are obtained. Using simulation study, mean, relative bias, scaled root mean square error for maximum likelihood estimators are obtained. Also, Confidence intervals for unknown two parameters are calculated.

**Keywords:**Hazard function, Maximum likelihood estimation, Modified Weibull Distribution, Moment, The (P-A-L) family, Variance-covariance matrix.

## 1 Introduction

Many companies and organizations have too much data and the amount of data available is growing. One of the ways to turn these data into useful information is to have a statistical model for the way the data are generated. On the other hand the industrial processes as well as the way the components of a business interact are getting more and more complicated. For these reasons, no matter how many distributional models become available there arise the needs for new models.

Several methods have been suggested in the statistical literature to construct new statistical distributions from existing ones. Among these methods there is the formula suggested by Marshall and Olkin (1997) that became known by their

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name. There is also the rank transmutation map due to Shaw and Buckley (2007). Of interest to us here is the (P-A-L) extension suggested by Pappas et al. (2012). Lee et al. (2013) present a good review of some of the most families methods for generating new families of distributions from existing ones.

In this article, the method known as the (P-A-L) is utilized to create a new family of distributions from the Modified Weibull distribution of Sarahan and Zaindin (2009). This distribution called The (P-A-L) Modified Weibull Distribution with four parameters  $\alpha, \beta, \gamma, p$  where it denotes as PMWD  $(\alpha, \beta, \gamma, p)$ . PMWD  $(\alpha, \beta, \gamma, p)$  introduced in section 2 and present properties of this distribution in section 3. We estimate four parameters for PMWD by the maximum likelihood method of estimation. Finally, we analyze real data for PMWD.

## 2 The (P-A-L) Modified Weibull Distribution

The survival function of the (P-A-L) family is

$$s(x) = \frac{\ln\{1 - (1 - p) s_0(x)\}}{\ln p}, \quad x \in R_+, p \in R_+ - \{0\} \quad (1)$$

Where  $s_0$  is the survival function of the base distribution and it should be noted that if  $p \rightarrow 1$ , then  $s \rightarrow s_0$ . The probability density function and hazard function take the form

$$f(x) = \frac{(p - 1) f_0(x)}{\{1 - (1 - p) s_0(x)\} \ln p}, \quad (2)$$

$$h(x) = \frac{(p - 1) s_0(x) h_0(x)}{\{1 - (1 - p) s_0(x)\} \ln\{1 - (1 - p) s_0(x)\}}, \quad (3)$$

Where  $f_0$  and  $h_0$  are the probability density function and hazard function of the base distribution. There are several distributions used by the (P-A-L) family, such

as the (P-A-L) extended modified Weibull studied by Pappas et al. (2012), the (P-A-L) extended Weibull distribution introduced by Al-Zahrani et al. (2016) and The (P-A-L) Generalized Exponential Distribution presented by Mahmoud and Mandouh (2017).

Now, we put  $s_0(x; \alpha, \beta, \gamma) = [e^{-\alpha x - \beta x^\gamma}]$  is a survival function for the Modified Weibull distribution presented by Sarahan and Zaindin (2009).

By substituting  $s_0$  in (2.1), we get

$$S(x) = \frac{\ln(1 - (1 - p)e^{-\alpha x - \beta x^\gamma})}{\ln(p)} \quad ; \quad x > 0, \gamma > 0, \alpha, \beta \geq 0, p \in R_+ - \{0\} \quad (4)$$

where  $\alpha$  is scale parameter and  $\beta, \gamma, p$  are shape parameter. Hence

$$f(x) = \frac{(p - 1)(\alpha + \beta \gamma x^{\gamma-1})e^{-\alpha x - \beta x^\gamma}}{(1 - (1 - p)e^{-\alpha x - \beta x^\gamma}) \ln(p)} \quad (5)$$

Then the hazard function takes the form

$$h(x) = \frac{(p - 1)e^{-\alpha x - \beta x^\gamma} [\alpha + \beta \gamma x^{\gamma-1}]}{[1 - (1 - p)e^{-\alpha x - \beta x^\gamma}] \ln(1 - (1 - p)e^{-\alpha x - \beta x^\gamma})} \quad (6)$$

We note that  $h(x)$  can be constant, increasing or decreasing depending on the parameter values. For example, if  $p \rightarrow 1$ ,  $\beta=1$  and  $\gamma=1$  then  $h(x)=\alpha+1$  is constant, whereas if  $p \rightarrow 1$  and  $\beta=1$ , then  $h(x) = \alpha + \gamma x^{\gamma-1}$ , which is increasing for  $\gamma > 1$  and decreasing for  $\gamma < 1$ .

Figures 1 and 2 show different shapes for the probability density function and the hazard function of The (P-A-L) Modified Weibull Distribution for different values of the parameters  $\alpha, \beta, \gamma$  and  $p$ .

To find the raw moments, we have

$$E(X^r) = \int_0^\infty x^r f(x) dx$$

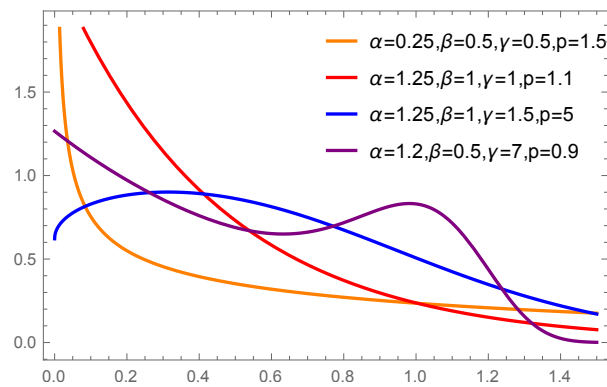


Figure 1: Probability density function of (P-A-L)-MWD for different values of the parameters

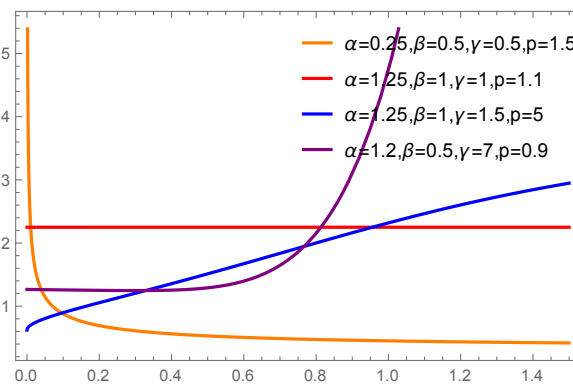


Figure 2: Hazard function of (P-A-L)-MWD for different values of the parameters

from equation(2.5), we get

$$E(X^r) = \int_0^{\infty} x^r \frac{(p-1)(\alpha + \beta\gamma x^{\gamma-1})e^{-\alpha x - \beta x^{\gamma}}}{(1 - (1-p)e^{-\alpha x - \beta x^{\gamma}}) \ln(1 - (1-p))} dx \quad (7)$$

To calculate the  $r^{th}$  raw moments of The (P-A-L) Modified Weibull Distribution, we can use Numerical integration steps. We can use the Binomial expansion to

get the following expression:

$$E(X^r) = \sum_{n=0}^{\infty} \frac{(-1)^n (p-1)^{(n+1)}}{\log(1-(1-p))} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{(n+1)^i \beta^i}{(n+1)^{r+i\gamma}} \left[ \frac{\Gamma(i\gamma + r + 1)}{(n+1)\alpha^{i\gamma+r}} + \frac{\beta\gamma}{(n+1)^\gamma \alpha^{\gamma+i\gamma+r}} \Gamma(\gamma + i\gamma + k) \right] \quad (8)$$

Let  $\alpha = 1.1, \beta = 2, \gamma = 3, p = 2$  and  $r = 1$ , one can easily check that equation (2.8) and numerical integration of (2.7) take the same value, i.e.  $E(X^r) = 0.53$ .

### 3 Random Number Generation and Estimation of the Parameters

Using the inversion method, one can generate random from The (P-A-L) Modified Weibull Distribution with the following formula

$$u = F(x) \quad (9)$$

By substituting  $F(x)$  in (3.1), we get

$$u = 1 - \frac{\log(1 - (1-p)e^{-\alpha x - \beta x^\gamma})}{\log(p)} \quad (10)$$

Where  $u \in (0, 1)$ . We take log

$$\beta x^\gamma + \alpha x + \log \frac{1 - (1 - (1-p))^{(1-u)}}{-(p-1)} = 0 \quad (11)$$

The equation (3.3) has no closed form solution in  $x$  so, we have to use numerical technique.

Now, we will study parameter estimation by using maximum likelihood method.

### 3.1 Maximum Likelihood Estimation

Let  $x_1, x_2, x_3, \dots, x_n$  is a random sample follows The (P-A-L) Modified Weibull Distribution. The likelihood function is given by

$$L(x; \alpha, \beta, \gamma, p) = \prod_{i=1}^n \frac{(p-1)(\alpha + \beta\gamma x_i^{\gamma-1})e^{-\alpha x_i - \beta x_i^\gamma}}{(1 - (1-p)e^{-\alpha x_i - \beta x_i^\gamma}) \log(p)} \quad (12)$$

The log likelihood is

$$\begin{aligned} \ln L = n[\ln(p-1)] + \sum_{i=1}^n \ln(\alpha + \beta\gamma x_i^{\gamma-1}) + \sum_{i=1}^n (-\alpha x_i - \beta x_i^\gamma) \\ - n[\ln(\ln(p))] - \sum_{i=1}^n \ln(1 - (1-p)e^{-\alpha x_i - \beta x_i^\gamma}) \end{aligned} \quad (13)$$

Differentiating (3.5) with respect to  $\alpha, \beta, \gamma$  and  $p$ , we have

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^n \frac{1}{\alpha + \beta\gamma x_i^{\gamma-1}} - \sum_{i=1}^n x_i + (p-1) \sum_{i=1}^n \frac{x_i e^{-\alpha x_i - \beta x_i^\gamma}}{1 - (1-p)e^{-\alpha x_i - \beta x_i^\gamma}} \quad (14)$$

$$\frac{\partial l}{\partial \beta} = \gamma \sum_{i=1}^n \frac{x_i^{\gamma-1}}{\alpha + \beta\gamma x_i^{\gamma-1}} - \sum_{i=1}^n x_i^\gamma + (p-1) \sum_{i=1}^n \frac{x_i^\gamma e^{-\alpha x_i - \beta x_i^\gamma}}{1 - (1-p)e^{-\alpha x_i - \beta x_i^\gamma}} \quad (15)$$

$$\begin{aligned} \frac{\partial l}{\partial \gamma} = \sum_{i=1}^n -\beta \ln(x_i) x_i^\gamma - \sum_{i=1}^n \frac{e^{-\alpha x_i - \beta x_i^\gamma} (1-p) \beta \ln(x_i) x_i^\gamma}{1 - e^{-\alpha x_i - \beta x_i^\gamma} (1-p)} \\ + \sum_{i=1}^n \frac{\beta x_i^{-1+\gamma} + \beta \gamma \ln(x_i) x_i^{-1+\gamma}}{\alpha + \beta\gamma x_i^{-1+\gamma}} \end{aligned} \quad (16)$$

$$\frac{\partial l}{\partial p} = \frac{n}{p-1} - \frac{n}{p \ln(p)} - \sum_{i=1}^n \frac{e^{-\alpha x_i - \beta x_i^\gamma}}{1 - (1-p)e^{-\alpha x_i - \beta x_i^\gamma}} \quad (17)$$

Equating the derivatives in (3.6), (3.7), (3.8) and (3.9) to zero, then we solve the four nonlinear equations by numerically, we get the maximum likelihood estimators  $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$  and  $\hat{p}$ .

Hence

The second derivatives of the log likelihood function are given by

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha^2} = -I_{11} = & -\sum_{i=1}^n \left( -\frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} (p-1)^2 x_i^2}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} + \frac{e^{-\alpha x_i - \beta x_i^\gamma} (p-1) x_i^2}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right) \\ & + \sum_{i=1}^n -\frac{1}{(\alpha + \beta \gamma x_i^{\gamma-1})^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta^2} = -I_{22} = & -\sum_{i=1}^n \frac{\gamma^2 x_i^{-2-2\gamma}}{(\alpha + \beta \gamma x_i^{-1+\gamma})^2} - \sum_{i=1}^n \left( -\frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} (p-1)^2 x_i^{2\gamma}}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} \right. \\ & \left. + \frac{e^{-\alpha x_i - \beta x_i^\gamma} (p-1) x_i^{2\gamma}}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \gamma^2} = -I_{33} = & \sum_{i=1}^n -\beta \log[x_i]^2 x_i^\gamma + \sum_{i=1}^n \left[ -\frac{(\beta x_i^{-1+\gamma} + \beta \gamma \log[x_i] x_i^{-1+\gamma})^2}{(\alpha + \beta \gamma x_i^{-1+\gamma})^2} \right. \\ & \left. + \frac{2\beta \log[x_i] x_i^{-1+\gamma} + \beta \gamma \log[x_i]^2 x_i^{-1+\gamma}}{\alpha + \beta \gamma x_i^{-1+\gamma}} \right] - \sum_{i=1}^n \left[ -\frac{e^{-\alpha x_i - \beta x_i^\gamma} \beta (p-1) \log[x_i]^2 x_i^\gamma}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right. \\ & \left. - \frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} \beta^2 (p-1)^2 \log[x_i]^2 x_i^{2\gamma}}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} + \frac{e^{-\alpha x_i - \beta x_i^\gamma} \beta^2 (p-1) \log[x_i]^2 x_i^{2\gamma}}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial p^2} = -I_{44} = & -\frac{n}{(p-1)^2} + \frac{n}{(1 - (1-p))^2 \log[1 - (1-p)]^2} \\ & + \frac{n}{(1 - (1-p))^2 \log[1 - (1-p)]} - \sum_{i=1}^n \frac{-e^{-2\alpha x_i - 2\beta x_i^\gamma}}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} \end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} &= -I_{12} = \sum_{i=1}^n \frac{-\gamma x_i^{-1+\gamma}}{(\alpha + \beta \gamma x_i^{-1+\gamma})^2} - \sum_{i=1}^n \left( -\frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} (p-1)^2 x_i^{1+\gamma}}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} \right. \\ &\quad \left. + \frac{e^{\alpha x_i - \beta x_i^\gamma} (p-1) x_i^{1+\gamma}}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right) \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \gamma} &= -I_{13} = \sum_{i=1}^n -\frac{\beta x_i^{-1+\gamma} + \beta \gamma \log[x_i] x_i^{-1+\gamma}}{(\alpha + \beta \gamma x_i^{-1+\gamma})^2} - \sum_{i=1}^n \left[ -\frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} \beta (p-1)^2 \log[x_i] x_i^{1+\gamma}}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} \right. \\ &\quad \left. + \frac{e^{-\alpha x_i - \beta x_i^\gamma} \beta (p-1) \log[x_i] x_i^{1+\gamma}}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right] \\ \frac{\partial^2 \ln L}{\partial \alpha \partial p} &= -I_{14} = -\sum_{i=1}^n \left( \frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} (p-1) x_i}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} - \frac{e^{-\alpha x_i - \beta x_i^\gamma} x_i}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right) \\ \frac{\partial^2 \ln L}{\partial \beta \partial \gamma} &= -I_{23} = \sum_{i=1}^n -\log[x_i] x_i^\gamma - \sum_{i=1}^n \left[ -\frac{e^{-\alpha x_i - \beta x_i^\gamma} (p-1) \log[x_i] x_i^\gamma}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right. \\ &\quad \left. - \frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} \beta (p-1)^2 \log[x_i] x_i^{2\gamma}}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} + \frac{-e^{-\alpha x_i - \beta x_i^\gamma} \beta (p-1) \log[x_i] x_i^{2\gamma}}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right] + \sum_{i=1}^n \left[ \frac{x_i^{-1+\gamma}}{\alpha + \beta \gamma x_i^{-1+\gamma}} \right. \\ &\quad \left. + \frac{\gamma \log[x_i] x_i^{-1+\gamma}}{\alpha + \beta \gamma x_i^{-1+\gamma}} - \frac{\gamma x_i^{-1+\gamma} (\beta x_i^{-1+\gamma} + \beta \gamma \log[x_i] x_i^{-1+\gamma})}{(\alpha + \beta \gamma x_i^{-1+\gamma})^2} \right] \\ \frac{\partial^2 \ln L}{\partial \beta \partial p} &= -I_{24} = -\sum_{i=1}^n \left( \frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} (p-1) x_i^\gamma}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} - \frac{e^{-\alpha x_i - \beta x_i^\gamma} x_i^\gamma}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right) \\ \frac{\partial^2 \ln L}{\partial \gamma \partial p} &= -I_{34} = -\sum_{i=1}^n \left( \frac{e^{-2\alpha x_i - 2\beta x_i^\gamma} \beta (p-1) \log[x_i] x_i^\gamma}{(1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1))^2} - \frac{e^{-\alpha x_i - \beta x_i^\gamma} \beta \log[x_i] x_i^\gamma}{1 + e^{-\alpha x_i - \beta x_i^\gamma} (p-1)} \right)\end{aligned}$$

Then the observed information matrix is given by

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}$$

So that the variance-covariance matrix may be approximated as

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}^{-1}$$

It is known that the asymptotic distribution of the MLE  $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{p})$  is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \\ \hat{p} \end{pmatrix} \sim N \left[ \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ p \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix} \right] \quad (18)$$

Since V involves the parameters  $\alpha, \beta, \gamma, p$ , we replace the parameters by the corresponding MLE's in order to obtain an estimate of V, which is denoted by

$$\hat{V} = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}^{-1} \quad (19)$$

Where  $\hat{I}_{ij}$  when  $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{p})$  replaces  $(\alpha, \beta, \gamma, p)$ .

By using (3.10), approximate  $100(1 - \theta)\%$  confidence intervals for  $\alpha, \beta, \gamma, p$  are determined, respectively, as

$$\hat{\alpha} \pm z_{\theta/2} \sqrt{\hat{V}_{11}}, \quad \hat{\beta} \pm z_{\theta/2} \sqrt{\hat{V}_{22}}, \quad \hat{\gamma} \pm z_{\theta/2} \sqrt{\hat{V}_{33}}, \quad \hat{p} \pm z_{\theta/2} \sqrt{\hat{V}_{44}} \quad (20)$$

Where  $z_{\theta}$  is the upper  $\theta$ -th percentile of the standard normal distribution.

## 4 Simulation Study

A simulation study was carried to check the performance of the accuracy of point and interval estimates for several cases, of which estimates two param-

eters, estimates three parameters and finally, estimates four parameters of (P-A-L)-MWD  $(\alpha, \beta, \gamma, p)$  for  $m=1000$ , the sample size  $n$  are 50, 100, 150, 200, 250, 300 and different parameter value. The following steps were followed to obtain the results:

1. Specify initial values for parameters  $\alpha, \beta, \gamma$  and  $p$ .
2. Specify the sample size  $n$ .
3. Generate  $m$  times of random sample with size  $n$  from (P-A-L)-MWD $(\alpha, \beta, \gamma, p)$ .
4. Obtain the maximum likelihood estimates for  $\alpha, \beta, \gamma, p$  for different sample sizes.
5. Obtain the mean, bias and relative bias, mean squared error (MSE), root mean squared error(RMSE) and scaled root mean square error(scaled RMSE) for each estimator for different sample size. Where[scaled RMSE = RMSE/initial value]
6. Repeat 1:5 for several values of  $\alpha, \beta, \gamma$  and  $p$ .

By the software Mathematica 10 we tried to do simulation and estimate the parameters  $(\gamma, p)$  with respect to  $(\alpha, \beta)$  are known, The result is shown in Table 1 and we estimate the parameters  $(\beta, p)$  with respect to  $(\alpha, \gamma)$  are known as shown in Table 2 Similarly, we estimate the parameters  $(\beta, \gamma, p)$  with respect to  $\alpha$  are known as shown in Table 3 Finally, we estimate the parameters  $(\alpha, \beta, \gamma, p)$  as shown in Table 4.

Based on the values we obtained from Simulation study, the results are shown in Table 1, Table 2, Table 3, Table 4.

By results of the simulation study, show that the bias for any estimator is decreased when the sample size increases also the relative bias is decreased when the sample size increases. Mean square error (MSE), root mean square error, scaled root mean square error decrease when the sample size increases. Calculate LL(lower limit) and UL(upper limit).

## 5 Application for Real Data

These following data from Murthy *et al.* (2004) used to show the applicability of the distribution and estimation method

Data Set: Failure Times of 50 Components:

0.013, 0.065, 0.111, 0.111, 0.163, 0.309, 0.426, 0.535, 0.684, 0.747, 0.997, 1.284, 1.304, 1.647, 1.829, 2.336, 2.838, 3.269, 3.977, 3.981, 4.520, 4.789, 4.849, 5.202, 5.291, 5.349, 5.911, 6.018, 6.427, 6.456, 6.572, 7.023, 7.087, 7.291, 7.787, 8.596, 9.388, 10.261, 10.713, 11.658, 13.006, 13.388, 13.842, 17.152, 17.283, 19.418, 23.471, 24.777, 32.795, 48.105

Parameters of (P-A-L)-MWD are estimated by maximum likelihood estimation follows

$$\hat{\alpha} = 0.0656, \quad \hat{\beta} = 0.0157, \quad \hat{\gamma} = 1.0586, \quad \hat{p} = 0.1762$$

Now, Chi-Square is studied, where the following hypothesis is:

$H_0$ : the data come from The (P-A-L) Modified Weibull Distribution.

$H_1$ : the data does not come from The (P-A-L) Modified Weibull Distribution.

The data were classified in 8 class intervals as shown in Table 5

The calculated statistic value of Chi-square test (7.4686) is less than the tabulated statistic value (7.8147). Therefore the null hypothesis do not reject that the data came from The (P-A-L) Modified Weibull Distribution. We use software Mathematica 10 to get the goodness of fit test.

## 6 Conclusions

In this paper new lifetime distribution is presented which called The (P-A-L) Modified Weibull Distribution (PMWD). The hazard shape for The (P-A-L) Modified Weibull Distribution has different types of shapes such as decreasing and increasing hazard function. For example, if  $p \rightarrow 1$ ,  $\beta=1$  and  $\gamma=1$  then  $h(x)=\alpha+1$  is constant, whereas if  $p \rightarrow 1$  and  $\beta=1$ , then  $h(x) = \alpha + \gamma x^{\gamma-1}$ , which is increasing for  $\gamma > 1$  and decreasing for  $\gamma < 1$ . The maximum likelihood estimates of the parameters and their variance covariance matrix were derived using software Mathematica 10 for numerically solving equations have no analytical solutions. By application on real data, we found that the  $PMWD(\alpha, \beta, \gamma, p)$  fits the data.

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Table 1: MLE's and confidence intervals of two parameter ( $\gamma$  and  $p$ ) in the case of (P-A-L)-MWD

| 1. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 1.5, p_0 = 1.98$ |          |       |               |          |      |             |        |       |
|-----------------------------------------------------------------|----------|-------|---------------|----------|------|-------------|--------|-------|
|                                                                 |          | Mean  | Relative bias | variance | RMSE | Scaled RMSE | LL     | UL    |
| n=50                                                            | $\gamma$ | 1.64  | -0.09         | 0.25     | 0.52 | 0.35        | 1.03   | 2.25  |
|                                                                 | $p$      | 2.41  | -0.22         | 3.52     | 1.92 | 0.97        | -0.83  | 5.64  |
| n=75                                                            | $\gamma$ | 1.60  | -0.7          | 0.22     | 0.48 | 0.32        | 0.88   | 2.70  |
|                                                                 | $p$      | 2.38  | -0.20         | 2.52     | 1.64 | 0.83        | -2.47  | 7.24  |
| n=100                                                           | $\gamma$ | 1.59  | -0.06         | 0.18     | 0.44 | 0.27        | 1.08   | 0.39  |
|                                                                 | $p$      | 2.30  | -0.16         | 1.96     | 1.44 | 0.62        | 2.10   | 4.20  |
| n=150                                                           | $\gamma$ | 1.57  | -0.05         | 0.12     | 0.35 | 0.23        | 1.18   | 1.96  |
|                                                                 | $p$      | 2.19  | -0.11         | 1.26     | 1.14 | 0.52        | 0.8131 | 3.58  |
| 2. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 0.5, p_0 = 0.9$  |          |       |               |          |      |             |        |       |
|                                                                 |          | Mean  | Relative bias | variance | RMSE | Scaled RMSE | LL     | UL    |
| n=50                                                            | $\gamma$ | 0.54  | -0.04         | 0.007    | 0.09 | 0.18        | 0.06   | 1.02  |
|                                                                 | $p$      | 0.98  | -0.08         | 0.49     | 0.71 | 0.72        | -3.47  | 5.45  |
| n=75                                                            | $\gamma$ | 0.53  | -0.06         | 0.005    | 0.08 | 0.15        | 0.41   | 0.65  |
|                                                                 | $p$      | 0.97  | -0.08         | 0.35     | 0.60 | 0.62        | -0.56  | 2.52  |
| n=100                                                           | $\gamma$ | 0.52  | -0.05         | 0.003    | 0.06 | 0.12        | 0.46   | 0.59  |
|                                                                 | $p$      | 0.938 | -0.04         | 0.18     | 0.43 | 0.46        | -0.05  | 1.92  |
| n=150                                                           | $\gamma$ | 0.51  | -0.03         | 0.002    | 0.05 | 0.09        | 0.29   | 0.75  |
|                                                                 | $p$      | 0.935 | -0.042        | 0.13     | 0.36 | 0.39        | -1.36  | 3.24  |
| 3. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 7, p_0 = 5$      |          |       |               |          |      |             |        |       |
|                                                                 |          | Mean  | Relative bias | variance | RMSE | Scaled RMSE | LL     | UL    |
| n=50                                                            | $\gamma$ | 7.18  | -0.03         | 2.87     | 1.71 | 0.23        | 4.24   | 10.13 |
|                                                                 | $p$      | 5.85  | -0.17         | 10.25    | 3.31 | 0.56        | 0.41   | 11.31 |
| n=75                                                            | $\gamma$ | 7.34  | -0.04         | 2.45     | 1.60 | 0.2183      | 4.76   | 9.93  |
|                                                                 | $p$      | 5.47  | -0.09         | 6.85     | 2.66 | 0.48        | 0.89   | 10.05 |
| n=100                                                           | $\gamma$ | 7.19  | -0.02         | 1.69     | 1.32 | 0.18        | 2.63   | 11.76 |
|                                                                 | $p$      | 5.46  | -0.09         | 5.87     | 2.46 | 0.45        | -6.57  | 17.49 |
| n=150                                                           | $\gamma$ | 7.09  | -0.013        | 1.04     | 1.02 | 0.144       | 4.76   | 9.42  |
|                                                                 | $p$      | 5.26  | -0.05         | 3.46     | 1.88 | 0.36        | 1.68   | 8.86  |

Table 2: MLE's and confidence intervals of two parameter ( $\beta$  and  $p$ ) in the case of (P-A-L)-MWD

| 1. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 1.5, p_0 = 1.98$ |         |      |               |          |      |             |        |       |
|-----------------------------------------------------------------|---------|------|---------------|----------|------|-------------|--------|-------|
|                                                                 |         | Mean | Relative bias | variance | RMSE | Scaled RMSE | LL     | UL    |
| n=50                                                            | $\beta$ | 2.93 | -0.17         | 0.89     | 1.04 | 0.42        | 0.92   | 4.95  |
|                                                                 | p       | 2.49 | -0.26         | 15.35    | 3.95 | 1.99        | -3.88  | 8.88  |
| n=75                                                            | $\beta$ | 2.79 | -0.12         | 0.49     | 0.76 | 0.30        | 0.92   | 4.66  |
|                                                                 | p       | 1.93 | 0.02          | 2.41     | 1.55 | 0.78        | -2.25  | 6.12  |
| n=100                                                           | $\beta$ | 2.74 | -0.09         | 0.35     | 0.64 | 0.14        | 1.59   | 3.88  |
|                                                                 | p       | 1.78 | 0.09          | 1.43     | 1.21 | 0.61        | 0.14   | 3.45  |
| n=150                                                           | $\beta$ | 2.67 | -0.07         | 0.24     | 0.52 | 0.21        | 1.17   | 4.17  |
|                                                                 | p       | 1.59 | 0.19          | 0.47     | 0.78 | 0.39        | -0.75  | 3.94  |
| 2. $\alpha_0 = 1.25, \beta_0 = 0.5, \gamma_0 = 1.5, p_0 = 2$    |         |      |               |          |      |             |        |       |
|                                                                 |         | Mean | Relative bias | variance | RMSE | Scaled RMSE | LL     | UL    |
| n=50                                                            | $\beta$ | 0.69 | -0.39         | 0.08     | 0.35 | 0.72        | -0.67  | 2.07  |
|                                                                 | p       | 3.43 | -0.72         | 22.32    | 4.93 | 2.46        | -11.83 | 18.83 |
| n=75                                                            | $\beta$ | 0.65 | -0.30         | 0.06     | 0.28 | 0.57        | -0.23  | 1.53  |
|                                                                 | p       | 2.92 | -0.46         | 9.53     | 3.22 | 1.61        | -3.71  | 9.55  |
| n=100                                                           | $\beta$ | 0.61 | -0.21         | 0.03     | 0.21 | 0.41        | 0.15   | 1.05  |
|                                                                 | p       | 2.49 | -0.24         | 1.95     | 1.47 | 0.74        | -0.01  | 4.98  |
| n=150                                                           | $\beta$ | 0.58 | -0.16         | 0.02     | 0.17 | 0.35        | 0.23   | 0.94  |
|                                                                 | p       | 2.34 | -0.17         | 0.88     | 0.99 | 0.49        | 0.54   | 4.13  |

Table 2 (continued )

| 3. $\alpha_0 = 1.25, \beta_0 = 1.5, \gamma_0 = 1.5, p_0 = 0.5$ |         |      |               |          |      |             |       |      |
|----------------------------------------------------------------|---------|------|---------------|----------|------|-------------|-------|------|
|                                                                |         | Mean | Relative bias | variance | RMSE | Scaled RMSE | LL    | UL   |
| n=50                                                           | $\beta$ | 1.69 | -0.13         | 0.46     | 0.71 | 0.47        | 0.11  | 3.27 |
|                                                                | p       | 0.84 | -0.68         | 1.05     | 1.07 | 2.16        | -0.74 | 2.43 |
| n=75                                                           | $\beta$ | 1.64 | -0.09         | 0.25     | 0.52 | 0.35        | 0.68  | 2.59 |
|                                                                | p       | 0.72 | -0.44         | 0.21     | 0.51 | 1.02        | 0.08  | 1.35 |
| n=100                                                          | $\beta$ | 1.59 | -0.06         | 0.17     | 0.42 | 0.28        | 0.07  | 3.11 |
|                                                                | p       | 0.65 | -0.29         | 0.09     | 0.33 | 0.67        | -0.67 | 1.97 |
| n=150                                                          | $\beta$ | 1.55 | -0.04         | 0.11     | 0.33 | 0.22        | 0.75  | 2.36 |
|                                                                | p       | 0.61 | -0.21         | 0.04     | 0.22 | 0.43        | 0.05  | 1.16 |

Table 3: MLE's and mean square error of three parameters ( $\beta, \gamma$  and  $p$ ) in the case of (P-A-L)-MWD

| 1. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 0.5, p_0 = 0.9$ |          |       |               |          |       |             |
|----------------------------------------------------------------|----------|-------|---------------|----------|-------|-------------|
|                                                                |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
| n=50                                                           | $\beta$  | 2.738 | -0.095        | 0.568    | 0.791 | 0.316       |
|                                                                | $\gamma$ | 0.501 | -0.002        | 0.005    | 0.071 | 0.143       |
|                                                                | p        | 1.285 | -0.428        | 0.784    | 0.966 | 1.073       |
| n=100                                                          | $\beta$  | 2.670 | -0.068        | 0.312    | 0.584 | 0.234       |
|                                                                | $\gamma$ | 0.496 | 0.007         | 0.003    | 0.054 | 0.107       |
|                                                                | p        | 1.216 | -0.351        | 0.457    | 0.746 | 0.829       |
| n=150                                                          | $\beta$  | 2.627 | -0.051        | 0.167    | 0.427 | 0.171       |
|                                                                | $\gamma$ | 0.496 | 0.006         | 0.002    | 0.048 | 0.097       |
|                                                                | p        | 1.142 | -0.268        | 0.320    | 0.615 | 0.684       |
| n=200                                                          | $\beta$  | 2.627 | -0.051        | 0.167    | 0.427 | 0.171       |
|                                                                | $\gamma$ | 0.499 | 0.002         | 0.002    | 0.043 | 0.087       |
|                                                                | p        | 1.120 | -0.245        | 0.306    | 0.596 | 0.662       |

Table 3 (continued )

|                                                               |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
|---------------------------------------------------------------|----------|-------|---------------|----------|-------|-------------|
| n=250                                                         | $\beta$  | 2.587 | -0.035        | 0.149    | 0.395 | 0.158       |
|                                                               | $\gamma$ | 0.498 | 0.004         | 0.001    | 0.038 | 0.076       |
|                                                               | $p$      | 1.088 | -0.209        | 0.268    | 0.551 | 0.612       |
| n=300                                                         | $\beta$  | 2.578 | -0.031        | 0.133    | 0.373 | 0.149       |
|                                                               | $\gamma$ | 0.498 | 0.004         | 0.001    | 0.035 | 0.071       |
|                                                               | $p$      | 1.076 | -0.196        | 0.245    | 0.525 | 0.584       |
| 2. $\alpha_0 = 0.2, \beta_0 = 2.2, \gamma_0 = 2.8, p_0 = 0.8$ |          |       |               |          |       |             |
|                                                               |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
| n=50                                                          | $\beta$  | 2.529 | -0.149        | 0.305    | 0.625 | 0.292       |
|                                                               | $\gamma$ | 2.719 | 0.029         | 0.211    | 0.466 | 0.167       |
|                                                               | $p$      | 1.461 | -0.0826       | 1.042    | 1.216 | 1.520       |
| n=100                                                         | $\beta$  | 2.428 | -0.104        | 0.149    | 0.449 | 0.204       |
|                                                               | $\gamma$ | 2.720 | 0.028         | 0.151    | 0.397 | 0.142       |
|                                                               | $p$      | 1.294 | -0.618        | 0.707    | 0.975 | 1.219       |
| n=150                                                         | $\beta$  | 2.388 | -0.085        | 0.104    | 0.374 | 0.169       |
|                                                               | $\gamma$ | 2.722 | 0.028         | 0.109    | 0.339 | 0.121       |
|                                                               | $p$      | 1.230 | -0.538        | 0.581    | 0.875 | 1.094       |
| n=200                                                         | $\beta$  | 2.359 | -0.072        | 0.086    | 0.333 | 0.151       |
|                                                               | $\gamma$ | 2.701 | 0.035         | 0.082    | 0.303 | 0.108       |
|                                                               | $p$      | 1.176 | -0.470        | 0.362    | 0.709 | 0.887       |
| n=250                                                         | $\beta$  | 2.342 | -0.065        | 0.076    | 0.311 | 0.141       |
|                                                               | $\gamma$ | 2.739 | 0.022         | 0.081    | 0.292 | 0.104       |
|                                                               | $p$      | 1.125 | -0.406        | 0.399    | 0.711 | 0.888       |
| n=300                                                         | $\beta$  | 2.343 | -0.065        | 0.065    | 0.293 | 0.133       |
|                                                               | $\gamma$ | 2.719 | 0.029         | 0.071    | 0.278 | 0.099       |
|                                                               | $p$      | 1.127 | -0.409        | 0.320    | 0.654 | 0.817       |

Table 3 (continued )

| 3. $\alpha_0 = 1.2, \beta_0 = 2, \gamma_0 = 5, p_0 = 6.5$ |          |        |               |          |        |             |
|-----------------------------------------------------------|----------|--------|---------------|----------|--------|-------------|
|                                                           |          | Mean   | Relative bias | variance | RMSE   | Scaled RMSE |
| n=50                                                      | $\beta$  | 2.353  | -0.176        | 0.428    | 0.744  | 0.372       |
|                                                           | $\gamma$ | 5.134  | -0.026        | 3.262    | 1.811  | 0.362       |
|                                                           | p        | 12.865 | -0.979        | 141.06   | 13.475 | 2.073       |
| n=100                                                     | $\beta$  | 2.237  | -0.118        | 0.267    | 0.569  | 0.284       |
|                                                           | $\gamma$ | 4.984  | 0.033         | 1.414    | 1.189  | 0.238       |
|                                                           | p        | 10.503 | -0.616        | 71.433   | 9.352  | 1.439       |
| n=150                                                     | $\beta$  | 2.158  | -0.079        | 0.195    | 0.469  | 0.235       |
|                                                           | $\gamma$ | 4.953  | 0.009         | 0.903    | 0.952  | 0.190       |
|                                                           | p        | 9.652  | -0.485        | 53.834   | 7.985  | 1.229       |
| n=200                                                     | $\beta$  | 2.141  | -0.070        | 0.167    | 0.433  | 0.216       |
|                                                           | $\gamma$ | 4.987  | 0.003         | 0.691    | 0.832  | 0.166       |
|                                                           | p        | 9.101  | -0.400        | 49.919   | 7.529  | 1.158       |
| n=250                                                     | $\beta$  | 2.084  | -0.042        | 0.126    | 0.364  | 0.182       |
|                                                           | $\gamma$ | 5.043  | -0.008        | 0.608    | 0.781  | 0.156       |
|                                                           | p        | 8.212  | -0.263        | 29.810   | 5.722  | 0.880       |
| n=300                                                     | $\beta$  | 2.065  | -0.033        | 0.092    | 0.309  | 0.155       |
|                                                           | $\gamma$ | 5.023  | -0.005        | 0.427    | 0.653  | 0.131       |
|                                                           | p        | 7.803  | -0.201        | 18.784   | 4.525  | 0.696       |

Table 3 (continued )

| 4. $\alpha_0 = 1.5, \beta_0 = 3.5, \gamma_0 = 5, p_0 = 7$ |          |        |               |          |              |             |
|-----------------------------------------------------------|----------|--------|---------------|----------|--------------|-------------|
|                                                           |          | Mean   | Relative bias | variance | RMSE         | Scaled RMSE |
| n=50                                                      | $\beta$  | 4.067  | -0.162        | 1.528    | 1.360        | 0.388       |
|                                                           | $\gamma$ | 4.972  | 0.005         | 2.854    | 1.689        | 0.338       |
|                                                           | p        | 12.800 | -0.828        | 93.549   | 11.278,1.611 |             |
| n=100                                                     | $\beta$  | 3.848  | -0.099        | 0.451    | 0.756        | 0.216       |
|                                                           | $\gamma$ | 4.878  | 0.024         | 1.212    | 1.107        | 0.222       |
|                                                           | p        | 11.407 | -0.629        | 53.752   | 8.554        | 1.222       |
| n=150                                                     | $\beta$  | 3.714  | -0.061        | 0.298    | 0.586        | 0.167       |
|                                                           | $\gamma$ | 4.951  | 0.009         | 0.872    | 0.935        | 0.187       |
|                                                           | p        | 9.507  | -0.358        | 27.697   | 5.829        | 0.833       |
| n=200                                                     | $\beta$  | 3.692  | -0.055        | 0.232    | 0.518        | 0.148       |
|                                                           | $\gamma$ | 4.894  | 0.021         | 0.725    | 0.857        | 0.172       |
|                                                           | p        | 9.726  | -0.389        | 29.232   | 6.055        | 0.865       |
| n=250                                                     | $\beta$  | 3.651  | -0.043        | 0.166    | 0.436        | 0.125       |
|                                                           | $\gamma$ | 4.941  | 0.012         | 0.567    | 0.756        | 0.151       |
|                                                           | p        | 8.947  | -0.278        | 17.054   | 4.566        | 0.652       |
| n=300                                                     | $\beta$  | 3.625  | -0.036        | 0.142    | 0.397        | 0.113       |
|                                                           | $\gamma$ | 4.956  | 0.008         | 0.484    | 0.697        | 0.139       |
|                                                           | p        | 8.723  | -0.246        | 15.366   | 4.282        | 0.612       |

Table 3 (continued )

| 5. $\alpha_0 = 0.8, \beta_0 = 1.5, \gamma_0 = 4.5, p_0 = 2.5$ |          |       |               |          |       |             |
|---------------------------------------------------------------|----------|-------|---------------|----------|-------|-------------|
|                                                               |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
| n=50                                                          | $\beta$  | 1.836 | -0.224        | 0.355    | 0.684 | 0.456       |
|                                                               | $\gamma$ | 4.322 | 0.039         | 2.111    | 1.464 | 0.325       |
|                                                               | p        | 5.882 | -1.353        | 36.172   | 6.900 | 2.760       |
| n=100                                                         | $\beta$  | 1.768 | -0.179        | 0.243    | 0.561 | 0.374       |
|                                                               | $\gamma$ | 4.274 | 0.050         | 1.103    | 1.075 | 0.239       |
|                                                               | p        | 5.228 | -1.091        | 32.090   | 6.287 | 2.515       |
| n=150                                                         | $\beta$  | 1.669 | -0.113        | 0.155    | 0.428 | 0.286       |
|                                                               | $\gamma$ | 4.389 | 0.025         | 0.741    | 0.868 | 0.193       |
|                                                               | p        | 4.167 | -0.667        | 12.798   | 3.946 | 1.578       |
| n=200                                                         | $\beta$  | 1.632 | -0.088        | 0.116    | 0.365 | 0.243       |
|                                                               | $\gamma$ | 4.399 | 0.023         | 0.566    | 0.759 | 0.169       |
|                                                               | p        | 3.639 | -0.456        | 7.757    | 3.009 | 1.204       |
| n=250                                                         | $\beta$  | 1.606 | -0.071        | 0.093    | 0.323 | 0.215       |
|                                                               | $\gamma$ | 4.402 | 0.022         | 0.444    | 0.674 | 0.149       |
|                                                               | p        | 3.343 | -0.337        | 4.943    | 2.378 | 0.951       |
| n=300                                                         | $\beta$  | 1.589 | -0.059        | 0.076    | 0.289 | 0.193       |
|                                                               | $\gamma$ | 4.433 | 0.015         | 0.369    | 0.612 | 0.136       |
|                                                               | p        | 3.246 | -0.298        | 4.161    | 2.172 | 0.869       |

Table 3 (continued )

| 6. $\alpha_0 = 0.7, \beta_0 = 3.5, \gamma_0 = 4, p_0 = 1.5$ |          |       |               |          |       |             |
|-------------------------------------------------------------|----------|-------|---------------|----------|-------|-------------|
|                                                             |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
| n=50                                                        | $\beta$  | 3.979 | -0.137        | 0.950    | 1.086 | 0.310       |
|                                                             | $\gamma$ | 3.916 | 0.021         | 1.086    | 1.045 | 0.261       |
|                                                             | p        | 3.051 | -1.034        | 8.104    | 3.242 | 2.161       |
| n=100                                                       | $\beta$  | 3.798 | -0.085        | 0.429    | 0.720 | 0.206       |
|                                                             | $\gamma$ | 3.853 | 0.037         | 0.600    | 0.789 | 0.197       |
|                                                             | p        | 2.554 | -0.703        | 4.171    | 2.298 | 1.532       |
| n=150                                                       | $\beta$  | 3.716 | -0.062        | 0.266    | 0.559 | 0.159       |
|                                                             | $\gamma$ | 3.861 | 0.035         | 0.382    | 0.633 | 0.158       |
|                                                             | p        | 2.317 | -0.545        | 2.186    | 1.689 | 1.126       |
| n=200                                                       | $\beta$  | 3.678 | -0.051        | 0.191    | 0.472 | 0.135       |
|                                                             | $\gamma$ | 3.925 | 0.018         | 0.320    | 0.571 | 0.143       |
|                                                             | p        | 2.090 | -0.394        | 1.530    | 1.371 | 0.914       |
| n=250                                                       | $\beta$  | 3.659 | -0.045        | 0.155    | 0.425 | 0.121       |
|                                                             | $\gamma$ | 3.944 | 0.014         | 0.277    | 0.529 | 0.132       |
|                                                             | p        | 1.996 | -0.331        | 1.313    | 1.249 | 0.832       |
| n=300                                                       | $\beta$  | 3.612 | -0.032        | 0.128    | 0.375 | 0.107       |
|                                                             | $\gamma$ | 3.911 | 0.022         | 0.235    | 0.493 | 0.123       |
|                                                             | p        | 1.942 | -0.295        | 0.959    | 1.075 | 0.717       |

Table 4: MLE's and mean square error of four parameters ( $\alpha, \beta, \gamma$  and  $p$ ) in the case of (P-A-L)-MWD

| 1. $\alpha_0 = 1.25, \beta_0 = 2.5, \gamma_0 = 0.5, p_0 = 0.9$ |          |       |               |          |        |             |
|----------------------------------------------------------------|----------|-------|---------------|----------|--------|-------------|
|                                                                |          | Mean  | Relative bias | variance | RMSE   | Scaled RMSE |
| n=50                                                           | $\alpha$ | 1.745 | -0.396        | 1.321    | 1.252  | 1.001       |
|                                                                | $\beta$  | 2.382 | 0.047         | 1.271    | 1.133  | 0.453       |
|                                                                | $\gamma$ | 0.488 | 0.024         | 0.009    | 0.099  | 0.198       |
|                                                                | $p$      | 2.654 | -1.949        | 1234.48  | 35.178 | 39.087      |
| n=100                                                          | $\alpha$ | 1.545 | -0.236        | 0.750    | 0.915  | 0.732       |
|                                                                | $\beta$  | 2.415 | 0.033         | 0.904    | 0.954  | 0.381       |
|                                                                | $\gamma$ | 0.485 | 0.029         | 0.004    | 0.068  | 0.136       |
|                                                                | $p$      | 1.217 | -0.352        | 0.786    | 0.941  | 1.046       |
| n=150                                                          | $\alpha$ | 1.436 | -0.149        | 0.712    | 0.864  | 0.691       |
|                                                                | $\beta$  | 2.439 | 0.024         | 0.959    | 0.981  | 0.392       |
|                                                                | $\gamma$ | 0.491 | 0.016         | 0.003    | 0.057  | 0.115       |
|                                                                | $p$      | 1.179 | -0.310        | 0.848    | 0.962  | 1.069       |
| n=200                                                          | $\alpha$ | 1.304 | -0.043        | 4.706    | 2.170  | 1.736       |
|                                                                | $\beta$  | 2.589 | -0.035        | 5.201    | 2.282  | 0.912       |
|                                                                | $\gamma$ | 0.490 | 0.019         | 0.004    | 0.0065 | 0.131       |
|                                                                | $p$      | 1.184 | -0.315        | 0.903    | 0.992  | 1.102       |
| n=250                                                          | $\alpha$ | 1.343 | -0.075        | 0.489    | 0.706  | 0.564       |
|                                                                | $\beta$  | 2.478 | 0.008         | 0.747    | 0.865  | 0.346       |
|                                                                | $\gamma$ | 0.490 | 0.019         | 0.001    | 0.045  | 0.090       |
|                                                                | $p$      | 1.114 | -0.237        | 0.511    | 0.746  | 0.829       |
| n=300                                                          | $\alpha$ | 1.317 | -0.053        | 0.480    | 0.696  | 0.556       |
|                                                                | $\beta$  | 2.495 | -0.001        | 0.731    | 0.855  | 0.342       |
|                                                                | $\gamma$ | 0.494 | 0.010         | 0.002    | 0.043  | 0.086       |
|                                                                | $p$      | 1.089 | -0.210        | 0.462    | 0.706  | 0.784       |

Table 4 (continued )

| 2. $\alpha_0 = 0.2, \beta_0 = 2.2, \gamma_0 = 2.8, p_0 = 0.8$ |          |       |               |          |        |             |
|---------------------------------------------------------------|----------|-------|---------------|----------|--------|-------------|
|                                                               |          | Mean  | Relative bias | variance | RMSE   | Scaled RMSE |
| n=50                                                          | $\alpha$ | 0.389 | -0.948        | 0.251    | 0.536  | 2.679       |
|                                                               | $\beta$  | 2.304 | -0.047        | 0.352    | 0.602  | 0.274       |
|                                                               | $\gamma$ | 2.958 | -0.056        | 0.417    | 0.665  | 0.237       |
|                                                               | $p$      | 1.408 | -0.760        | 7.616    | 2.679  | 3.533       |
| n=100                                                         | $\alpha$ | 0.263 | -0.316        | 0.053    | 0.238  | 1.193       |
|                                                               | $\beta$  | 2.284 | -0.038        | 0.139    | 0.382  | 0.174       |
|                                                               | $\gamma$ | 2.903 | -0.037        | 0.285    | 0.543  | 0.194       |
|                                                               | $p$      | 0.958 | -0.197        | 0.359    | 0.533  | 0.666       |
| n=150                                                         | $\alpha$ | 0.232 | -0.163        | 0.018    | 0.139  | 0.695       |
|                                                               | $\beta$  | 2.257 | 0.026         | 0.073    | 0.276  | 0.125       |
|                                                               | $\gamma$ | 2.831 | -0.011        | 0.094    | 0.308  | 0.110       |
|                                                               | $p$      | 0.940 | -0.175        | 0.152    | 0.414  | 0.517       |
| n=200                                                         | $\alpha$ | 0.219 | -0.098        | 0.013    | 0.116  | 0.584       |
|                                                               | $\beta$  | 2.255 | -0.025        | 0.054    | 0.0239 | 0.108       |
|                                                               | $\gamma$ | 2.808 | -0.002        | 0.073    | 0.271  | 0.096       |
|                                                               | $p$      | 0.930 | -0.163        | 0.120    | 0.370  | 0.463       |
| n=250                                                         | $\alpha$ | 0.215 | -0.079        | 0.010    | 0.101  | 0.507       |
|                                                               | $\beta$  | 2.274 | -0.033        | 0.051    | 0.237  | 0.108       |
|                                                               | $\gamma$ | 2.789 | 0.003         | 0.055    | 0.236  | 0.084       |
|                                                               | $p$      | 0.951 | -0.189        | 0.143    | 0.407  | 0.509       |

Table 4 (continued )

| 3. $\alpha_0 = 1.2, \beta_0 = 2, \gamma_0 = 5, p_0 = 6.5$ |          |       |               |          |       |             |
|-----------------------------------------------------------|----------|-------|---------------|----------|-------|-------------|
|                                                           |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
| n=50                                                      | $\alpha$ | 1.286 | -0.072        | 0.313    | 0.566 | 0.472       |
|                                                           | $\beta$  | 2.078 | -0.039        | 0.324    | 0.574 | 0.287       |
|                                                           | $\gamma$ | 5.710 | -0.142        | 4.127    | 2.152 | 0.430       |
|                                                           | $p$      | 8.412 | -0.294        | 30.485   | 5.843 | 0.898       |
| n=100                                                     | $\alpha$ | 1.222 | -0.018        | 0.131    | 0.362 | 0.302       |
|                                                           | $\beta$  | 2.058 | -0.028        | 0.152    | 0.394 | 0.197       |
|                                                           | $\gamma$ | 5.263 | -0.053        | 1.237    | 1.143 | 0.228       |
|                                                           | $p$      | 7.522 | -0.157        | 16.419   | 4.179 | 0.643       |
| n=150                                                     | $\alpha$ | 1.195 | 0.004         | 0.076    | 0.275 | 0.229       |
|                                                           | $\beta$  | 2.049 | -0.025        | 0.107    | 0.330 | 0.165       |
|                                                           | $\gamma$ | 5.150 | -0.030        | 0.706    | 0.854 | 0.171       |
|                                                           | $p$      | 7.178 | -0.104        | 12.110   | 3.546 | 0.545       |
| n=200                                                     | $\alpha$ | 1.190 | 0.008         | 0.052    | 0.229 | 0.191       |
|                                                           | $\beta$  | 2.022 | -0.011        | 0.076    | 0.277 | 0.139       |
|                                                           | $\gamma$ | 5.125 | -0.025        | 0.484    | 0.706 | 0.141       |
|                                                           | $p$      | 6.943 | -0.068        | 6.926    | 2.668 | 0.411       |
| n=250                                                     | $\alpha$ | 1.212 | -0.010        | 0.043    | 0.209 | 0.123       |
|                                                           | $\beta$  | 2.023 | -0.011        | 0.060    | 0.246 | 0.123       |
|                                                           | $\gamma$ | 5.094 | -0.018        | 0.422    | 0.657 | 0.131       |
|                                                           | $p$      | 7.040 | -0.083        | 0.6.256  | 2.559 | 0.393       |
| n=300                                                     | $\alpha$ | 1.203 | -0.002        | 0.034    | 0.186 | 0.155       |
|                                                           | $\beta$  | 2.017 | -0.008        | 0.048    | 0.221 | 0.110       |
|                                                           | $\gamma$ | 5.104 | -0.020        | 0.313    | 0.569 | 0.113       |
|                                                           | $p$      | 6.759 | -0.039        | 4.554    | 2.149 | 0.330       |

Table 4 (continued )

| 4. $\alpha_0 = 1.5, \beta_0 = 3.5, \gamma_0 = 5, p_0 = 7$ |          |       |               |          |       |             |
|-----------------------------------------------------------|----------|-------|---------------|----------|-------|-------------|
|                                                           |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
| n=50                                                      | $\alpha$ | 1.643 | -0.095        | 0.402    | 0.649 | 0.433       |
|                                                           | $\beta$  | 3.837 | -0.096        | 1.165    | 1.131 | 0.323       |
|                                                           | $\gamma$ | 5.747 | -0.149        | 3.624    | 2.045 | 0.409       |
|                                                           | $p$      | 9.398 | -0.343        | 29.794   | 5.962 | 0.852       |
| n=100                                                     | $\alpha$ | 1.850 | -0.053        | 0.204    | 0.459 | 0.306       |
|                                                           | $\beta$  | 3.689 | -0.054        | 0.420    | 0.675 | 0.193       |
|                                                           | $\gamma$ | 5.277 | -0.055        | 1.177    | 1.119 | 0.224       |
|                                                           | $p$      | 8.740 | -0.248        | 18.488   | 4.634 | 0.662       |
| n=150                                                     | $\alpha$ | 1.561 | -0.041        | 0.126    | 0.360 | 0.240       |
|                                                           | $\beta$  | 3.656 | -0.044        | 0.263    | 0.536 | 0.153       |
|                                                           | $\gamma$ | 5.136 | -0.027        | 0.263    | 0.536 | 0.153       |
|                                                           | $p$      | 8.401 | -0.200        | 10.928   | 3.590 | 0.513       |
| n=200                                                     | $\alpha$ | 1.304 | -0.043        | 4.706    | 2.170 | 1.736       |
|                                                           | $\beta$  | 2.589 | -0.035        | 5.201    | 2.282 | 0.912       |
|                                                           | $\gamma$ | 0.490 | 0.019         | 0.004    | 0.065 | 0.131       |
|                                                           | $p$      | 1.184 | -0.315        | 0.903    | 0.992 | 1.102       |
| n=250                                                     | $\alpha$ | 1.343 | -0.075        | 0.0489   | 0.706 | 0.564       |
|                                                           | $\beta$  | 2.478 | 0.008         | 0.747    | 0.865 | 0.346       |
|                                                           | $\gamma$ | 0.490 | 0.019         | 0.001    | 0.045 | 0.090       |
|                                                           | $p$      | 1.114 | -0.237        | 0.511    | 0.746 | 0.829       |
| n=300                                                     | $\alpha$ | 1.317 | -0.053        | 0.480    | 0.696 | 0.556       |
|                                                           | $\beta$  | 2.495 | -0.001        | 0.731    | 0.855 | 0.342       |
|                                                           | $\gamma$ | 0.494 | 0.010         | 0.002    | 0.043 | 0.086       |
|                                                           | $p$      | 1.089 | -0.210        | 0.462    | 0.706 | 0.784       |

Table 4 (continued )

| 5. $\alpha_0 = 0.8, \beta_0 = 1.5, \gamma_0 = 4.5, p_0 = 2.5$ |          |       |               |          |       |             |
|---------------------------------------------------------------|----------|-------|---------------|----------|-------|-------------|
|                                                               |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
| n=50                                                          | $\alpha$ | 0.933 | -0.166        | 0.212    | 0.479 | 0.598       |
|                                                               | $\beta$  | 1.519 | -0.013        | 0.196    | 0.443 | 0.295       |
|                                                               | $\gamma$ | 4.989 | -0.108        | 2.046    | 1.512 | 0.335       |
|                                                               | $p$      | 3.532 | -0.412        | 5.242    | 2.511 | 1.004       |
| n=100                                                         | $\alpha$ | 0.853 | -0.067        | 0.101    | 0.323 | 0.403       |
|                                                               | $\beta$  | 1.537 | -0.025        | 0.082    | 0.290 | 0.194       |
|                                                               | $\gamma$ | 4.676 | -0.039        | 0.689    | 0.848 | 0.188       |
|                                                               | $p$      | 3.178 | -0.271        | 3.193    | 1.912 | 0.765       |
| n=150                                                         | $\alpha$ | 0.848 | -0.016        | 0.074    | 0.277 | 0.346       |
|                                                               | $\beta$  | 1.516 | -0.011        | 0.046    | 0.215 | 0.143       |
|                                                               | $\gamma$ | 4.621 | -0.268        | 0.407    | 0.649 | 0.144       |
|                                                               | $p$      | 3.035 | -0.214        | 2.122    | 1.552 | 0.621       |
| n=200                                                         | $\alpha$ | 0.853 | -0.066        | 0.052    | 0.235 | 0.294       |
|                                                               | $\beta$  | 1.533 | -0.023        | 0.038    | 0.199 | 0.133       |
|                                                               | $\gamma$ | 4.563 | -0.014        | 0.284    | 0.536 | 0.119       |
|                                                               | $p$      | 3.062 | -0.225        | 1.776    | 1.446 | 0.578       |
| n=250                                                         | $\alpha$ | 0.839 | -0.049        | 0.043    | 0.212 | 0.265       |
|                                                               | $\beta$  | 1.523 | -0.015        | 0.033    | 0.183 | 0.122       |
|                                                               | $\gamma$ | 4.540 | -0.008        | 0.218    | 0.469 | 0.104       |
|                                                               | $p$      | 2.949 | -0.179        | 1.396    | 1.264 | 0.506       |
| n=300                                                         | $\alpha$ | 0.835 | -0.044        | 0.039    | 0.202 | 0.253       |
|                                                               | $\beta$  | 1.527 | -0.018        | 0.025    | 0.162 | 0.108       |
|                                                               | $\gamma$ | 4.539 | -0.008        | 0.199    | 0.448 | 0.099       |
|                                                               | $p$      | 2.922 | -0.168        | 1.265    | 1.201 | 0.480       |

Table 4 (continued )

| 6. $\alpha_0 = 0.7, \beta_0 = 3.5, \gamma_0 = 4, p_0 = 1.5$ |          |       |               |          |       |             |
|-------------------------------------------------------------|----------|-------|---------------|----------|-------|-------------|
|                                                             |          | Mean  | Relative bias | variance | RMSE  | Scaled RMSE |
| n=50                                                        | $\alpha$ | 0.861 | -0.230        | 0.276    | 0.549 | 0.785       |
|                                                             | $\beta$  | 3.853 | -0.101        | 1.099    | 1.106 | 0.316       |
|                                                             | $\gamma$ | 4.391 | -0.097        | 1.759    | 1.382 | 0.346       |
|                                                             | $p$      | 2.213 | -0.475        | 2.171    | 1.637 | 1.091       |
| n=100                                                       | $\alpha$ | 0.765 | -0.093        | 0.105    | 0.330 | 0.472       |
|                                                             | $\beta$  | 3.649 | -0.043        | 0.344    | 0.605 | 0.173       |
|                                                             | $\gamma$ | 4.135 | -0.034        | 0.411    | 0.655 | 0.164       |
|                                                             | $p$      | 1.871 | -0.247        | 0.881    | 1.009 | 0.673       |
| n=150                                                       | $\alpha$ | 0.762 | -0.089        | 0.066    | 0.264 | 0.378       |
|                                                             | $\beta$  | 3.655 | -0.044        | 0.219    | 0.494 | 0.141       |
|                                                             | $\gamma$ | 4.035 | -0.008        | 0.247    | 0.498 | 0.125       |
|                                                             | $p$      | 1.884 | -0.256        | 0.889    | 1.018 | 0.678       |
| n=200                                                       | $\alpha$ | 0.750 | -0.072        | 0.051    | 0.232 | 0.331       |
|                                                             | $\beta$  | 3.605 | -0.030        | 0.164    | 0.419 | 0.119       |
|                                                             | $\gamma$ | 4.054 | -0.013        | 0.187    | 0.436 | 0.109       |
|                                                             | $p$      | 1.816 | -0.211        | 0.617    | 0.847 | 0.565       |
| n=250                                                       | $\alpha$ | 0.743 | -0.062        | 0.042    | 0.209 | 0.299       |
|                                                             | $\beta$  | 3.587 | -0.025        | 0.144    | 0.389 | 0.111       |
|                                                             | $\gamma$ | 4.039 | -0.009        | 0.146    | 0.384 | 0.096       |
|                                                             | $p$      | 1.754 | -0.169        | 0.429    | 0.703 | 0.469       |
| n=300                                                       | $\alpha$ | 0.749 | -0.071        | 0.037    | 0.200 | 0.286       |
|                                                             | $\beta$  | 3.582 | -0.024        | 0.121    | 0.358 | 0.102       |
|                                                             | $\gamma$ | 3.992 | 0.002         | 0.120    | 0.347 | 0.087       |
|                                                             | $p$      | 1.800 | -0.201        | 0.458    | 0.741 | 0.494       |

Table 5: For goodness of fit test

| Class intervals | $O_i$ | $E_i$  | $\frac{(O_i - E_i)^2}{E_i}$ |
|-----------------|-------|--------|-----------------------------|
| 0-              | 6     | 3.977  | 1.028                       |
| 0.4-            | 5     | 4.985  | 0.00004                     |
| 1-              | 9     | 15.223 | 2.5438                      |
| 4-              | 7     | 5.821  | 0.2388                      |
| 6-              | 8     | 4.159  | 3.547                       |
| 8-              | 5     | 5.501  | 0.0455                      |
| 12-             | 5     | 4.585  | 0.0376                      |
| 18-49           | 5     | 5.384  | 0.0275                      |
| Total           |       |        | 7.468                       |

# Estimation of Distribution Function based on Ranked Set Sampling: Missing Data Approach

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## Abstract

In many studies, it is a well-established fact that estimation under either ranked set sample (RSS) or its variations is much better than other popular sampling techniques. In this article, missing data approach is adopted to present new estimators for distribution function under RSS setup. The consistency of the suggested estimators is also discussed. It merges from Monte Carlo simulations that the proposed estimators have a good performance as compared with their competitors. An empirical data set is also used for illustrative purposes.

**Keywords:** Cumulative distribution function; Concomitant variable; EM algorithm; Ranked set sample.

## 1. Introduction and Preliminaries

Ranked set sample (RSS) introduced by McIntyre (1952) is a cost-efficient sampling technique can be well implemented in the situations where ranking the sampling items is much easier than their actual quantification. The philosophy behind RSS is to use the ranking information for selecting the experimental units which are nearly spanning the full range of the interested variable's values  $Y$ . The ranking process can be proceeded by eye inspection or using an auxiliary variable frequently named as a concomitant variable  $X$  which is inexpensive to be measured. Here RSS based on  $X$  is adopted whose steps can be described as follows.

First, draw  $k$  random samples each of size  $k$  of the bivariate variables  $(X, Y)$ . Second, for each sample,  $X$  values are fully measured and sorted in ascending way. Then, measure

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only the  $Y$  values associated with the  $i^{th}$  smallest observation of  $X$  corresponding to the  $i^{th}$  sample ( $i = 1 \dots k$ ). Finally, repeat the preceding steps  $m$  times (cycles) in order to obtain  $n = km$  values of  $Y$ . If the number of the selected items across the cycles equals, RSS is hence called balanced RSS. Otherwise, it is called unbalanced RSS. It should be emphasized that we will confine ourselves that  $(X, Y)$  are continuous variables drawn through balanced RSS. To clarify the notation, let  $x_{i(j(i))}$  be the  $i^{th}$  smallest observation from  $j^{th}$  sample corresponding to  $l^{th}$  cycle. And also, let  $y_{i(i|l)}$  be the interested variable's values associated with  $x_{i(j(i))}$  ( $(i, j) = 1 \dots k, l = 1 \dots m$ ). Further, Perfect (Imperfect) ranking refers that the rankings are done without (with) errors denoted by rounded (squared) bracket.

Nonparametric estimation methods are becoming increasingly adopted in several statistical aspects. This is caused by either the population distribution is most likely unknown and needs to be correctly specified or a slight contamination of the observations might lead to erroneous results. Therefore, it is required to estimate the interested characteristics about the population in a more direct way without depending on any distributional assumption or prior information.

Literature on RSS, mainly in nonparametric estimation, has rapidly grown and several estimators have been provided. These Publications appear generally to be diverging into two classes. The first direction, which is the popular, relies completely on the exact measured items generated by RSS during operating the estimation process. Zamanzade and Mahdizadeh (2018a) and Ozturk (2018) are recent examples articles in this class. Whilst the second class considers the unmeasured items of  $Y$  as missing values to be estimated, then hence using the measured as well as the unmeasured items for estimation purposes. Zamanzade and Vock (2015) and Zamanzade and Mohammadi (2016) are famous examples published in this class. In this study, Henceforth the second approach will be adopted for estimating the population cumulative distribution function (CDF).

CDF estimation occupied a special position in a nonparametric analysis because through CDF, one can know a lot about the population properties, such as the skewness, kurtosis, entropy...etc. Generally speaking, there are two types of CDF under RSS setup. The first one is the CDF within each artificial stratum known as in-stratum CDF  $F_{[i]}(t)$ .

Whilst, the other type is the CDF for the entire sample called overall CDF  $F(t)$ . It is important to note that RSS is called a consistent process if:

$$F(t) = \frac{1}{k} \sum_{i=1}^k F_{[i]}(t).$$

In a fundamental article, Stokes and Sager (1988) suggested to estimate  $F(t)$  under RSS by the following estimator:

$$\hat{F}_S(t) = \frac{1}{n} \sum_{i=1}^k \sum_{l=1}^m I(y_{li[i]} < t),$$

where  $I(\cdot)$  is the indicator function. The authors have shown analytically that  $\hat{F}_S(t)$  is an unbiased estimator for  $F(t)$  with its efficiency greater than or equal unity in comparison to its analog in simple random sample (SRS) regardless the ranking errors. Additionally, they examined that the simulated critical values of Kolmogorov-Smirnov (KS) based on  $\hat{F}_S(t)$  is uniformly smaller than those under SRS set up. More discussion about the properties of  $\hat{F}_S(t)$  can be found in Duembgen and Zamanzade (2018). Naturally, the in-stratum CDF associated to  $\hat{F}_S(t)$  can be expressed as:

$$\hat{F}_{S[i]}^*(t) = \frac{1}{m} \sum_{l=1}^m I(y_{li[i]} < t).$$

Ozturk (2007) argued that  $\hat{F}_{S[i]}^*(t)$  cannot be always dependable as  $\hat{F}_{S[i]}^*(t)$  might violate a sensible condition named as stochastic order (SO) constrain which states that:

$$\hat{F}_{S[1]}^*(t) \geq \hat{F}_{S[2]}^*(t) \geq \dots \geq \hat{F}_{S[k]}^*(t).$$

Alternatively the author suggested to use the isotonized values of  $\hat{F}_{S[i]}^*(t)$  generated by pool-adjacent-violators algorithm (PAVA) denoted by  $\hat{F}_{S[i]}(t)$ . For more explanation about PAVA, see Silvapulle and Sen (2005). It is important to indicate that  $\hat{F}_S(t)$  is not absolutely affected by whichever either  $\hat{F}_{S[i]}^*(t)$  or  $\hat{F}_{S[i]}(t)$  is adopted under balanced RSS. In more precise,  $\hat{F}_S(t)$  can be rewritten as:

$$\hat{F}_S(t) = \frac{1}{k} \sum_{i=1}^k \hat{F}_{S[i]}^*(t) = \frac{1}{k} \sum_{i=1}^k \hat{F}_{S[i]}(t).$$

Additionally, Amiri et al. (2017) proved that  $\hat{F}_{S[i]}^*(t)$  is a consistent estimator for  $B_{i,k-i+1}(F(t))$  under the perfect ranking assumption and unbalanced RSS, where  $B_{a,b}(t)$  is the CDF of the Beta distribution with parameters  $a$  and  $b$  at the point  $t$

The main motivation behind this work is to incorporate the information supported by the unmeasured items for suggesting new overall and in-stratum CDF estimators. The remainder of this manuscript is structured as follows. Section 2 discusses the estimation of the CDF based on iterative EM algorithm. Section 3 illustrates the importance of the concomitant variable for CDF estimation. Section 4 exhibits our proposed CDF estimators. The numerical comparisons using simulated and empirical data are summarized and organized respectively in Section 5 and 6, and lastly Section 7 shows our concluding points and some possible extension works. Proofs are postponed to an appendix.

## 2. EM algorithm-based CDF Estimators

Kvam and Samaniego (1994) was the first who discussed the idea of using the CDF estimator under RSS based on missing data approach. Their idea is to use the ranking information supported by the measured units for estimating the expected count of the unmeasured items less than  $t$  under the perfectness situation. For instance at a certain  $l$ , if we observed that the observation  $y_{li[i]}$  exactly is less than or equal  $t$ , then it is reasonable to think that:

$$p(y_{li[j]} \leq t) = 1 \quad \forall j = 1 \dots i - 1, \quad (1)$$

whilst for the remaining items:

$$p(y_{li[j]} \leq t | y_{li[j]} > y_{li[i]}) = \frac{\int_{y_{li[i]}}^t f(y) dy}{\int_{y_{li[i]}}^{\infty} f(y) dy} = \frac{F(t) - F(y_{li[i]})}{1 - F(y_{li[i]})} \quad (2)$$

$$\forall j = i + 1 \dots k,$$

By combining (1) and (2), we can get:

$$F_1(t | y_{li[i]}, F) = i + (k - i) \frac{F(t) - F(y_{li[i]})}{1 - F(y_{li[i]})}, \quad (3)$$

where  $f(y)$  and  $F(y)$  are respectively the parent pdf and CDF of  $Y$ . In contradiction, if we observed that the observation  $y_{li[i]}$  is actually greater than  $t$ , then it is logically to claim that:

$$p(y_{li[j]} \leq t) = 0 \quad \forall j = i + 1 \dots k, \quad (4)$$

but for the remaining items

$$p(y_{li[j]} \leq t | y_{li[j]} < y_{li[i]}) = \frac{\int_{-\infty}^t f(y) dy}{\int_{-\infty}^{y_{li[i]}} f(y) dy} = \frac{F(t)}{F(y_{li[i]})} \quad (5)$$

$$\forall j = 1 \dots i - 1.$$

By combining (4) and (5), we can get:

$$F_2(t | y_{li[i]}, F) = (i - 1) \frac{F(t)}{F(y_{li[i]})} \quad (6)$$

By applying (3) and (6) to the entire RSS items  $\{y_{li[i]}, i = 1 \dots k; l = 1 \dots m\}$ , this enabled the authors to suggest a new CDF estimator given by:

$$\hat{F}_K^*(t) = \frac{1}{nk} \sum_{l=1}^m \sum_{i=1}^k \left( (F_1(t | y_{li[i]}, F)) I(y_{li[i]} \leq t) + (F_2(t | y_{li[i]}, F)) I(y_{li[i]} > t) \right). \quad (7)$$

It is worth observing that (2) and (5) are respectively the lower and the upper truncated CDF of  $y_{li[j]}$ . The authors stated that  $\hat{F}_K^*(t)$  is not estimable function as it simply depends on  $F(t)$ . Consequently, the authors claimed that the EM algorithm supported by Dempster et al. (1977) is convenience for solving this dilemma. The idea behind using EM algorithm is to use the expectation step to estimate the missing values in the sample, then applying the maximization step that updates these missing values. These steps can be explained in some details as follows:

- 1- Let  $\hat{F}_K^{*(0)}(t)$  be  $\hat{F}_K^*(t)$  after estimating the unknown CDFs in (7) with  $\hat{F}_S(t)$ .
- 2- Set  $r = 0$ .
- 3- Apply (7) to all the sampling items, i.e. compute the following equation:

$$\hat{F}_K^{*(r+1)}(y_{\mathcal{L}\ell[\ell]}) = \frac{1}{nk} \sum_{l=1}^m \sum_{i=1}^k \left( \left( F_1(y_{\mathcal{L}\ell[\ell]} | y_{li[i]}, \hat{F}_K^{*(r)}) \right) I(y_{li[i]} \leq y_{\mathcal{L}\ell[\ell]}) \right. \\ \left. + \left( F_2(y_{\mathcal{L}\ell[\ell]} | y_{li[i]}, \hat{F}_K^{*(r)}) \right) I(y_{li[i]} > y_{\mathcal{L}\ell[\ell]}) \right).$$

$$\mathcal{L} = 1 \dots m \text{ and } \ell = 1 \dots k.$$

4- Set  $r = r + 1$ .

5- Repeat steps (3-4) until stopping rule satisfies. i.e.

$$\max \left( \left| \hat{F}_K^{*(r)}(y_{\mathcal{L}\ell[\ell]}) - \hat{F}_K^{*(r+1)}(y_{\mathcal{L}\ell[\ell]}) \right| \right) \leq .001.$$

6- Finally the suggested estimator denoted by  $\hat{F}_K(t)$  given by:

$$\hat{F}_K(t) = \frac{1}{nk} \sum_{l=1}^m \sum_{i=1}^k \left( \left( i + (k-i) \frac{\hat{F}_S(t) - \hat{F}_K^{*(r)}(y_{li[i]})}{1 - \hat{F}_K^{*(r)}(y_{li[i]})} \right) I(y_{li[i]} \leq t) + \left( (i-1) \frac{\hat{F}_S(t)}{\hat{F}_K^{*(r)}(y_{li[i]})} \right) I(y_{li[i]} > t) \right).$$

Kvam and Samaniego (1994) examined analytically the properties of  $\hat{F}_K(t)$  and proved that the suggested algorithm converges to a unique consistent solution as long as the initial CDF value generated by a consistent estimator. Further, they make a numerical comparison study between  $\hat{F}_S(t)$  and  $\hat{F}_K(t)$  concluding the superiority of the latter estimator. Additionally, they exhibited that the KS statistics based on  $\hat{F}_K(t)$  has uniformly smaller critical values than its analogue based on  $\hat{F}_S(t)$ , which refers that the goodness of fit tests based on  $\hat{F}_K(t)$  is expectedly more powerful than those based on  $\hat{F}_S(t)$ . It is interesting to remark that we have numerically investigated that the above algorithm is not sensitive to the seed estimates used in step 1. Therefore adopting any consistent CDF estimator addressed in the literature, this finally would lead to the same result. It should be also indicated that Ozturk (2010) used also EM algorithm to submit a modified version of  $\hat{F}_K(t)$ . Unfortunately, this algorithm suffers from heavy computations and time consuming (see Frey and Zhang (2018)). Therefore, the estimators addressed in Ozturk (2010) will not considered in this work.

### 3. Concomitant-based CDF Estimators

Since it is well-investigated by several studies from exploiting the concomitant information is raising the precision of the inferential procedures, Zamanzade and Mahdizadeh (2018b), most recently, used the concomitant items to modify  $\hat{F}_S(t)$  for

getting another overall CDF estimator. Their idea is essentially based on the two well-known facts expressed as:

$$E(g(Y)) = E(E(g(Y)|X)) \text{ and } Var(g(Y)) \geq Var(E(g(Y)|X)), \quad (8)$$

where  $g(Y)$  is a proper function such that  $E(g(Y))$  exists. Taking  $g(Y)$  as  $I(Y < t)$  in (8) yielding another overall CDF estimator given by:

$$F_X(t) = E(E(I(Y < t)|X)).$$

In view of (8), one intuitively expects that the plug-in estimator of  $F_X(t)$  has less variance than the plug-in estimator of  $E(I(Y < t))$  which is  $\hat{F}_S(t)$ . In order to obtain the plug-in estimator for  $F_X(t)$  denoted by  $\hat{F}_Z(t)$ , it is required that  $I(Y < t)$  and  $X$  have the same number of observations which does not agree with the RSS strategy. Hence, Zamanzade and Mahdizadeh (2018b) decided to overcome this dilemma by interpolating the unmeasured values of  $I(Y < t)$  using their corresponding values of  $X$ . Of course, these interpolated values affect negatively on the efficiency of  $\hat{F}_Z(t)$ . It is also logically to expect that if the relation between  $Y$  and  $X$  is not strong, the interpolated values will no longer be accurate. Fortunately, Zamanzade and Mahdizadeh (2018b) investigated numerically that  $\hat{F}_Z(t)$  is still reliable even when the rankings are done completely random, i.e.  $Y$  and  $X$  are independent variables, as the efficiency-reduction than  $\hat{F}_S(t)$  is negligible. Without loss of generality, we will assume positive relation between  $Y$  and  $X$ . Hence, the steps of calculating  $\hat{F}_Z(t)$  can be summarized in the following steps.

- 1- Combining  $y_{l(i)}$  and their corresponding values of  $x_{l(i)}$  into two new variables  $(y_z^*, x_z^*, z = 1 \dots n)$  respectively.
- 2- Sorting ascending  $(y_z^*, x_z^*)$  according to  $x^*$  values yielding  $(y_{[z]}^*, x_{(z)}^*)$ .
- 3- Obtain the isotonized values for  $I(y_{[z]}^* \leq t)$  and save these values in  $\hat{F}_z^{iso}(t)$ .
- 4- For each  $x_{l(j(i))}$ , obtain the corresponding  $\hat{F}_z^{iso}(t)$  by utilizing the linear interpolation formula given by:

$$\hat{F}_x(t) = \begin{cases} \hat{F}_1^{iso}(t) & x \leq x_{(1)}^* \\ \hat{F}_z^{iso}(t) + \frac{\hat{F}_{z+1}^{iso}(t) - \hat{F}_z^{iso}(t)}{x_{(z+1)}^* - x_{(z)}^*} [x - x_{(z)}^*] & x_{(z)}^* \leq x < x_{(z+1)}^* \quad z = 1 \dots n-1. \\ \hat{F}_n^{iso}(t) & x \geq x_{(n)}^* \end{cases}$$

5. Lastly,  $\hat{F}_Z(t)$  can be directly get as:

$$\hat{F}_Z(t) = \frac{1}{kn} \sum_{i=1}^k \sum_{j=1}^k \sum_{l=1}^m \hat{F}_{x_{lj(i)}}(t).$$

In the same sequel, Ashour and Abdallah (2018) proposed the corresponding in-stratum CDF estimator to  $\hat{F}_Z(t)$  which can naturally be defined as:

$$\hat{F}_{Z[i]}(t) = \frac{1}{n} \sum_{j=1}^k \sum_{l=1}^m \hat{F}_{x_{lj(i)}}(t).$$

The authors proved the consistency of  $\hat{F}_Z(t)$  as well as  $\hat{F}_{Z[i]}(t)$  and stated that  $\hat{F}_{Z[i]}(t)$  is uniformly satisfying SO condition, further they utilized  $\hat{F}_{Z[i]}(t)$  for producing new perfectness tests.

#### 4. The Proposed CDF Estimators

In this part, two new CDF estimators are proposed using the same ideas explained in the last two sections. Our first estimator consists by just replacing the lower and the upper truncated CDF of  $y_{li[j]}$  expressed respectively in (2) and (5) with the conditional CDF order statistics, (see Arnold et al. (2008)), which can be shown as:

- If  $y_{li[j]} > y_{li[i]}$

$$p(y_{li[j]} \leq t | y_{li[j]} > y_{li[i]}) = \int_{y_{li[i]}}^t f_{j,i}^1(y | y_{li[i]}) dy = B_{j-i, k-j+1} \left( \frac{F(t) - F(y_{li[i]})}{1 - F(y_{li[i]})} \right), \quad (9)$$

- If  $y_{li[j]} < y_{li[i]}$

$$p(y_{li[j]} \leq t | y_{li[j]} < y_{li[i]}) = \int_{-\infty}^t f_{j,i}^2(y | y_{li[i]}) dy = B_{j,i-j} \left( \frac{F(t)}{F(y_{li[i]})} \right) \quad (10)$$

where  $f_{j,i}^1(y | y_{li[i]})$  and  $f_{j,i}^2(y | y_{li[i]})$  are the conditional order statistics pdfs respectively expressed as:

$$f_{j,i}^1(y | w) = \frac{(k-i)!}{(j-i-1)!(k-j)!} \left( \frac{F(y) - F(w)}{1 - F(w)} \right)^{j-i-1} \left( 1 - \frac{F(y) - F(w)}{1 - F(w)} \right)^{k-j} \frac{f(y)}{1 - F(w)} \quad w \leq y \leq \infty,$$

and

$$f_{j,i}^2(y|w) = \frac{(i-1)!}{(j-1)!(i-j-1)!} \left( \frac{F(y)}{F(w)} \right)^{j-1} \left( 1 - \frac{F(y)}{F(w)} \right)^{i-j-1} \frac{f(y)}{F(w)} \quad -\infty \leq y \leq w.$$

By inserting (9) and (10) in (7), another overall CDF estimator can be formulated after slight modifications which is given by:

$$\begin{aligned} \hat{F}_{P1}^*(t) = \frac{1}{nk} \sum_{l=1}^m \left( \sum_{i=1}^k i I(y_{li[i]} \leq t) + \sum_{i=1}^{k-1} \sum_{j=i+1}^k B_{j-i,k-j+1} \left( \frac{F(t) - F(y_{li[i]})}{1 - F(y_{li[i]})} \right) I(y_{li[i]} \leq t) \right. \\ \left. + \sum_{i=2}^k \sum_{j=1}^{i-1} B_{j,i-j} \left( \frac{F(t)}{F(y_{li[i]})} \right) I(y_{li[i]} > t) \right). \end{aligned} \quad (11)$$

Due to the difficulty of straightforwardly implementing  $\hat{F}_{P1}^*(t)$  as  $F(t)$  appears in the right hand side of (11), we also resort to use EM algorithm with the similar manner addressed in Kvam and Samaniego (1994) whose steps can be summarized as:

- 1- Let  $\hat{F}_{P1}^{*(0)}(t)$  be  $\hat{F}_{P1}^*(t)$  after estimating the unknown CDFs in (11) with  $\hat{F}_S(t)$ .
- 2- Set  $r = 0$ .
- 3- Apply (11) to all the sampling items, i.e. compute the following equation:

$$\begin{aligned} \hat{F}_{P1}^{*(r+1)}(y_{\mathcal{L}\ell[\ell]}) = \frac{1}{nk} \sum_{l=1}^m \left( \sum_{i=1}^k i I(y_{li[i]} \leq y_{\mathcal{L}\ell[\ell]}) \right. \\ \left. + \sum_{i=1}^{k-1} \sum_{j=i+1}^k B_{j-i,k-j+1} \left( \frac{\hat{F}_{P1}^{*(r)}(y_{\mathcal{L}\ell[\ell]}) - \hat{F}_{P1}^{*(r)}(y_{li[i]})}{1 - \hat{F}_{P1}^{*(r)}(y_{li[i]})} \right) I(y_{li[i]} \leq y_{\mathcal{L}\ell[\ell]}) \right. \\ \left. + \sum_{i=2}^k \sum_{j=1}^{i-1} B_{j,i-j} \left( \frac{\hat{F}_{P1}^{*(r)}(y_{\mathcal{L}\ell[\ell]})}{\hat{F}_{P1}^{*(r)}(y_{li[i]})} \right) I(y_{li[i]} > y_{\mathcal{L}\ell[\ell]}) \right). \end{aligned}$$

$$\mathcal{L} = 1 \dots m \text{ and } \ell = 1 \dots k.$$

- 4- Set  $r = r + 1$ .
- 5- Repeat steps (3-4) until stopping rule satisfies. i.e.

$$\max \left( \left| \hat{F}_{P1}^{*(r)}(y_{\mathcal{L}\ell[\ell]}) - \hat{F}_{P1}^{*(r+1)}(y_{\mathcal{L}\ell[\ell]}) \right| \right) \leq .001.$$

- 6- Finally the suggested estimator denoted by  $\hat{F}_{P1}(t)$  given by:

$$\begin{aligned}\hat{F}_{P1}(t) = & \frac{1}{nk} \sum_{l=1}^m \left( \sum_{i=1}^k i I(y_{li[i]} \leq t) + \sum_{i=1}^{k-1} \sum_{j=i+1}^k B_{j-i, k-j+1} \left( \frac{\hat{F}_S(t) - \hat{F}_{P1}^{*(r)}(y_{li[i]})}{1 - \hat{F}_{P1}^{*(r)}(y_{li[i]})} \right) I(y_{li[i]} \leq t) \right. \\ & \left. + \sum_{i=2}^k \sum_{j=1}^{i-1} B_{j, i-j} \left( \frac{\hat{F}_S(t)}{\hat{F}_{P1}^{*(r)}(y_{li[i]})} \right) I(y_{li[i]} > t) \right).\end{aligned}$$

The main advantageous of  $\hat{F}_{P1}(t)$  over  $\hat{F}_K(t)$  is it incorporates the order of the unmeasured items  $y_{li[j]}$  during the CDF estimation process which enables us to provide the associated in-stratum CDF estimators formulated as:

$$\begin{aligned}\hat{F}_{P1[i]}(t) = & \frac{1}{n} \sum_{l=1}^m \left( \sum_{j=i}^k I(y_{lj[j]} \leq t) + \sum_{j=1}^{i-1} B_{i-j, k-i+1} \left( \frac{\hat{F}_S(t) - \hat{F}_{P1}(y_{li[i]})}{1 - \hat{F}_{P1}(y_{li[i]})} \right) I(y_{lj[j]} \leq t) \right. \\ & \left. + \sum_{j=i+1}^k B_{i, j-i} \left( \frac{\hat{F}_S(t)}{\hat{F}_{P1}(y_{li[i]})} \right) I(y_{lj[j]} > t) \right).\end{aligned}$$

The properties of our first proposed overall and in-stratum CDF estimator can be presented by the following theorem.

**Theorem 1.** Let  $\{y_{li[i]}: i = 1 \dots k; l = 1 \dots m\}$  be a RSS of size  $n$ . Then:

- $\hat{F}_{P1}(t)$  is exactly the same as  $\hat{F}_K(t)$ .
- RSS based on  $\hat{F}_{P1[i]}(t)$  is a consistent process.
- $\hat{F}_{P1[i]}(t)$  is a consistent estimator to  $B_{i, k-i+1}(F(t))$  under the perfectness assumption.

Since it is investigated by Zamanzade and Mahdizadeh (2018b) that  $\hat{F}_Z(t)$  is at least as efficient as  $\hat{F}_S(t)$ , this motivates us to create another CDF estimator by replacing  $\hat{F}_S(t)$  with  $\hat{F}_Z(t)$  in the EM algorithm during computing  $\hat{F}_{P1}(t)$ . Accordingly, our second proposed overall CDF estimator can be accomplished as:

$$\begin{aligned}\hat{F}_{P2}(t) = & \frac{1}{nk} \sum_{l=1}^m \left( \sum_{i=1}^k i I(y_{li[i]} \leq t) + \sum_{i=1}^{k-1} \sum_{j=i+1}^k B_{j-i, k-j+1} \left( \frac{\hat{F}_Z(t) - \hat{F}_{P1}(y_{li[i]})}{1 - \hat{F}_{P1}(y_{li[i]})} \right) I(y_{li[i]} \leq t) \right. \\ & \left. + \sum_{i=2}^k \sum_{j=1}^{i-1} B_{j, i-j} \left( \frac{\hat{F}_Z(t)}{\hat{F}_{P1}(y_{li[i]})} \right) I(y_{li[i]} > t) \right).\end{aligned}$$

with corresponding in-stratum CDF estimators given by:

$$\begin{aligned}\hat{F}_{P2[i]}(t) = & \frac{1}{n} \sum_{l=1}^m \left( \sum_{j=i}^k I(y_{lj[j]} \leq t) + \sum_{j=1}^{i-1} B_{i-j,k-i+1} \left( \frac{\hat{F}_Z(t) - \hat{F}_{P1}(y_{li[l]})}{1 - \hat{F}_{P1}(y_{li[l]})} \right) I(y_{lj[j]} \leq t) \right. \\ & \left. + \sum_{j=i+1}^k B_{i,j-i} \left( \frac{\hat{F}_Z(t)}{\hat{F}_{P1}(y_{li[l]})} \right) I(y_{lj[j]} > t) \right).\end{aligned}$$

It should be pointed out that  $\hat{F}_{P2}(t)$  enjoys with a nice feature that it makes efficient using of the information supported by the unmeasured items as well as the concomitant variable. The properties of  $\hat{F}_{P2}(t)$  and  $\hat{F}_{P2[i]}(t)$  can be shown by the next theorem.

**Theorem 2.** Let  $\{y_{li[l]}: i = 1 \dots k; l = 1 \dots m\}$  be a RSS of size  $n$ . Then:

- $\hat{F}_{P2}(t)$  is a consistent estimator to  $F(t)$ .
- RSS based on  $\hat{F}_{P2[i]}(t)$  is a consistent process.
- $\hat{F}_{P2[i]}(t)$  is a consistent estimator to  $B_{i,k-i+1}(F(t))$  under the perfectness assumption.

## 5. Efficiency Comparison using the Simulated Data Set.

In this part, we conducted a simulation study based on the Dell and Clutter (1972) model which assuming  $(Y, X)$  has a standard bivariate normal distribution with correlation coefficient  $\rho$ . The chosen values of  $\rho$  are:  $\rho = 1$ , for perfect ranking,  $\rho = 0.90$  for nearly perfect ranking,  $\rho = 0.50$  for nearly imperfect ranking and  $\rho = 0.00$  for random ranking. To illustrate the effect of set and sample sizes, we considered four different scenarios:  $(k, m) = (2, 5), (2, 10), (5, 2)$  and  $(5, 4)$ . Then we take  $Y$ ,  $\phi_n(Y)$ ,  $\phi_E^{-1}(\phi_n(Y))$  and  $\phi_G^{-1}(\phi_n(Y))$  as the target variables, where  $\phi_n(\cdot)$ ,  $\phi_E(\cdot)$ ,  $\phi_G(\cdot)$  and  $\phi_l^{-1}(\cdot)$  are respectively the CDF for the normal, exponential, the gumbel distributions and the inverse CDF (quantile function) for  $\phi_l(\cdot)$ . Therefore, this enables us to examine the performance of the aforementioned CDF estimators under the relation between the target variable and the concomitant variable to be linear as well as non-linear.

**Table 1:** Calculated REs for  $\hat{F}_{P2[i]}(t)$ , and  $\hat{F}_{Z[i]}(t)$  to  $\hat{F}_{P1[i]}(t)$  based on the Simulated Data

|             |          |         | $\rho = 1$           |                     | $\rho = 0.90$        |                     | $\rho = 0.50$        |                     | $\rho = 0.00$        |                     |
|-------------|----------|---------|----------------------|---------------------|----------------------|---------------------|----------------------|---------------------|----------------------|---------------------|
|             | $(k, m)$ | $i$     | $\hat{F}_{P2[i]}(t)$ | $\hat{F}_{Z[i]}(t)$ | $\hat{F}_{P2[i]}(t)$ | $\hat{F}_{Z[i]}(t)$ | $\hat{F}_{P2[i]}(t)$ | $\hat{F}_{Z[i]}(t)$ | $\hat{F}_{P2[i]}(t)$ | $\hat{F}_{Z[i]}(t)$ |
| Normal      | (2,5)    | 1       | 1.137                | 1.077               | 1.055                | 0.943               | 1.003                | 0.853               | 0.995                | 0.728               |
|             |          | overall | 1.157                | 1.207               | 1.058                | 1.016               | 0.998                | 0.941               | 0.984                | 0.931               |
|             | (2,10)   | 1       | 1.241                | 1.194               | 1.098                | 1.003               | 1.028                | 0.903               | 0.993                | 0.525               |
|             |          | overall | 1.262                | 1.375               | 1.102                | 1.072               | 1.032                | 1.017               | 0.993                | 0.956               |
|             | (5,2)    | 1       | 1.188                | 1.087               | 1.041                | 0.895               | 0.973                | 0.629               | 0.954                | 0.554               |
|             |          | 2       | 1.442                | 1.335               | 1.094                | 0.989               | 0.983                | 0.847               | 0.923                | 0.754               |
|             |          | 3       | 1.441                | 1.271               | 1.097                | 1.010               | 0.983                | 0.887               | 0.935                | 0.823               |
|             |          | overall | 1.403                | 1.434               | 1.092                | 1.051               | 0.975                | 0.912               | 0.928                | 0.859               |
|             | (5,4)    | 1       | 1.478                | 1.303               | 1.129                | 0.926               | 1.023                | 0.925               | 0.986                | 0.415               |
|             |          | 2       | 1.661                | 1.394               | 1.135                | 1.012               | 1.044                | 0.999               | 0.984                | 0.554               |
|             |          | 3       | 1.609                | 1.278               | 1.112                | 0.958               | 1.039                | 0.987               | 0.983                | 0.861               |
|             |          | Overall | 1.689                | 1.737               | 1.138                | 1.084               | 1.023                | 0.992               | 0.970                | 0.956               |
| Uniform     | (2,5)    | 1       | 1.178                | 1.079               | 1.079                | 0.967               | 0.999                | 0.805               | 0.997                | 0.690               |
|             |          | overall | 1.198                | 1.285               | 1.071                | 1.042               | 0.999                | 0.913               | 0.999                | 0.893               |
|             | (2,10)   | 1       | 1.226                | 1.116               | 1.108                | 1.023               | 0.994                | 0.705               | 0.995                | 0.481               |
|             |          | overall | 1.265                | 1.393               | 1.104                | 1.088               | 0.997                | 0.906               | 0.990                | 0.877               |
|             | (5,2)    | 1       | 1.284                | 1.111               | 1.058                | 0.885               | 0.971                | 0.522               | 0.980                | 0.366               |
|             |          | 2       | 1.320                | 1.218               | 1.059                | 0.967               | 0.988                | 0.805               | 0.986                | 0.672               |
|             |          | 3       | 1.366                | 1.241               | 1.064                | 0.927               | 0.998                | 0.930               | 0.988                | 0.963               |
|             |          | overall | 1.368                | 1.390               | 1.070                | 1.043               | 0.986                | 0.890               | 0.988                | 0.868               |
|             | (5,4)    | 1       | 1.577                | 1.358               | 1.081                | 0.806               | 0.987                | 0.533               | 0.978                | 0.420               |
|             |          | 2       | 1.638                | 1.388               | 1.074                | 0.938               | 0.975                | 0.630               | 0.977                | 0.485               |
|             |          | 3       | 1.581                | 1.313               | 1.084                | 0.972               | 0.972                | 0.844               | 0.979                | 0.916               |
|             |          | overall | 1.708                | 1.788               | 1.129                | 1.098               | 0.976                | 0.879               | 0.970                | 0.863               |
| Gumbel      | (2,5)    | 1       | 1.190                | 1.078               | 1.064                | 0.966               | 1.004                | 0.791               | 0.997                | 0.637               |
|             |          | 2       | 1.158                | 1.151               | 1.075                | 1.019               | 1.003                | 0.892               | 0.995                | 0.799               |
|             |          | overall | 1.186                | 1.301               | 1.077                | 1.088               | 1.005                | 0.968               | 0.996                | 0.980               |
|             | (2,10)   | 1       | 1.241                | 1.128               | 1.091                | 1.000               | 1.015                | 0.695               | 0.993                | 0.485               |
|             |          | 2       | 1.230                | 1.208               | 1.094                | 1.036               | 1.012                | 0.811               | 0.986                | 0.577               |
|             |          | overall | 1.251                | 1.374               | 1.095                | 1.092               | 1.014                | 0.984               | 0.987                | 0.982               |
|             | (5,2)    | 1       | 1.237                | 1.115               | 1.019                | 0.863               | 0.964                | 0.576               | 0.978                | 0.394               |
|             |          | 2       | 1.441                | 1.264               | 1.060                | 0.980               | 0.970                | 0.794               | 0.973                | 0.672               |
|             |          | 3       | 1.441                | 1.287               | 1.069                | 0.970               | 0.966                | 0.888               | 0.979                | 0.935               |
|             |          | 4       | 1.426                | 1.294               | 1.087                | 0.963               | 0.976                | 0.846               | 0.976                | 0.787               |
|             |          | 5       | 1.163                | 1.070               | 1.035                | 0.908               | 0.987                | 0.642               | 0.982                | 0.453               |
|             | (5,4)    | overall | 1.395                | 1.434               | 1.069                | 1.040               | 0.972                | 0.961               | 0.972                | 0.934               |
|             |          | 1       | 1.486                | 1.268               | 1.095                | 0.873               | 0.983                | 0.594               | 0.985                | 0.423               |
|             |          | 2       | 1.659                | 1.401               | 1.097                | 0.955               | 0.981                | 0.624               | 0.985                | 0.500               |
|             |          | 3       | 1.649                | 1.331               | 1.085                | 0.966               | 0.993                | 0.811               | 0.985                | 0.866               |
|             |          | 4       | 1.669                | 1.409               | 1.115                | 0.974               | 0.992                | 0.732               | 0.985                | 0.603               |
|             |          | 5       | 1.468                | 1.312               | 1.101                | 0.904               | 0.981                | 0.548               | 0.987                | 0.425               |
|             |          | overall | 1.675                | 1.726               | 1.112                | 1.102               | 0.987                | 0.950               | 0.970                | 1.015               |
| Exponential | (2,5)    | 1       | 1.154                | 1.085               | 1.077                | 1.006               | 0.999                | 0.791               | 0.997                | 0.682               |
|             |          | 2       | 1.152                | 1.122               | 1.056                | 0.989               | 1.006                | 0.904               | 0.991                | 0.777               |
|             |          | overall | 1.164                | 1.263               | 1.068                | 1.071               | 1.003                | 0.965               | 0.992                | 0.966               |
|             | (2,10)   | 1       | 1.218                | 1.101               | 1.080                | 0.960               | 1.013                | 0.696               | 0.996                | 0.474               |
|             |          | 2       | 1.223                | 1.203               | 1.093                | 1.032               | 1.006                | 0.788               | 0.991                | 0.566               |
|             |          | overall | 1.239                | 1.345               | 1.091                | 1.075               | 1.010                | 0.978               | 0.991                | 0.978               |
|             | (5,2)    | 1       | 1.363                | 1.156               | 1.090                | 0.871               | 0.994                | 0.506               | 0.994                | 0.340               |
|             |          | 2       | 1.441                | 1.241               | 1.084                | 0.970               | 0.982                | 0.733               | 0.987                | 0.625               |
|             |          | 3       | 1.411                | 1.297               | 1.073                | 1.000               | 0.975                | 0.917               | 0.977                | 0.965               |
|             |          | 4       | 1.436                | 1.334               | 1.113                | 1.023               | 0.977                | 0.893               | 0.970                | 0.850               |
|             |          | 5       | 1.180                | 1.084               | 1.046                | 0.908               | 0.976                | 0.654               | 0.981                | 0.458               |
|             | (5,4)    | overall | 1.401                | 1.453               | 1.093                | 1.068               | 0.979                | 0.944               | 0.975                | 0.977               |
|             |          | 1       | 1.469                | 1.188               | 1.105                | 0.832               | 0.995                | 0.324               | 0.994                | 0.194               |
|             |          | 2       | 1.657                | 1.363               | 1.105                | 0.955               | 0.999                | 0.563               | 0.989                | 0.439               |
|             |          | 3       | 1.656                | 1.297               | 1.105                | 0.994               | 1.009                | 0.854               | 0.980                | 0.868               |
|             |          | 4       | 1.714                | 1.444               | 1.134                | 1.004               | 0.997                | 0.798               | 0.975                | 0.680               |
|             |          | 5       | 1.488                | 1.333               | 1.103                | 0.908               | 0.977                | 0.452               | 0.983                | 0.258               |
|             |          | overall | 1.696                | 1.760               | 1.126                | 1.095               | 0.998                | 1.023               | 0.967                | 0.969               |

Overall indicates Overall-CDF.

For each combination of  $\rho$ ,  $k$  and  $m$ , we generate 5,000 samples under RSS. For each sample, all the aforesaid estimators were computed for all the target variables. A frequent

way to compare between two estimators is obtaining the ratio of their integrated MSE (IMSE) which the latter can be expressed as (see Ozturk (2007)):

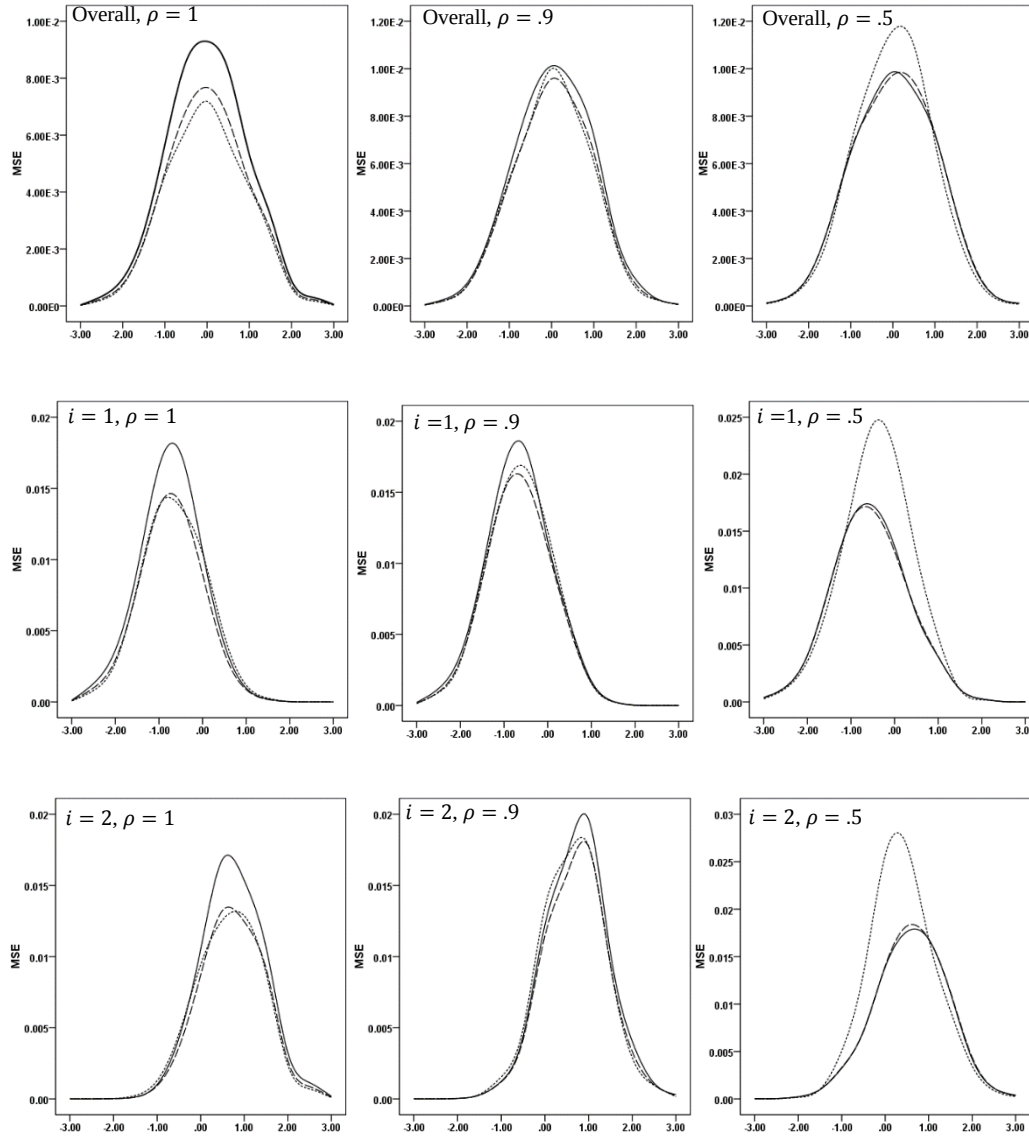


**Fig. 1:** MSE for  $\hat{F}_{Z[i]}$ ,  $\hat{F}_{P2[i]}$  and  $\hat{F}_{P1[i]}$  represented by dotted, dashed and solid respectively.

$$IMSE(\hat{F}) = \int (\hat{F}(t) - F(t))^2 dt.$$

For instance, the relative efficiency (RE) of  $\hat{F}_{Z[i]}$  to  $\hat{F}_{P1[i]}$  can be expressed as:

$$RE = \frac{IMSE(\hat{F}_{P1[i]})}{IMSE(\hat{F}_{Z[i]})}.$$

It is clear that if RE greater than one, this refers to the superiority of  $\hat{F}_{Z[i]}$  over  $\hat{F}_{P1[i]}$  and vice versa. Interestingly, IMSE can be approximated by the trapezoid rule given by:

$$IMSE(\hat{F}) \cong \frac{b-a}{2L} \left[ \sum_{i=0}^L MSE(\hat{F}(q_i)) + \sum_{i=1}^{L-1} MSE(\hat{F}(q_i)) \right],$$

where  $a$  and  $b$  are the lower and upper bounds of the approximated integral respectively,  $L$  is the number of the points within the interval  $[a, b]$  used for approximating the integral, and  $q_i = a + \frac{i(b-a)}{L}$ . We take  $a$  and  $b$  equal respectively the .05<sup>th</sup> and .95<sup>th</sup> quantile values of the target population, and the interval width equals 0.2. According to Frey (2012), in the case of the symmetric distributions, the RE of  $F_{[i]}(t)$  is equivalent to the RE of  $F_{[k-i+1]}(t)$ . Therefore, we do not need to exhibit the RE related to  $i$  greater than 1(3) at  $k$  equals 2 (5) corresponding to normal and uniform distributions.

From Table 1, we can observe the following: 1- For a fixed sample size, higher REs associated to the overall-CDF estimators can be achieved by increasing the number of cycles rather than the set size except if the rankings are perfect done, increasing the set size is preferable. 2- For a fixed set size increasing the sample size has a positive effect on the REs related to all the CDF estimators particularly that the relation between  $X$  and  $Y$  are reasonable strong. 3- There is observable effect of the values of  $\rho$  on the REs, as increasing the values of  $\rho$  raises the values of REs. 4-  $\hat{F}_{P2[i]}$  is the best in-stratum CDF estimator with similar behavior to  $\hat{F}_{P1[i]}$  when the ranking tends to the randomization. 5- The overall CDF estimator  $\hat{F}_{P2}$  outperforms uniformly  $\hat{F}_Z$  except if the rankings are completely perfect,  $\hat{F}_Z$  has a slight better behavior. However,  $\hat{F}_{P1}$  is the best overall CDF estimator in the case of the ranking tends to the imperfect.

For further detailed results concerned to one special case are given by Fig.1, which presents MSEs for  $\hat{F}_{Z[i]}$ ,  $\hat{F}_{P2[i]}$  and  $\hat{F}_{P1[i]}$  under  $\rho = 1, .9$  and  $.5$  with  $k = 2$  and  $m = 10$  from the standard normal distribution. According to Fig.1, it is apparent that there is a high agreement between the results shown by Fig.1 and those displayed in Table 1. As  $\hat{F}_{P2[i]}$  can be observed as the best in-stratum CDF estimator as long as the rankings tends to the perfectness. However,  $\hat{F}_Z$  has the least MSE among overall CDF estimators in the only case of the completely perfectness. It is also evident that all CDF estimators have the same MSEs at the tail of the data.

## 6. Efficiency Comparison using the Empirical Data Set.

In what follows, we will illustrate the applicability of the aforementioned CDF by using real data set known as tree height data set. This data set of size 396 observations is provided by Platt et al. (1988) and can be found at Chen et al.(2004). It also has been used by many authors (e.g. Lim et al. (2014) and Zamanzade and Mahdizadeh (2018b)).

**Table 2:** Calculated REs for  $\hat{F}_{Z[i]}(t)$ , and  $\hat{F}_{P2[i]}(t)$  to  $\hat{F}_{P1[i]}(t)$  based on the Real Data

| $i$     | (2,5)               |                      | (2,10)              |                      | (5,2)               |                      | (5,4)               |                      |
|---------|---------------------|----------------------|---------------------|----------------------|---------------------|----------------------|---------------------|----------------------|
|         | $\hat{F}_{Z[i]}(t)$ | $\hat{F}_{P2[i]}(t)$ | $\hat{F}_{Z[i]}(t)$ | $\hat{F}_{P2[i]}(t)$ | $\hat{F}_{Z[i]}(t)$ | $\hat{F}_{P2[i]}(t)$ | $\hat{F}_{Z[i]}(t)$ | $\hat{F}_{P2[i]}(t)$ |
| 1       | 1.101               | 1.027                | 1.113               | 1.047                | 1.046               | 0.931                | 1.137               | 0.985                |
| 2       | 1.092               | 1.057                | 1.158               | 1.119                | 1.180               | 1.030                | 1.197               | 0.965                |
| 3       | ---                 | ---                  | ---                 | ---                  | 1.219               | 1.099                | 1.249               | 1.059                |
| 4       | ---                 | ---                  | ---                 | ---                  | 1.199               | 1.112                | 1.246               | 1.136                |
| 5       | ---                 | ---                  | ---                 | ---                  | 1.062               | 0.973                | 1.107               | 0.984                |
| Overall | 1.152               | 1.162                | 1.195               | 1.203                | 1.177               | 1.183                | 1.211               | 1.222                |

We will consider this data set as the target population with overall and in-stratum CDF are respectively given by

$$F(t) = \frac{1}{396} \sum_{i=1}^{396} I(Y_i \leq t) \quad \text{and} \quad F_{[i]}(t) = B_{i,k-i+1}(F(t)) \quad i = 1 \dots k, 1 < k \leq 396.$$

Similar to Zamanzade and Mahdizadeh (2018b), We only consider two variables:  $X$  as the diameter in centimeters at breast height, and  $Y$  as the entire height in feet. These two variables are highly correlated as their Pearson correlation coefficient is 90%, thus one can expect that the ranking process will be nearly perfect. For the same values of  $(k, m)$  determined in the preceding Section, 5,000 samples with replacement were selected from the target population. For each sample, all the considered CDF estimators were estimated and their REs were also obtained and reported in the Tables 2. It should be mentioned that the IMSE are approximated over the interval between the .05<sup>th</sup> and .95<sup>th</sup> quantile values of the tree height data with interval width 20. One can easily deduce that the results exhibited in Table 2 are consistent with those shown in Table 1. As, due to the strength of the relation between  $X$  and  $Y$ ,  $\hat{F}_Z$  and  $\hat{F}_{P2}$  have a quite similar performance with observable superiority than  $\hat{F}_{P1}$ . Further  $\hat{F}_{P2[i]}$  outperforms uniformly  $\hat{F}_{Z[i]}$  which the latter works better than  $\hat{F}_{P1[i]}$  in almost cases. Finally it should be informed that all simulation studies in this work are programmed using R statistical software and available under request from the second author.

## 7. Conclusion

In this avenue, new CDF estimators under RSS based on missing data approach is provided. The proposed estimators attempt to make greater use of the ranking information supported by the concomitant variable and the unmeasured items. Based on the results provided by our simulation, we can claim that the suggested CDF estimators can be dependable and useful particularly when the perfectness assumption is hold.

Concerning to the subsequent researches, there is a plenty of work can be done. For instance, the CDF estimators discussed above can be utilized for improving the accuracy of the confidence interval for  $F(t)$  (see Frey (2012)). Further these estimators can also be used for increasing the power of the goodness of fit tests (see Frey and Wang (2014)). Additionally it also can be helpful for improving the behavior of the entropy estimators (see Zamanzade and Mahdizadeh (2017)).

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## Appendix

### Proof of Theorem 1

a) Using the binominal expansion, one can easily investigate that:

$$\frac{1}{k-i} \sum_{j=i+1}^k B_{j-i, k-j+1}(b) = \frac{1}{i-1} \sum_{j=1}^{i-1} B_{j, i-j}(b) = b, \quad b > 0. \quad (12)$$

Applying (12) to (11), we will get immediately (7) and hence the proof (a) is completed.

b)

$$\begin{aligned} \frac{1}{k} \sum_{i=1}^k \hat{F}_{P1[i]}(t) &= \frac{1}{nk} \sum_{l=1}^m \left( \sum_{i=1}^k i I(y_{li[i]} \leq t) + \sum_{i=2}^k \sum_{j=1}^{i-1} B_{i-j, k-i+1} \left( \frac{\hat{F}_S(t) - \hat{F}_{P1}(y_{li[i]})}{1 - \hat{F}_{P1}(y_{li[i]})} \right) I(y_{lj[j]} \leq t) \right. \\ &\quad \left. + \sum_{i=1}^{k-1} \sum_{j=i+1}^k B_{i, j-i} \left( \frac{\hat{F}_S(t)}{\hat{F}_{P1}(y_{li[i]})} \right) I(y_{lj[j]} > t) \right). \end{aligned}$$

By changing the order of the summation, we will get:

$$\begin{aligned} \frac{1}{k} \sum_{i=1}^k \hat{F}_{P1[i]}(t) &= \frac{1}{nk} \sum_{l=1}^m \left( \sum_{i=1}^k I(y_{li[i]} \leq t) + \sum_{j=1}^{k-1} \sum_{i=j+1}^k B_{i-j, k-i+1} \left( \frac{\hat{F}_S(t) - \hat{F}_{P1}(y_{li[i]})}{1 - \hat{F}_{P1}(y_{li[i]})} \right) I(y_{lj[j]} \leq t) \right. \\ &\quad \left. + \sum_{j=2}^k \sum_{i=1}^{j-1} B_{i, j-i} \left( \frac{\hat{F}_S(t)}{\hat{F}_{P1}(y_{li[i]})} \right) I(y_{lj[j]} > t) \right) = \hat{F}_{P1}(t). \end{aligned}$$

Hence the proof (b) is established.

c)

Using the fact that for large  $m$  and under the perfectness assumption that:

$$\begin{aligned} \frac{1}{m} \sum_{l=1}^m I(y_{li(i)} \leq t) &\cong P(y_{li(i)} \leq t) = \int_{-\infty}^t \frac{(k)!}{(i-1)! (k-i)!} (F(y))^{i-1} (1-F(y))^{k-i} f(y) dy \\ &= \int_0^{F(t)} f_i(u) d(u) \cong \int_0^{\hat{F}_{P1}(t)} f_i(u) d(u) = B_{i, k-i+1}(\hat{F}_{P1}(t)), \end{aligned} \quad (13)$$

where  $f_i(u)$  is the pdf of  $i^{th}$  order statistics from parent standard uniform distribution. Also by applying the law of large numbers and under the perfectness assumption, we can get:

$$\frac{1}{m} \sum_{l=1}^m g(y_{li(i)}) I(y_{li(i)} \leq t) \cong \int_0^{\hat{F}_{P1}(t)} h(u) f_i(u) d(u). \quad (14)$$

In the light of (13) and (14),  $\hat{F}_{P1[i]}$  can be expressed as:

$$\begin{aligned} \hat{F}_{P1[i]}(t) &= \frac{1}{k} \left( \sum_{j=1}^{i-1} \int_0^{\hat{F}_{P1}(t)} B_{i-j, k-i+1} \left( \frac{\hat{F}_S(t) - u}{1 - u} \right) f_j(u) d(u) \right. \\ &\quad \left. + \sum_{j=i+1}^k \int_{\hat{F}_{P1}(t)}^1 B_{i, j-i} \left( \frac{\hat{F}_S(t)}{u} \right) f_j(u) d(u) + \sum_{j=i}^k B_{j, k-j+1}(\hat{F}_{P1}(t)) \right). \end{aligned}$$

Due to the consistency of  $\hat{F}_S(t)$  and  $\hat{F}_{P1}(t)$ ,  $\hat{F}_{P1[i]}$  can be rewritten for  $m \rightarrow \infty$  as:

$$\begin{aligned} \hat{F}_{P1[i]}(t) &= \frac{1}{k} \left( \sum_{j=1}^{i-1} \int_0^{F(t)} B_{i-j, k-i+1} \left( \frac{F(t) - u}{1 - u} \right) f_j(u) d(u) \right. \\ &\quad \left. + \sum_{j=i+1}^k \int_{F(t)}^1 B_{i, j-i} \left( \frac{F(t)}{u} \right) f_j(u) d(u) + \sum_{j=i}^k B_{j, k-j+1}(F(t)) \right). \\ &= \frac{1}{k} (A + B + C). \end{aligned} \quad (15)$$

where the three sums in (15) will be referred respectively by  $A, B$  and  $C$ . In the light of the well-known identity that (see Paris (2010)):

$$B_{a,b}(x) = \sum_{j=a}^{a+b-1} \frac{(a+b-1)!}{j! (a+b-1-j)!} x^j (1-x)^{a+b-1-j}. \quad (16)$$

we can simplify the three sums  $A, B$  and  $C$  as below. The first term  $A$  can be addressed as:

$$\begin{aligned} A &= \sum_{j=1}^{i-1} \int_0^{F(t)} B_{i-j,k-i+1} \left( \frac{F(t)-u}{1-u} \right) f_j(u) d(u). \\ &= \sum_{j=1}^{i-1} \int_0^{F(t)} \sum_{q=i-j}^{k-j} \frac{(k-j)!}{q! (k-j-q)!} \left( \frac{F(t)-u}{1-u} \right)^q \left( 1 - \left( \frac{F(t)-u}{1-u} \right) \right)^{k-j-q} f_j(u) d(u). \\ &= \sum_{j=1}^{i-1} \int_0^{F(t)} \sum_{q=i-j}^{k-j} \frac{(k-j)!}{q! (k-j-q)!} \left( \frac{F(t)-u}{1-u} \right)^q \left( \frac{1-F(t)}{1-u} \right)^{k-j-q} \frac{(k)!}{(j-1)! (k-j)!} (u)^{j-1} (1-u)^{k-j} d(u) \\ &= \sum_{j=1}^{i-1} \sum_{q=i-j}^{k-j} \frac{(k)!}{q! (k-j-q)! (j-1)!} (1-F(t))^{k-j-q} \int_0^{F(t)} (F(t)-u)^q (u)^{j-1} d(u). \end{aligned}$$

Taking the transformation  $w = \frac{u}{F(t)}$ , we will get:

$$\begin{aligned} A &= \sum_{j=1}^{i-1} \sum_{q=i-j}^{k-j} \frac{(k)!}{q! (k-j-q)! (j-1)!} F(t)^{q+j} (1-F(t))^{k-j-q} \int_0^1 (1-w)^q (w)^{j-1} d(w). \\ &= \sum_{j=1}^{i-1} \sum_{q=i-j}^{k-j} \frac{(k)!}{(q+j)! (k-j-q)!} F(t)^{q+j} (1-F(t))^{k-j-q}. \end{aligned}$$

or equivalently as:

$$A = \sum_{j=1}^{i-1} \sum_{h=i}^k \frac{(k)!}{(h)! (k-h)! (j-1)!} F(t)^h (1-F(t))^{k-h}.$$

In the light of the identity in (16),  $A$  can be simplified as:

$$A = \sum_{j=1}^{i-1} B_{i,k-i+1}(F(t)) = (i-1)B_{i,k-i+1}(F(t)). \quad (17)$$

Secondly  $B$  can be studied as follows:

$$\begin{aligned} B &= \sum_{j=i+1}^k \int_{F(t)}^1 B_{i,j-i} \left( \frac{F(t)}{u} \right) f_j(u) d(u). \\ &= \sum_{j=i+1}^k \int_{F(t)}^1 \sum_{q=i}^{j-1} \frac{(j-1)!}{q! (j-1-q)!} \left( \frac{F(t)}{u} \right)^q \left( 1 - \frac{F(t)}{u} \right)^{j-1-q} f_j(u) d(u). \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=i+1}^k \sum_{q=i}^{j-1} \frac{(k)!}{q! (j-1-q)! (k-j)!} (F(t))^q \int_{F(t)}^1 (u-F(t))^{j-1-q} (1-u)^{k-j} d(u). \\
&= \sum_{q=i}^{k-1} \frac{(k)!}{q! (k-1-q)!} (F(t))^q \int_{F(t)}^1 (u-F(t))^{k-1-q} d(u) \\
&\quad + \sum_{j=i+1}^{k-1} \sum_{q=i}^{j-1} \frac{(k)!}{q! (j-1-q)! (k-j)!} (F(t))^q \int_{F(t)}^1 (u-F(t))^{j-1-q} (1-u)^{k-j} d(u).
\end{aligned}$$

The first integration can easily be solved. Yet, the second one can be solved by integration by parts sequentially  $j-1-q$  times, we will hence obtain:

$$\begin{aligned}
B &= \sum_{q=i}^{k-1} \frac{(k)!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q} + \sum_{j=i+1}^{k-1} \sum_{q=i}^{j-1} \frac{(k)!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q}. \\
&= \sum_{j=i+1}^k \sum_{q=i}^{j-1} \frac{(k)!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q}. \tag{18}
\end{aligned}$$

Finally  $C$  can be rewritten as follows:

$$C = \sum_{j=i}^k B_{j,k-j+1}(F(t)) = \sum_{j=i}^k \sum_{q=j}^k \frac{k!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q}. \tag{19}$$

By substituting (17), (18) and (19) in (15) we can find that:

$$\begin{aligned}
\hat{F}_{P1[i]} &= \frac{1}{k} \left( (i-1)B_{i,k-i+1}(F(t)) + \sum_{j=i+1}^k \sum_{q=i}^{j-1} \frac{(k)!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q} \right. \\
&\quad \left. + \sum_{j=i}^k \sum_{q=j}^k \frac{k!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q} \right). \\
\hat{F}_{P1[i]} &= \frac{1}{k} \left( (i-1)B_{i,k-i+1}(F(t)) + \sum_{j=i+1}^k \sum_{q=i}^{j-1} \frac{(k)!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q} \right. \\
&\quad + \sum_{j=i+1}^k \sum_{q=j}^k \frac{k!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q} \\
&\quad \left. + \sum_{q=i}^k \frac{k!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q} \right).
\end{aligned}$$

$$= \frac{1}{k} \left( (i-1)B_{i,k-i+1}(F(t)) + \sum_{j=i+1}^k \sum_{q=i}^k \frac{(k)!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q} + \sum_{q=i}^k \frac{k!}{q! (k-q)!} (F(t))^q (1-F(t))^{k-q} \right).$$

By using again the identity in (16), we can get:

$$\hat{F}_{P1[i]} = \frac{1}{k} \left( (i-1)B_{i,k-i+1}(F(t)) + \sum_{j=i+1}^k B_{i,k-i+1}(F(t)) + B_{i,k-i+1}(F(t)) \right).$$

$$= B_{i,k-i+1}(F(t)). \text{ Which completes the proof (c).}$$

### **Proof of Theorem 2**

Due to the consistency of  $\hat{F}_S(t)$  as well as  $\hat{F}_Z(t)$  to  $F(t)$ , one can expect that  $|\hat{F}_S(t) - \hat{F}_Z(t)| \rightarrow 0$  as  $n \rightarrow \infty$  leading to  $|\hat{F}_{P1}(t) - \hat{F}_{P2}(t)| \rightarrow 0$  as  $n \rightarrow \infty$ . Hence from Theorem 1, (a) , (b) and (c) can be straightforwardly proved.

# Parametric Estimation Methods based on Ranked Set Sampling: Missing Data Approach

Samir K. Ashour, and Mohamed S. Abdallah

## Abstract

The problem of parameters estimation plays a significant role in various areas of academic researches. In this article, we propose three different methods of estimation for the parameters of location-scale family under ranked set sampling in the view of missing data mechanism. Through a series of Monte Carlo simulations, it is well investigated that the proposed methods are relatively robust from violating the perfect ranking condition and provide better performance over their competitors using bias and MSE (mean square error) criteria. An empirical data set is also used for illustrative purposes.

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**Keywords:** Cramér-von-Mises; Concomitant variable; EM algorithm; Estimation methods ; Imperfect Models; Missing Data Approach; Ranked set sampling.

## 1. Introduction

The problem of parameters estimation occupied an important position in various areas of researches such as agricultural, biological, ecological, engineering, medical, physical, and social sciences. As parameters estimation can be considered as an essential step during performing many statistical aspects (e.g. testing hypotheses, goodness of fit, reliability analysis, ... etc.). With this motivation, many efforts are devoted to produce several estimation methods which are available in the literature. However, the appeal of the estimation methods vary from researcher to another. For instance, one may prefer to use the moment estimators (MEs) rather than the maximum likelihood estimators (MLEs) in the case that the parent distribution is a more complicated leading the computations may not converge. For more discussion about different types of methods estimation, one can refer to Mittelhammer (2013).

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In the real life, there are several situations that the variable of interest  $Y$  can be more easily ranked by either visual inspection or using a concomitant variable  $X$  than quantified, ranked set sample (RSS) introduced by McIntyre (1952) is widely recommended to be adopted in such these situations. The attractive property of RSS is that it provides an informative image of the population in a more efficient way than other traditional sampling techniques such as simple random sample (SRS). The reason is that RSS depends on the information supported by not only the exact quantification of the  $Y$ 's items but also their rankings. The basic procedures of concomitant-based RSS can be briefly described as follows:

First, draw  $k$  random samples each of size  $k$  of the bivariate variables  $(X, Y)$ . Second, for each sample,  $X$  values are fully measured and sorted in ascending way. Then, measure only the  $Y$  values associated with the  $i^{th}$  smallest observation of  $X$  corresponding to the  $i^{th}$  sample ( $i = 1 \dots k$ ). Finally, repeat the preceding steps  $m$  times (cycles) in order to obtain  $n = km$  values of  $Y$ . If the number of the selected items across the cycles equals, RSS is hence called balanced RSS. Otherwise, it is called unbalanced RSS. It should be emphasized that we will confine ourselves that  $(X, Y)$  are continuous variables drawn through balanced RSS. In order to ease the notation, let  $x_{lj(i)}$  be the  $i^{th}$  smallest observation from  $j^{th}$  sample corresponding to  $l^{th}$  cycle. And also, let  $y_{li[j]}$  be the interested variable's values associated with  $x_{lj(i)}$  ( $(i, j) = 1 \dots k, l = 1 \dots m$ ). Further, Perfect (Imperfect) ranking refers that the rankings are done without (with) errors denoted by rounded (squared) bracket.

Although RSS actually is a nonparametric in its nature, a large number of publications for parametric estimation has been produced in the literature. From those, Abu-Dayyeh et al. (2013) used different estimation methods for estimating the shape and scale parameters of Pareto distribution based on RSS and SRS. Sarikavanij et al. (2014) made a comparison study between SRS and RSS for estimating the location and scale parameters of a two-parameter exponential distribution. Hassan (2013) did a study about exponentiated exponential distribution using both SRS and RSS techniques based on maximum likelihood and Bayesian estimation methods. Yousef and Al-Subh (2014) compared between the efficiency of RSS and SRS in estimating the location and scale parameters of the Gumbel distribution using different estimation method. Dey et al. (2017) addressed the estimation of the parameter of Rayleigh distribution using different methods of

frequentist and Bayesian estimation approaches based on SRS and RSS and other sampling schemes. For other interesting works related to RSS, see Arslan and Ozturk (2013).

It is well established by many publications that the superiority of RSS depends heavily on the quality of ranking information. As this superiority increases (decreases) in the case of perfect (imperfect) ranking. Therefore it is expected that the statistical procedures which assume the perfect assumption have a poor performance when this assumption is violated. One way to overcome this problem is to construct such these statistical procedures under imperfect ranking models. This motivated us to derive parametric methods for estimating the location and scale parameters which implicitly use the stochastic order property and be less sensitive to the departure of the perfect ranking as well. In this article, three different methods of estimation of the parameters are proposed. The first one is derived under Fraction-of-random rankings model, the others are constructed based on Pool-Adjacent-Violators Algorithm (PAVA).

The remainder of this manuscript is structured as follows. Section 2 exhibits several popular methods of parameter estimation. Section 3 presents the three proposed estimation methods. The numerical comparisons using Monte Carlo simulation are summarized and organized in Section 4. In Section 5, real data set is used to present the applicability of the estimation methods, and lastly Section 6 shows our concluding points and some possible extension works.

## 2. The Estimation Methods

Let  $\{y_{li}; i = 1 \dots k; l = 1 \dots m\}$  be a concomitant-based RSS of size  $n$  drawn from a location and scale family of distribution  $F\left(\frac{y-\mathcal{M}}{\delta}\right)$ , where  $F(\cdot)$  is the cumulative distribution function (CDF) having mean zero and scale 1 of the standardized random variable  $\frac{Y-\mathcal{M}}{\delta}$ , and the support of  $F(\cdot)$  does not depend on  $\theta = (\mathcal{M}, \delta)$ . Although several methods are available to estimate  $\theta$ , there is no universal agreement about the most efficient method. In this part, four estimation procedures are briefly discussed one by one followed by our three novel methods.

### 2.1 Moments Estimators

This method is firstly used by Karl Pearson (see Rajagopalan and Dhanavathan (2012)) which is based on the well-known fact that the population moments are functions of the unknown parameters and because of the consistency of the sample moments to the population moments, this method works through solving a system of equations constructing by equating the first  $M$  raw moments for the population distribution with their corresponding raw sampling moments. Mathematically, the system of equations can be expressed as:

$$E(Y^r) = \hat{\mu}^r = \frac{1}{n} \sum_{i=1}^k \sum_{l=1}^m y_{li[i]}^r \quad r = 1, 2 \dots M, \quad (1)$$

where  $M$  is the number of the unknown parameters. Here  $M = 2$  and the estimates obtained by solving (1) will be denoted as  $\hat{\theta}_M = (\hat{\mathcal{M}}_M, \hat{\delta}_M)$ .

## 2.2 Maximum Likelihood Estimators

Stokes (1995) utilized the fact the items of RSS are, under the perfectness, a random sample of  $i^{th}$  order statistics of size  $k$  with alternatively likelihood function given by:

$$\begin{aligned} L_S(\mathcal{M}, \delta | Y) = & \sum_{l=1}^m \sum_{i=1}^k \log \left( f_i \left( \frac{y_{li[i]} - \mathcal{M}}{\delta} \right) \right) \propto -n \log(\delta) \\ & + \sum_{l=1}^m \sum_{i=1}^k \log \left( f \left( \frac{y_{li[i]} - \mathcal{M}}{\delta} \right) \right) + \sum_{l=1}^m \sum_{i=1}^k (i-1) \log \left( F \left( \frac{y_{li[i]} - \mathcal{M}}{\delta} \right) \right) \\ & + \sum_{l=1}^m \sum_{i=1}^k (k-i) \log \left( 1 - F \left( \frac{y_{li[i]} - \mathcal{M}}{\delta} \right) \right), \end{aligned}$$

where  $f_i(\cdot)$  is the probability density function (pdf) of  $i^{th}$  order statistics. The MLE of  $\mathcal{M}$  and  $\delta$  can be obtained by either maximizing  $L_S(\mathcal{M}, \delta | Y)$  or solving simultaneously the following two equations:

$$\left. \begin{aligned} \frac{\partial L_S(\mathcal{M}, \delta | Y)}{\partial \mathcal{M}} &= \sum_{l=1}^m \sum_{i=1}^k \frac{f_i' \left( \frac{y_{li[i]} - \mathcal{M}}{\delta} \right)}{f_i \left( \frac{y_{li[i]} - \mathcal{M}}{\delta} \right)} = 0 \\ \frac{\partial L_S(\mathcal{M}, \delta | Y)}{\partial \delta} &= \sum_{l=1}^m \sum_{i=1}^k \frac{f_i' \left( \frac{y_{li[i]} - \mathcal{M}}{\delta} \right)}{f_i \left( \frac{y_{li[i]} - \mathcal{M}}{\delta} \right)} = 0 \end{aligned} \right\}. \quad (2)$$

where  $f'^a(\cdot)$  is the first derivative of  $f(\cdot)$  with respect to  $a$ . The corresponding estimates of solving (2) will be denoted as  $\hat{\theta}_L = (\hat{\mathcal{M}}_L, \hat{\delta}_L)$ .

### 2.3 Cramér-von-Mises Estimators

Another class of parameters estimation method can be obtained by using the well-known goodness of fit statistics such as Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling statistics. The idea is to select the estimates that minimize the distance between the sampling distribution function and the parent CDF. Interestingly, Macdonald (1971) provided empirical evidence that the bias of the estimates based on Cramér–von Mises is smaller than those based on the other goodness of fit statistics. This motivated us to adopt Cramér–von Mises statistic which can be expressed as:

$$L_C^*(\mathcal{M}, \delta|Y) = n \int_{-\infty}^{\infty} \left( F\left(\frac{y - \mathcal{M}}{\delta}\right) - \hat{F}(y) \right)^2 dF\left(\frac{y - \mathcal{M}}{\delta}\right), \quad (3)$$

where  $\hat{F}(y)$  is the sampling CDF frequently be estimated by the empirical distribution function (EDF) leading (3) takes a simpler formula given by:

$$L_C(\mathcal{M}, \delta|Y) = \frac{1}{12n} + \frac{1}{n} \sum_{i=1}^n \left( F\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) - \frac{(2i-1)}{2n} \right)^2.$$

Where  $y_i^*$ 's are the pooled values of  $y_{l[i]}$ 's across all the cycles,  $y_{(i)}^*$ 's are ordered values of  $y_i^*$ 's. Hence Cramér–von Mises estimators can be obtained either by minimizing  $L_C(\mathcal{M}, \delta|Y)$  or equivalently solving simultaneously the following equations:

$$\begin{aligned} \frac{\partial L_C(\mathcal{M}, \delta|Y)}{\partial \mathcal{M}} &= \sum_{i=1}^n \left( F\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) - \frac{(2i-1)}{2n} \right) F'_{\mathcal{M}}\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) = 0 \\ \frac{\partial L_C(\mathcal{M}, \delta|Y)}{\partial \delta} &= \sum_{i=1}^n \left( F\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) - \frac{(2i-1)}{2n} \right) F'_{\delta}\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) = 0 \end{aligned}$$

The Cramér–von Mises estimators will be denoted as  $\hat{\theta}_C = (\hat{\mathcal{M}}_C, \hat{\delta}_C)$ . For interesting properties about Cramér–von Mises estimators, reader can refer to Ozturk and Hettmansperger (1997).

## 2.4 Maximum Product Of Spacings Estimators

Mahdizadeh and Tamandi (2016) suggested recently to adopt the maximum product of spacing (MPS) Method, proposed by Cheng and Amin (1983), for getting the parameter estimates rather than using MLE approach proposed by Stokes (1995) under RSS. MPS is based on the idea of expressing the likelihood function by differences between the values of the CDF at consecutive data points which can be expressed as:

$$L_T(\mathcal{M}, \delta | Y) = \sum_{i=1}^{n+1} \log \left( F \left( \frac{y_{(i)}^* - \mathcal{M}}{\delta} \right) - F \left( \frac{y_{(i-1)}^* - \mathcal{M}}{\delta} \right) \right),$$

where  $F \left( \frac{y_{(0)}^* - \mathcal{M}}{\delta} \right) = 0$  and  $F \left( \frac{y_{(n+1)}^* - \mathcal{M}}{\delta} \right) = 1$ . Hence the estimates of  $\theta = (\mathcal{M}, \delta)$  can be obtained either by maximizing  $L_T(\mathcal{M}, \delta | Y)$  or equivalently solving simultaneously the equations in (4):

$$\left. \begin{aligned} \frac{\partial L_T(\mathcal{M}, \delta | Y)}{\partial \mathcal{M}} &= \sum_{i=1}^{n+1} \frac{F' \mathcal{M} \left( \frac{y_{(i)}^* - \mathcal{M}}{\delta} \right) - F' \mathcal{M} \left( \frac{y_{(i-1)}^* - \mathcal{M}}{\delta} \right)}{F \left( \frac{y_{(i)}^* - \mathcal{M}}{\delta} \right) - F \left( \frac{y_{(i-1)}^* - \mathcal{M}}{\delta} \right)} = 0 \\ \frac{\partial L_T(\mathcal{M}, \delta | Y)}{\partial \delta} &= \sum_{i=1}^{n+1} \frac{F' \delta \left( \frac{y_{(i)}^* - \mathcal{M}}{\delta} \right) - F' \delta \left( \frac{y_{(i-1)}^* - \mathcal{M}}{\delta} \right)}{F \left( \frac{y_{(i)}^* - \mathcal{M}}{\delta} \right) - F \left( \frac{y_{(i-1)}^* - \mathcal{M}}{\delta} \right)} = 0 \end{aligned} \right\}. \quad (4)$$

The corresponding estimates of  $\theta = (\mathcal{M}, \delta)$  for MPS method will be denoted as  $\hat{\theta}_T = (\hat{\mathcal{M}}_T, \hat{\delta}_T)$ .

## 3. The Proposed Estimation Methods

In this part three novel parametric methods of estimation under RSS are suggested and presented one by one as follows. Our idea is to incorporate the information supported by the unmeasured items as well as the measured items for suggesting new parametric method.

### 3.1 Iterative Moments Estimators

Our first proposed estimation method is based on incorporating the stochastic relationship between the measured items and the unmeasured items to estimate  $\theta = (\mathcal{M}, \delta)$ . In more precise, our strategy is to impute the unmeasured items in the view of the measured items, then estimate  $\theta = (\mathcal{M}, \delta)$  using (1) based on the information provided by both the measured items and the imputed unmeasured items, finally use the parameters'

estimates to upgrade these imputed unmeasured items. By repeating these steps iteratively until the convergence of the parameters' estimates occurs. Imputing the unmeasured items will be achieved in a parametric fashion using the conditional order statistics pdf as explained below:

- If  $y_{lj[i]} > y_{li[i]}$ , then:

$$E(y^r | y > y_{li[i]}) = \int_{y_{li[i]}}^{\infty} y^r f_{j,i}^1(y, \mathcal{M}, \delta | y_{li[i]}) dy. \quad (5)$$

- In contradiction, if  $y_{lj[i]} < y_{li[i]}$ , then:

$$E(y^r | y < y_{li[i]}) = \int_{-\infty}^{y_{li[i]}} y^r f_{j,i}^2(y, \mathcal{M}, \delta | y_{li[i]}) dy. \quad (6)$$

where  $f_{j,i}^1(y | y_{li[i]})$  and  $f_{j,i}^2(y | y_{li[i]})$  are the conditional order statistics pdfs respectively expressed as:

$$f_{j,i}^1(y | w) = \frac{(k-i)!}{(j-i-1)!(k-j)!} \left( \frac{F(y, \mathcal{M}, \delta) - F(w, \mathcal{M}, \delta)}{1 - F(w, \mathcal{M}, \delta)} \right)^{j-i-1} \left( 1 - \frac{F(y, \mathcal{M}, \delta) - F(w, \mathcal{M}, \delta)}{1 - F(w, \mathcal{M}, \delta)} \right)^{k-j} \frac{f(y, \mathcal{M}, \delta)}{1 - F(w, \mathcal{M}, \delta)}$$

$$w \leq y \leq \infty$$

and

$$f_{j,i}^2(y | w) = \frac{(i-1)!}{(j-1)!(i-j-1)!} \left( \frac{F(y, \mathcal{M}, \delta)}{F(w, \mathcal{M}, \delta)} \right)^{j-1} \left( 1 - \frac{F(y, \mathcal{M}, \delta)}{F(w, \mathcal{M}, \delta)} \right)^{i-j-1} \frac{f(y, \mathcal{M}, \delta)}{F(w, \mathcal{M}, \delta)},$$

$$-\infty \leq y \leq w.$$

By applying (5) and (6) to the entire sample, we will get:

$$\mu_E^{r*}(\mathcal{M}, \delta) = \frac{1}{kn} \sum_{l=1}^m \sum_{i=1}^k \left( y_{li[i]}^r + E(y^r | y > y_{li[i]}) + E(y^r | y < y_{li[i]}) \right).$$

It is observant that  $\mu_E^{r*}(\mathcal{M}, \delta)$  is dependable only if the perfectness assumption is hold. However when this condition is violated,  $\mu_E^{r*}(\mathcal{M}, \delta)$  may give illogical values. To tackle this dilemma, we resort to use fraction-of-random-rankings model proposed by Frey et al.

(2007) to modify  $\mu_E^{r*}(\mathcal{M}, \delta)$  as being more robust to the departure from perfect rankings as presented below:

$$\mu_E^r(\mathcal{M}, \delta) = \frac{1}{kn} \sum_{l=1}^m \sum_{i=1}^k \left( (1 - \lambda) \left( y_{li[i]}^r + E(y^r | y > y_{li[i]}) + E(y^r | y < y_{li[i]}) \right) + \lambda E(y^r) \right).$$

Since  $\mu_E^r(\mathcal{M}, \delta)$  can not be directly computed as it simply relies on the unknown parameters  $\mathcal{M}, \delta$  and  $\lambda$ , EM algorithm is adopted to overcome this problem whose steps are described below.

- 1- Let  $(\widehat{\mathcal{M}}^{(0)}, \widehat{\delta}^{(0)})$  be the seed estimates for  $\mathcal{M}, \delta$ .
- 2- Set  $p = 0$ .
- 3- Estimate the parameter  $\lambda$  by the corresponding likelihood function of fraction-of-random-rankings model (Frey and Zhang (2018) and Ashour and Abdallah (2018a)):

$$\hat{\lambda} = \underset{\lambda \in [0,1]}{\text{Max}} \prod_{i=1}^k \prod_{l=1}^h \left( (1 - \lambda) b_{i,k-i+1}(F(y_{[i,l]}), \widehat{\mathcal{M}}^{(p)}, \widehat{\delta}^{(p)}) + \lambda \right).$$

where  $b_{a,b}(t)$  is the pdf of the Beta distribution with parameters  $a$  and  $b$  at the point  $t$ .

- 4- In the light of (5) and (6), compute the sampling raw moments by the following equation:

$$\hat{\mu}_E^r(\widehat{\mathcal{M}}^{(p)}, \widehat{\delta}^{(p)}) = \frac{1}{kn} \sum_{l=1}^m \sum_{i=1}^k \left( (1 - \hat{\lambda}) \left( y_{li[i]}^r + E(y^r | y > y_{li[i]}) + E(y^r | y < y_{li[i]}) \right) + \hat{\lambda} E(y^r) \right).$$

- 5- Use  $\hat{\mu}_E^1(\mathcal{M}, \delta)$  and  $\hat{\mu}_E^2(\mathcal{M}, \delta)$  for estimating  $\mathcal{M}, \delta$  by solving the following system:

$$E(Y^r) = \hat{\mu}_E^r(\mathcal{M}, \delta). \quad r = 1, 2 \quad (7)$$

Denote the solution of (7) by  $(\widehat{\mathcal{M}}^{(p+1)}, \widehat{\delta}^{(p+1)})$ .

- 6- Set  $p = p + 1$ .
- 7- Repeat steps (3 – 6) until stopping rule satisfies. i.e.

$$\text{Max}(|\widehat{\mathcal{M}}^{(p)} - \widehat{\mathcal{M}}^{(p-1)}|, |\widehat{\delta}^{(p)} - \widehat{\delta}^{(p-1)}|) \leq .001.$$

- 8- The final estimates are  $\hat{\theta}_E = (\widehat{\mathcal{M}}_E, \widehat{\delta}_E) = (\widehat{\mathcal{M}}^{(p)}, \widehat{\delta}^{(p)})$ .

**Remark 1.** In the light of the condition (10) of Wu (1983), the suggested EM algorithm will always converge to a local maximum as long as  $F(\cdot)$  belongs to the exponential family. It is hence advisable to adopt  $\hat{\theta}_E$  in the case of  $F(\cdot)$  as a member of the exponential family.

**Remark 2.** Under the perfectness assumption and in the light of the law of large numbers, one can show that  $\hat{\mu}_E^r(\hat{\mathcal{M}}_E, \hat{\delta}_E)$  is a consistent estimator to  $E(Y^r)$  causing  $\hat{\theta}_E$  to be a consistent estimator to  $\theta$ .

### 3.2 Modified Moments Estimators

The second modification of  $\hat{\theta}_M$  consists of replacing  $\hat{\mu}^r$  in (1) by another more efficient estimator. As opposite to  $\hat{\theta}_E$ , one can also impute the unmeasured items in a nonparametric way using the information ranking supported by  $X$ . In the light of Zamanzade and Mohammadi (2016),  $E(Y^r)$  can be estimated through applying the following steps:

- 1- Combining  $y_{l(i)}$  and their corresponding values of  $x_{l(i)}$  into two new variables  $(y_z^*, x_z^*, z = 1 \dots n)$  respectively.
- 2- Sorting ascending  $(y_z^*, x_z^*)$  according to  $x^*$  values yielding  $(y_{[z]}^*, x_{(z)}^*)$ .
- 3- Computing the  $r^{\text{th}}$  raw sampling moments as:

$$\hat{\mu}_X^r = \frac{1}{nk} \sum_{t=1}^{nk} h_1^r(x_{(t)}^*). \quad r = 1, 2.$$

where

$$h_1^r(x) = \begin{cases} y_{[1]}^{r, iso} & x < x_{(1)}^* \\ y_{[t]}^{r, iso} + \frac{y_{[t+1]}^{r, iso} - y_{[t]}^{r, iso}}{x_{(t+1)}^* - x_{(t)}^*} (x - x_{(t)}^*) & x_{(t)}^* < x < x_{(t+1)}^* \quad t = 1 \dots n-1 \\ y_{[n]}^{r, iso} & x_{(n)}^* < x \\ y_{[t]}^{r, iso} & x = x_{(t)}^* \quad t = 1 \dots n \end{cases},$$

and  $y^{r, iso}$  is the isotonized values of  $y^r$  obtained by the PAVA. For more discussion about PAVA, one can refer to Ozturk (2007).

By replacing  $\hat{\mu}$  with  $\hat{\mu}_X^r$  in (1), we will get that:

$$E(Y^r) = \hat{\mu}_X^r = \frac{1}{nk} \sum_{t=1}^{nk} h_1^r(x_{(t)}^*). \quad r = 1, 2 \quad (8)$$

The estimates produced by solving (8) will be denoted as  $\hat{\theta}_X = (\hat{\mathcal{M}}_X, \hat{\delta}_X)$ .

### 3.3 Modified Cramér–Von Mises Estimators

Due to a plethora CDF estimators provided in the literature particularly in the case of RSS, it is quite reasonable to replace  $\hat{F}(y)$  with an efficient CDF estimator rather than the EDF in (4). Most recently, Zamanzade and Mahdizadeh (2018) suggested a novel CDF estimator based on a concomitant-based RSS. The authors showed numerically that their estimator is at least as efficient as EDF which can be expressed as:

$$\hat{F}_X(t) = \frac{1}{kn} \sum_{i=1}^k \sum_{j=1}^k \sum_{l=1}^m \hat{F}_{x_{lj(i)}}(t).$$

where

$$\hat{F}_X(t) = \begin{cases} \hat{F}_1^{iso}(t) & x \leq x_{(1)}^* \\ \hat{F}_z^{iso}(t) + \frac{\hat{F}_{z+1}^{iso}(t) - \hat{F}_z^{iso}(t)}{x_{(z+1)}^* - x_{(z)}^*} [x - x_{(z)}^*] & x_{(z)}^* \leq x < x_{(z+1)}^* \quad z = 1 \dots n-1, \\ \hat{F}_n^{iso}(t) & x \geq x_{(n)}^* \end{cases}$$

where  $\hat{F}_z^{iso}(t)$  is the isotonized values of  $I(y_{[z]}^* \leq t)$  and  $I(\cdot)$  is the indicator function. For a more discussion about  $\hat{F}_X(t)$ , see Ashour and Abdallah (2018 a,b). One can rewrite the Cramér–von Mises statistic into another general formula as:

$$\begin{aligned} L_F^*(\mathcal{M}, \delta | Y) &= n \sum_{i=0}^n \int_{y_{(i)}^*}^{y_{(i+1)}^*} \left( F\left(\frac{y - \mathcal{M}}{\delta}\right) - \hat{F}(y_{[i]}^*) \right)^2 dF\left(\frac{y - \mathcal{M}}{\delta}\right) \\ &= \frac{n}{3} \sum_{i=0}^n \left( \left( F\left(\frac{y_{(i+1)}^* - \mathcal{M}}{\delta}\right) - \hat{F}(y_{(i)}^*) \right)^3 - \left( F\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) - \hat{F}(y_{(i)}^*) \right)^3 \right). \quad (9) \end{aligned}$$

where  $y_{(0)}^* = -\infty$ ,  $y_{(n+1)}^* = \infty$ . By replacing  $\hat{F}(y_{[i]}^*)$  with  $\hat{F}_X(y_{(i)}^*)$  yields to:

$$L_F(\mathcal{M}, \delta | Y) = \frac{n}{3} \sum_{i=0}^n \left( \left( F\left(\frac{y_{(i+1)}^* - \mathcal{M}}{\delta}\right) - \hat{F}_X(y_{(i)}^*) \right)^3 - \left( F\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) - \hat{F}_X(y_{(i)}^*) \right)^3 \right). \quad (10)$$

Hence the modified Cramér–von Mises estimators can be obtained either by minimizing  $L_F(\mathcal{M}, \delta|Y)$  or equivalently solving simultaneously the following equations:

$$\left. \begin{aligned} \frac{\partial L_F(\mathcal{M}, \delta|Y)}{\partial \mathcal{M}} &= \sum_{i=0}^n \left( \left( F\left(\frac{y_{(i+1)}^* - \mathcal{M}}{\delta}\right) - \hat{F}_x(y_{(i)}^*) \right)^2 F'^{\mathcal{M}}\left(\frac{y_{(i+1)}^* - \mathcal{M}}{\delta}\right) - \left( F\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) - \hat{F}_x(y_{(i)}^*) \right)^2 F'^{\mathcal{M}}\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) \right) = 0 \\ \frac{\partial L_F(\mathcal{M}, \delta|Y)}{\partial \delta} &= \sum_{i=0}^n \left( \left( F\left(\frac{y_{(i+1)}^* - \mathcal{M}}{\delta}\right) - \hat{F}_x(y_{(i)}^*) \right)^2 F'^{\delta}\left(\frac{y_{(i+1)}^* - \mathcal{M}}{\delta}\right) - \left( F\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) - \hat{F}_x(y_{(i)}^*) \right)^2 F'^{\delta}\left(\frac{y_{(i)}^* - \mathcal{M}}{\delta}\right) \right) = 0 \end{aligned} \right\}. \quad (11)$$

The estimates based on solving (11) will be denoted as  $\hat{\theta}_F = (\hat{\mathcal{M}}_F, \hat{\delta}_F)$ . To close this section, it should be indicated that Ozturk (2011) used EM algorithm to suggest a new estimation method under RSS. However, his strategy required heavy computations and time consuming causing to be omitted in this work. Another important point deserves to be mentioned is to obtain the variances estimates of the proposed estimators. Since we feel that deriving these variances seems not an easy mission, an approximation can be simply obtained by using bootstrap RSS by rows (BRSSR) method proposed by Modarres et al. (2006) similar to what is done in Mahdizadeh and Tamandi (2016).

#### 4. Efficiency Comparison using Simulated Data Set

This part exhibits the performance of the aforementioned estimation methods based on the simulated data generated through Dell and Clutter (1972) model which assuming  $(Y, X)$  has a standard bivariate normal distribution with a correlation coefficient  $\rho$ . The chosen values of  $\rho$  are:  $\rho = 1, .80$  and  $.50$ . To illustrate the effect of set and sample sizes, we considered four different configurations:  $(k, m) = (2, 5), (2, 10), (5, 2)$  and  $(5, 4)$ . Then we take  $Y, \phi_L^{-1}(\phi_n(Y)), \phi_G^{-1}(\phi_n(Y))$  and  $\phi_E^{-1}(\phi_n(Y))$  as the target variables, where  $\phi_n(\cdot)$  is the CDF for the standard normal and  $\phi_L^{-1}(\cdot), \phi_G^{-1}(\cdot)$  and  $\phi_E^{-1}(\cdot)$  are respectively the quantile function for logistic, gumbel and exponential distributions. For each combination of  $\rho, k$  and  $m$ , data sets are generated from the target variables based on 5,000 samples under RSS. The biases and the MSEs for  $\hat{\theta}_L, \hat{\theta}_T, \hat{\theta}_C, \hat{\theta}_E, \hat{\theta}_X$  and  $\hat{\theta}_F$  estimators were computed and reported as shown by Table 1.

In the context of the simulation results, the following points can be concluded: 1- For a fixed sample size, increasing the set size rather than the number of cycles has a positive effect on the biases and the MSEs of the estimators provided that the rankings are perfect done. 2- For a fixed set size increasing the sample size has a positive effect on the biases and the MSEs of the estimators regardless the quality ranking. 3- As expected, there is a

**Table 1:** Estimated biases and MSEs of the methods of estimation using the simulated data

|                   |        |                  | $\mathcal{M}$ |       |               |       |               |       | $\delta$   |       |               |       |               |       |
|-------------------|--------|------------------|---------------|-------|---------------|-------|---------------|-------|------------|-------|---------------|-------|---------------|-------|
| (k, m)            |        |                  | $\rho = 1$    |       | $\rho = 0.80$ |       | $\rho = 0.50$ |       | $\rho = 1$ |       | $\rho = 0.80$ |       | $\rho = 0.50$ |       |
|                   |        |                  | Bias          | MSE   | Bias          | MSE   | Bias          | MSE   | Bias       | MSE   | Bias          | MSE   | Bias          | MSE   |
| Standard Normal   | (2,5)  | $\hat{\theta}_L$ | 0.002         | 0.065 | 0.005         | 0.081 | 0.017         | 0.088 | 0.052      | 0.045 | 0.012         | 0.056 | 0.040         | 0.071 |
|                   |        | $\hat{\theta}_T$ | 0.003         | 0.066 | 0.005         | 0.083 | 0.015         | 0.087 | 0.222      | 0.089 | 0.234         | 0.098 | 0.241         | 0.099 |
|                   |        | $\hat{\theta}_C$ | 0.005         | 0.069 | 0.005         | 0.089 | 0.010         | 0.091 | 0.018      | 0.087 | 0.035         | 0.085 | 0.051         | 0.080 |
|                   |        | $\hat{\theta}_E$ | 0.002         | 0.065 | 0.005         | 0.081 | 0.017         | 0.087 | 0.065      | 0.046 | 0.053         | 0.057 | 0.051         | 0.054 |
|                   |        | $\hat{\theta}_X$ | 0.002         | 0.052 | 0.001         | 0.076 | 0.012         | 0.085 | 0.054      | 0.044 | 0.063         | 0.057 | 0.072         | 0.054 |
|                   |        | $\hat{\theta}_F$ | 0.005         | 0.053 | 0.001         | 0.085 | 0.004         | 0.091 | 0.033      | 0.048 | 0.055         | 0.085 | 0.071         | 0.092 |
|                   | (2,10) | $\hat{\theta}_L$ | 0.004         | 0.033 | 0.006         | 0.038 | 0.001         | 0.042 | 0.038      | 0.023 | 0.003         | 0.025 | 0.083         | 0.040 |
|                   |        | $\hat{\theta}_T$ | 0.004         | 0.033 | 0.008         | 0.038 | 0.001         | 0.042 | 0.142      | 0.043 | 0.152         | 0.045 | 0.144         | 0.043 |
|                   |        | $\hat{\theta}_C$ | 0.007         | 0.036 | 0.005         | 0.042 | 0.004         | 0.046 | 0.025      | 0.043 | 0.037         | 0.042 | 0.027         | 0.042 |
|                   |        | $\hat{\theta}_E$ | 0.004         | 0.033 | 0.007         | 0.038 | 0.002         | 0.041 | 0.046      | 0.024 | 0.027         | 0.024 | 0.001         | 0.028 |
|                   |        | $\hat{\theta}_X$ | 0.004         | 0.027 | 0.004         | 0.038 | 0.002         | 0.041 | 0.046      | 0.021 | 0.049         | 0.025 | 0.037         | 0.027 |
|                   |        | $\hat{\theta}_F$ | 0.003         | 0.029 | 0.002         | 0.037 | 0.008         | 0.045 | 0.033      | 0.021 | 0.043         | 0.036 | 0.028         | 0.044 |
|                   | (5,2)  | $\hat{\theta}_L$ | 0.007         | 0.036 | 0.017         | 0.060 | 0.003         | 0.079 | 0.039      | 0.031 | 0.066         | 0.062 | 0.282         | 0.194 |
|                   |        | $\hat{\theta}_T$ | 0.006         | 0.036 | 0.017         | 0.064 | 0.003         | 0.082 | 0.210      | 0.074 | 0.238         | 0.092 | 0.244         | 0.097 |
|                   |        | $\hat{\theta}_C$ | 0.001         | 0.038 | 0.016         | 0.066 | 0.004         | 0.089 | 0.008      | 0.061 | 0.031         | 0.076 | 0.056         | 0.083 |
|                   |        | $\hat{\theta}_E$ | 0.007         | 0.031 | 0.018         | 0.06  | 0.004         | 0.079 | 0.043      | 0.031 | 0.012         | 0.051 | 0.037         | 0.064 |
|                   |        | $\hat{\theta}_X$ | 0.008         | 0.022 | 0.022         | 0.055 | 0.002         | 0.084 | 0.055      | 0.031 | 0.07          | 0.046 | 0.082         | 0.051 |
|                   |        | $\hat{\theta}_F$ | 0.004         | 0.022 | 0.021         | 0.063 | 0.001         | 0.096 | 0.032      | 0.023 | 0.048         | 0.074 | 0.070         | 0.087 |
|                   | (5,4)  | $\hat{\theta}_L$ | 0.003         | 0.018 | 0.005         | 0.028 | 0.019         | 0.043 | 0.020      | 0.016 | 0.120         | 0.045 | 0.190         | 0.346 |
|                   |        | $\hat{\theta}_T$ | 0.005         | 0.019 | 0.005         | 0.028 | 0.019         | 0.043 | 0.124      | 0.031 | 0.136         | 0.040 | 0.043         | 0.142 |
|                   |        | $\hat{\theta}_C$ | 0.001         | 0.011 | 0.004         | 0.030 | 0.022         | 0.049 | 0.004      | 0.029 | 0.020         | 0.038 | 0.040         | 0.029 |
|                   |        | $\hat{\theta}_E$ | 0.003         | 0.011 | 0.006         | 0.028 | 0.02          | 0.043 | 0.021      | 0.016 | 0.074         | 0.033 | 0.052         | 0.052 |
|                   |        | $\hat{\theta}_X$ | 0.003         | 0.017 | 0.002         | 0.025 | 0.02          | 0.044 | 0.035      | 0.014 | 0.036         | 0.024 | 0.028         | 0.041 |
|                   |        | $\hat{\theta}_F$ | 0.002         | 0.016 | 0.001         | 0.027 | 0.022         | 0.052 | 0.018      | 0.001 | 0.031         | 0.038 | 0.044         | 0.039 |
| Standard Logistic | (2,5)  | $\hat{\theta}_L$ | 0.027         | 0.220 | 0.011         | 0.243 | 0.014         | 0.271 | 0.045      | 0.056 | 0.005         | 0.072 | 0.068         | 0.103 |
|                   |        | $\hat{\theta}_T$ | 0.025         | 0.223 | 0.011         | 0.244 | 0.015         | 0.271 | 0.214      | 0.094 | 0.233         | 0.104 | 0.249         | 0.108 |
|                   |        | $\hat{\theta}_C$ | 0.024         | 0.226 | 0.012         | 0.247 | 0.018         | 0.282 | 0.002      | 0.087 | 0.028         | 0.095 | 0.052         | 0.089 |
|                   |        | $\hat{\theta}_E$ | 0.032         | 0.225 | 0.009         | 0.261 | 0.015         | 0.271 | 0.084      | 0.062 | 0.065         | 0.072 | 0.069         | 0.073 |
|                   |        | $\hat{\theta}_X$ | 0.025         | 0.198 | 0.014         | 0.233 | 0.011         | 0.272 | 0.077      | 0.059 | 0.084         | 0.073 | 0.102         | 0.075 |
|                   |        | $\hat{\theta}_F$ | 0.020         | 0.174 | 0.017         | 0.232 | 0.010         | 0.281 | 0.039      | 0.047 | 0.049         | 0.098 | 0.071         | 0.094 |
|                   | (2,10) | $\hat{\theta}_L$ | 0.007         | 0.096 | 0.006         | 0.117 | 0.020         | 0.140 | 0.024      | 0.032 | 0.028         | 0.038 | 0.123         | 0.069 |
|                   |        | $\hat{\theta}_T$ | 0.008         | 0.096 | 0.008         | 0.118 | 0.019         | 0.140 | 0.131      | 0.047 | 0.130         | 0.048 | 0.131         | 0.050 |
|                   |        | $\hat{\theta}_C$ | 0.006         | 0.096 | 0.006         | 0.121 | 0.018         | 0.142 | 0.016      | 0.048 | 0.009         | 0.052 | 0.008         | 0.056 |
|                   |        | $\hat{\theta}_E$ | 0.006         | 0.106 | 0.009         | 0.12  | 0.023         | 0.15  | 0.041      | 0.038 | 0.017         | 0.036 | 0.011         | 0.045 |
|                   |        | $\hat{\theta}_X$ | 0.001         | 0.088 | 0.006         | 0.116 | 0.014         | 0.154 | 0.042      | 0.034 | 0.043         | 0.037 | 0.035         | 0.043 |
|                   |        | $\hat{\theta}_F$ | 0.004         | 0.077 | 0.011         | 0.110 | 0.010         | 0.141 | 0.021      | 0.024 | 0.026         | 0.045 | 0.011         | 0.057 |
|                   | (5,2)  | $\hat{\theta}_L$ | 0.014         | 0.100 | 0.011         | 0.206 | 0.013         | 0.243 | 0.043      | 0.045 | 0.123         | 0.093 | 0.371         | 0.352 |
|                   |        | $\hat{\theta}_T$ | 0.12          | 0.099 | 0.011         | 0.205 | 0.015         | 0.252 | 0.213      | 0.085 | 0.213         | 0.093 | 0.243         | 0.112 |
|                   |        | $\hat{\theta}_C$ | 0.014         | 0.099 | 0.013         | 0.209 | 0.015         | 0.263 | 0.006      | 0.069 | 0.023         | 0.093 | 0.042         | 0.123 |
|                   |        | $\hat{\theta}_E$ | 0.019         | 0.073 | 0.011         | 0.212 | 0.011         | 0.243 | 0.054      | 0.049 | 0.044         | 0.076 | 0.042         | 0.087 |
|                   |        | $\hat{\theta}_X$ | 0.019         | 0.105 | 0.023         | 0.203 | 0.009         | 0.253 | 0.081      | 0.048 | 0.068         | 0.074 | 0.092         | 0.073 |
|                   |        | $\hat{\theta}_F$ | 0.006         | 0.105 | 0.032         | 0.192 | 0.014         | 0.274 | 0.044      | 0.024 | 0.066         | 0.083 | 0.071         | 0.112 |
|                   | (5,4)  | $\hat{\theta}_L$ | 0.003         | 0.046 | 0.035         | 0.090 | 0.001         | 0.119 | 0.016      | 0.021 | 0.137         | 0.062 | 0.409         | 0.254 |
|                   |        | $\hat{\theta}_T$ | 0.004         | 0.047 | 0.035         | 0.089 | 0.001         | 0.121 | 0.122      | 0.036 | 0.134         | 0.045 | 0.139         | 0.047 |
|                   |        | $\hat{\theta}_C$ | 0.007         | 0.048 | 0.036         | 0.092 | 0.002         | 0.123 | 0.001      | 0.034 | 0.178         | 0.044 | 0.020         | 0.047 |
|                   |        | $\hat{\theta}_E$ | 0.008         | 0.047 | 0.037         | 0.094 | 0.001         | 0.122 | 0.026      | 0.022 | 0.078         | 0.043 | 0.126         | 0.07  |
|                   |        | $\hat{\theta}_X$ | 0.007         | 0.033 | 0.03          | 0.091 | 0.001         | 0.126 | 0.048      | 0.022 | 0.052         | 0.033 | 0.053         | 0.037 |
|                   |        | $\hat{\theta}_F$ | 0.003         | 0.030 | 0.028         | 0.083 | 0.002         | 0.128 | 0.018      | 0.009 | 0.018         | 0.044 | 0.026         | 0.052 |

**Table 1:** Continue

| $(k, m)$             |        |                  | $\mathcal{M}$ |       |               |       |               |       | $\delta$   |       |               |       |               |       |
|----------------------|--------|------------------|---------------|-------|---------------|-------|---------------|-------|------------|-------|---------------|-------|---------------|-------|
|                      |        |                  | $\rho = 1$    |       | $\rho = 0.80$ |       | $\rho = 0.50$ |       | $\rho = 1$ |       | $\rho = 0.80$ |       | $\rho = 0.50$ |       |
|                      |        |                  | Bias          | MSE   | Bias          | MSE   | Bias          | MSE   | Bias       | MSE   | Bias          | MSE   | Bias          | MSE   |
| Standard Gumbel      | (2,5)  | $\hat{\theta}_L$ | 0.049         | 0.089 | 0.009         | 0.084 | 0.009         | 0.109 | 0.066      | 0.059 | 0.010         | 0.060 | 0.048         | 0.096 |
|                      |        | $\hat{\theta}_T$ | 0.090         | 0.100 | 0.060         | 0.091 | 0.074         | 0.119 | 0.230      | 0.102 | 0.223         | 0.100 | 0.235         | 0.109 |
|                      |        | $\hat{\theta}_C$ | 0.045         | 0.099 | 0.016         | 0.094 | 0.036         | 0.123 | 0.016      | 0.100 | 0.015         | 0.110 | 0.031         | 0.110 |
|                      |        | $\hat{\theta}_E$ | 0.069         | 0.092 | 0.031         | 0.086 | 0.047         | 0.114 | 0.104      | 0.075 | 0.062         | 0.085 | 0.068         | 0.093 |
|                      |        | $\hat{\theta}_X$ | 0.056         | 0.069 | 0.042         | 0.08  | 0.057         | 0.115 | 0.114      | 0.059 | 0.087         | 0.077 | 0.099         | 0.099 |
|                      |        | $\hat{\theta}_F$ | 0.040         | 0.072 | 0.031         | 0.084 | 0.040         | 0.122 | 0.038      | 0.059 | 0.038         | 0.103 | 0.041         | 0.120 |
|                      | (2,10) | $\hat{\theta}_L$ | 0.006         | 0.037 | 0.014         | 0.046 | 0.005         | 0.064 | 0.035      | 0.025 | 0.007         | 0.031 | 0.064         | 0.088 |
|                      |        | $\hat{\theta}_T$ | 0.021         | 0.038 | 0.049         | 0.049 | 0.044         | 0.055 | 0.142      | 0.045 | 0.152         | 0.050 | 0.139         | 0.044 |
|                      |        | $\hat{\theta}_C$ | 0.003         | 0.043 | 0.029         | 0.053 | 0.024         | 0.062 | 0.028      | 0.045 | 0.037         | 0.049 | 0.020         | 0.044 |
|                      |        | $\hat{\theta}_E$ | 0.006         | 0.039 | 0.028         | 0.047 | 0.023         | 0.055 | 0.041      | 0.046 | 0.031         | 0.041 | 0.006         | 0.047 |
|                      |        | $\hat{\theta}_X$ | 0.009         | 0.029 | 0.029         | 0.042 | 0.035         | 0.053 | 0.057      | 0.032 | 0.067         | 0.041 | 0.056         | 0.048 |
|                      |        | $\hat{\theta}_F$ | 0.001         | 0.030 | 0.022         | 0.046 | 0.025         | 0.055 | 0.026      | 0.022 | 0.036         | 0.045 | 0.026         | 0.046 |
|                      | (5,2)  | $\hat{\theta}_L$ | 0.007         | 0.036 | 0.006         | 0.064 | 0.044         | 0.108 | 0.050      | 0.038 | 0.069         | 0.070 | 0.299         | 0.240 |
|                      |        | $\hat{\theta}_T$ | 0.045         | 0.039 | 0.075         | 0.071 | 0.087         | 0.116 | 0.218      | 0.081 | 0.234         | 0.100 | 0.238         | 0.102 |
|                      |        | $\hat{\theta}_C$ | 0.002         | 0.038 | 0.031         | 0.073 | 0.042         | 0.114 | 0.001      | 0.068 | 0.032         | 0.095 | 0.029         | 0.091 |
|                      |        | $\hat{\theta}_E$ | 0.02          | 0.036 | 0.026         | 0.066 | 0.026         | 0.105 | 0.068      | 0.058 | 0.02          | 0.09  | 0.058         | 0.131 |
|                      |        | $\hat{\theta}_X$ | 0.024         | 0.025 | 0.047         | 0.064 | 0.071         | 0.114 | 0.11       | 0.043 | 0.101         | 0.075 | 0.086         | 0.099 |
|                      |        | $\hat{\theta}_F$ | 0.010         | 0.023 | 0.034         | 0.070 | 0.026         | 0.118 | 0.034      | 0.025 | 0.065         | 0.081 | 0.035         | 0.098 |
|                      | (5,4)  | $\hat{\theta}_L$ | 0.007         | 0.020 | 0.025         | 0.034 | 0.082         | 0.056 | 0.027      | 0.018 | 0.013         | 0.045 | 0.341         | 0.196 |
|                      |        | $\hat{\theta}_T$ | 0.027         | 0.022 | 0.029         | 0.036 | 0.039         | 0.052 | 0.129      | 0.034 | 0.017         | 0.043 | 0.160         | 0.049 |
|                      |        | $\hat{\theta}_C$ | 0.004         | 0.022 | 0.008         | 0.039 | 0.015         | 0.054 | 0.011      | 0.033 | 0.014         | 0.041 | 0.043         | 0.045 |
|                      |        | $\hat{\theta}_E$ | 0.013         | 0.02  | 0.011         | 0.036 | 0.013         | 0.051 | 0.032      | 0.028 | 0.071         | 0.047 | 0.102         | 0.078 |
|                      |        | $\hat{\theta}_X$ | 0.018         | 0.012 | 0.021         | 0.034 | 0.026         | 0.048 | 0.071      | 0.019 | 0.066         | 0.033 | 0.076         | 0.045 |
|                      |        | $\hat{\theta}_F$ | 0.006         | 0.011 | 0.013         | 0.034 | 0.014         | 0.050 | 0.023      | 0.010 | 0.022         | 0.038 | 0.047         | 0.052 |
| Standard Exponential | (2,5)  | $\hat{\theta}_L$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.074      | 0.101 | 0.057         | 0.130 | 0.023         | 0.112 |
|                      |        | $\hat{\theta}_T$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.187      | 0.156 | 0.197         | 0.217 | 0.201         | 0.207 |
|                      |        | $\hat{\theta}_C$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.060      | 0.122 | 0.095         | 0.251 | 0.088         | 0.187 |
|                      |        | $\hat{\theta}_E$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.084      | 0.105 | 0.082         | 0.139 | 0.078         | 0.135 |
|                      |        | $\hat{\theta}_X$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.09       | 0.085 | 0.097         | 0.129 | 0.1           | 0.141 |
|                      |        | $\hat{\theta}_F$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.048      | 0.087 | 0.083         | 0.172 | 0.098         | 0.215 |
|                      | (2,10) | $\hat{\theta}_L$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.037      | 0.044 | 0.016         | 0.047 | 0.009         | 0.056 |
|                      |        | $\hat{\theta}_T$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.105      | 0.065 | 0.111         | 0.070 | 0.116         | 0.085 |
|                      |        | $\hat{\theta}_C$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.031      | 0.054 | 0.031         | 0.058 | 0.052         | 0.084 |
|                      |        | $\hat{\theta}_E$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.04       | 0.045 | 0.029         | 0.049 | 0.031         | 0.061 |
|                      |        | $\hat{\theta}_X$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.048      | 0.035 | 0.051         | 0.046 | 0.056         | 0.067 |
|                      |        | $\hat{\theta}_F$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.029      | 0.039 | 0.034         | 0.051 | 0.058         | 0.088 |
|                      | (5,2)  | $\hat{\theta}_L$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.043      | 0.043 | 0.019         | 0.067 | 0.015         | 0.099 |
|                      |        | $\hat{\theta}_T$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.152      | 0.077 | 0.185         | 0.134 | 0.221         | 0.204 |
|                      |        | $\hat{\theta}_C$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.044      | 0.068 | 0.070         | 0.116 | 0.095         | 0.173 |
|                      |        | $\hat{\theta}_E$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.045      | 0.043 | 0.038         | 0.073 | 0.064         | 0.124 |
|                      |        | $\hat{\theta}_X$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.066      | 0.031 | 0.093         | 0.086 | 0.132         | 0.151 |
|                      |        | $\hat{\theta}_F$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.028      | 0.028 | 0.064         | 0.109 | 0.112         | 0.203 |
|                      | (5,4)  | $\hat{\theta}_L$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.024      | 0.019 | 0.026         | 0.032 | 0.098         | 0.046 |
|                      |        | $\hat{\theta}_T$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.091      | 0.032 | 0.094         | 0.050 | 0.081         | 0.060 |
|                      |        | $\hat{\theta}_C$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.015      | 0.021 | 0.018         | 0.045 | 0.008         | 0.060 |
|                      |        | $\hat{\theta}_E$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.024      | 0.018 | 0.012         | 0.032 | 0.032         | 0.044 |
|                      |        | $\hat{\theta}_X$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.0947     | 0.014 | 0.044         | 0.033 | 0.023         | 0.047 |
|                      |        | $\hat{\theta}_F$ | ---           | ---   | ---           | ---   | ---           | ---   | 0.017      | 0.014 | 0.027         | 0.041 | 0.006         | 0.057 |

touchable effect of the values of  $\rho$  on the behavior of the considered estimators, as increasing the values of  $\rho$  decreases the biases and MSEs of all estimators and vice versa.

4- Concerned about estimating  $\mathcal{M}$ , it is clear to say that the differences between the performance of the considered estimators are quite close and can be ignorable. Therefore, it does not matter which estimation method should be adopted for estimating  $\mathcal{M}$ .

5- On the other hand, concerned about estimating  $\delta$ , it is easily to see that the behavior of the estimators has various patterns. Under the perfect ranking,  $\hat{\theta}_L$  and  $\hat{\theta}_F$  are the best estimators with respect to the bias and the MSE criteria with a few exceptions. Whereas this superiority decreases in favor to  $\hat{\theta}_X$  and  $\hat{\theta}_E$  as the quality of the ranking goes to the imperfect case.

6- Although both  $\hat{\theta}_L$  and  $\hat{\theta}_E$  deal with the RSS items as independent order statistics, it is apparent that  $\hat{\theta}_E$  is much less sensitive to the perfectness assumption than  $\hat{\theta}_L$ . The reason for this phenomenon is that  $\hat{\theta}_E$  uses the fraction-of-random-rankings model which reduces the effect of violating the perfectness assumption.

7- It is also evident that  $\hat{\theta}_X$  outperforms  $\hat{\theta}_C$  in all the situations as long as the rankings are done either perfect or nearly perfect.

## 5. Efficiency Comparison using Empirical Data Set

In what follows, we assess the performances of  $\hat{\theta}_L$ ,  $\hat{\theta}_T$ ,  $\hat{\theta}_C$ ,  $\hat{\theta}_E$ ,  $\hat{\theta}_X$  and  $\hat{\theta}_F$  estimators using the empirical data set known as Veterans' Administration Lung Cancer Study (WALCS) data set and is available in survival package of R statistical software. We will consider this data set as the target population which includes eight variables with 137 observations yet only two variables will be considered: the "karno" as the concomitant variable  $X$  and the "time" as the variable of interest  $Y$ . These two variables are not highly correlated as their Pearson correlation coefficient is only 37.5%, thus one can expect that if RSS is collected it will be nearly imperfect.

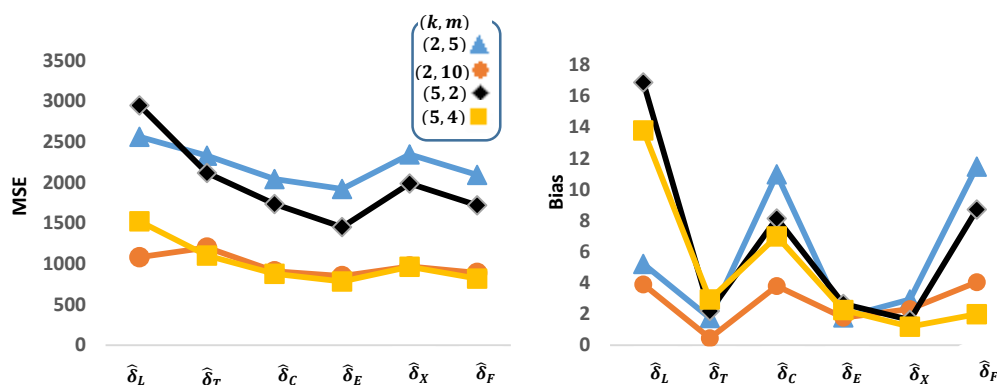


**Fig. 1:** Estimated biases and MSEs of the methods of estimation using the empirical data. This figure appears in color in the electronic version of this article.

In the light of the fact that Zamanzade (2018) demonstrated that  $Y$  follows the exponential distribution with  $\delta = 121.67$  using CDF based goodness of fit tests, another comparison study is performed by the following procedures. For the same values of  $(k, m)$  determined in the preceding Section, 5,000 concomitant-based RSS with replacement were selected from the target population. For each sample,  $\hat{\delta}_L$ ,  $\hat{\delta}_T$ ,  $\hat{\delta}_C$ ,  $\hat{\delta}_E$ ,  $\hat{\delta}_X$  and  $\hat{\delta}_F$  estimators were estimated and their biases and MSEs were also obtained and plotted as presented by Fig.1. One can easily deduce that there is a high agreement between the results shown by Fig.1 and those reported by Table 1. As due to the weakness of the relation between  $X$  and  $Y$ , for a fixed sample size, increasing the cycle size rather than the set size has a better effect on all the considered estimators particularly  $\hat{\delta}_L$  which is uniformly the worst estimator in terms of both the bias and MSE criteria. In contradiction,  $\hat{\delta}_E$  and  $\hat{\delta}_X$  have satisfied behavior particularly at the small sample sizes yet  $\hat{\delta}_F$  has relatively high bias in almost cases. In spite of the superiority of  $\hat{\delta}_T$  in terms of bias criterion, it has a slight higher MSE. Finally it should be informed that all simulation studies in this work are programmed using R statistical software and available under request from the second author.

## 6. Conclusion

In this study, we provide three novel methods for estimating location-scale parameters under RSS namely Iterative Moments, Modified Moments and Modified Cramér–Von Mises. The first method is based on EM algorithm, while the second and the third methods are derived under existing of the concomitant based-information. In the view of bias and MSE, the numerical findings recommended that the proposed methods are more efficient than their competitors in almost cases and they are relatively less sensitive to the perfectness assumption. Although we confine our attention on the location-scale family, the proposed methods can be straightforwardly extended to the non location-scale family either by directly applying the proposed methods or transforming to a distribution belongs to the location-scale family. For instance, if  $Y$  follows a Weibull distribution then  $\{-\log(Y)\}$  follows a Gumbel distribution (see Cabana and Quiroz (2005)). Recently, Ozturk et al. (2018) constructed a Rao-Blackwellized version of maximum likelihood

estimators (MLE) and best linear unbiased estimators (BLUE) under judgment post stratified (JPS) samples, therefore it may be a good potential point for future works to extend our proposed methods to JPS samples.

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# Estimation of Parameters of Generalized Inverted Exponential Distribution Under Progressive Type-I Censoring

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**Abstract** In this paper, we consider generalized inverted exponential distribution which is capable of modeling various shapes of failure rates and aging processes. Based on progressive type-I censored data, we consider the problem of estimation of parameters under classical and Bayesian approaches. In this regard, we obtain maximum likelihood estimates, and Bayes estimates under squared error loss function. We also compute 95% asymptotic confidence interval and highest posterior density interval estimates. Finally, we analyze a real data set and conduct a Monte Carlo simulation study to compare the performance of the various proposed estimators.

**Keywords** Asymptotic confidence interval, Bayesian estimation, Generalized inverted exponential distribution, Highest posterior density interval, Maximum likelihood estimation, Monte Carlo Markov Chain, Metropolis-Hasting algorithm, Progressive type-I censoring scheme.

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# 1 Introduction

Inverted distributions have been introduced to overcome some disadvantages of many widely used distributions in reliability and survival analysis. These disadvantages include constant hazard (failure) rates of an exponential distribution and non-closed form of some distribution functions such as gamma distribution. The inverted exponential (IE) distribution has been studied and used to overcome the restriction of the constant hazard rate. Lin *et al.* (1989) studied the properties of the IE distribution such as: reliability function, hazard rate, and estimation of the parameters by using maximum likelihood method. While, Dey (2007) studied the IE distribution from the Bayesian viewpoint depended on squared error and LINEX loss functions. The generalized inverted exponential (GIE) distribution was proposed as another useful two-parameter generalization of the inverted exponential distribution (Abouammoh and Alshingiti (2009)). This lifetime distribution is capable of modeling diverse shapes of failure rates, and thus various shapes of aging criteria.

The probability density function (pdf) and cumulative distribution function (cdf) of the GIE distribution are respectively given by

$$f(x; \alpha, \lambda) = \frac{\alpha\lambda}{x^2} \exp\left(\frac{-\lambda}{x}\right) \left[1 - \exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha-1}, \quad x > 0, \quad \lambda, \alpha > 0 \quad (1)$$

and

$$F(x; \alpha, \lambda) = 1 - \left[1 - \exp\left(\frac{-\lambda}{x}\right)\right]^\alpha, \quad (2)$$

where  $\alpha$  is the shape parameter and  $\lambda$  is the scale parameter. If  $\alpha = 1$  then GIE distribution reduces to IE distribution. Figure (1) illustrated the behavior of the GIE distribution at  $\lambda = 1$  and for some various values of  $\alpha$ .

The hazard rate function of GIE distribution can be obtained from

$$h(x) = \frac{\alpha\lambda}{x^2 \left(\exp\left(\frac{\lambda}{x}\right) - 1\right)}$$

and its shape is illustrated in Figure (2) at  $\lambda = 1$  and for some various values of  $\alpha$ .

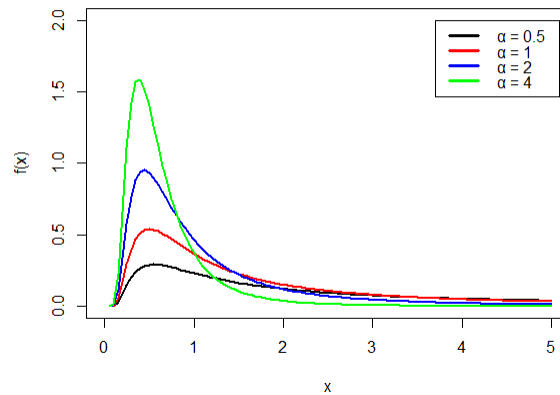


Figure 1: Density function of GIE distribution for some values of  $\alpha$

Several interesting characteristics and properties of GIE distribution have been studied by Abouammoh and Alshingiti (2009) in details. The hazard rate function of GIE distribution can be increasing in the beginning of the ageing and decreasing in the end of the ageing but never constant relying on the shape parameter. The GIE distribution has a unimodal and right skewed density function. Furthermore, they supply utilizations where the GIE distribution provides a better fit than gamma, Weibull, generalized exponential and IE distributions (see Abouammoh and Alshingiti (2009)). According to recent contributions to this distribution, Krishna and Kumar (2013), Dey and Dey (2013; 2014), Dey and Pradhan (2014), Dey *et al.* (2016), Singh *et al.* (2013), Singh *et al.* (2015), and Dube *et al.* (2015) should be mentioned.

In practical life testing experiment, censored data arise when the experiments including the lifetimes of test units have to be terminated before collecting complete observation. The censoring technique is common and unavoidable in practice, especially in reliability engineering, for many reasons such as time constraint and cost reduction. Various kinds of censoring have been discussed in the literature, with the most common censoring schemes being type-I censoring and type-II censoring. More recently, a generalized form of censoring named progressive censoring schemes has gained significant attention in

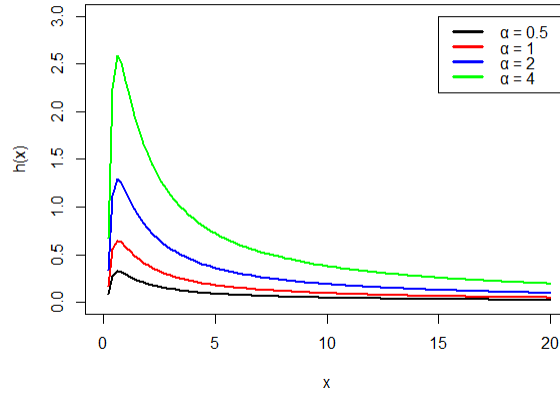


Figure 2: Hazard rate function of GIE distribution for some values of  $\alpha$

the literature because of its effective utilization of the available resources in comparison with traditional censoring designs. One of these progressive censoring schemes is called progressive type-I. This scheme is observed when a pre-fixed number of life test units are continuously removed through the experiment at the end of each of pre-specified time intervals. It provides both the practical feature of knowing the termination time and the larger flexibility to the experimenter in the phase of design by allowing eliminating the test units at non-terminal time points (Balakrishnan *et al.* (2011)).

Suppose  $n$  units are placed on a life testing experiment. Further, suppose  $X_1, X_2, \dots, X_n$  denote the lifetimes of these  $n$  units taken from a population with cdf  $F(x; \theta)$  and pdf  $f(x; \theta)$ , where  $\theta$  is a vector of unknown parameters of the distribution. Let  $x_{(1)} < x_{(2)} < \dots < x_{(n)}$  denote the corresponding ordered lifetimes observed from the life test. Progressive type-I censoring is observed when  $R_i$  items are removed from the survived items at the predetermined time of censoring  $T_{q_i}$  corresponding to the  $q_i$ th quantiles,  $i = 1, 2, \dots, m$ , where  $m$  is the number of stages in the test,  $T_{q_i} > T_{q_{i-1}}$  and  $n = r + \sum_{i=1}^m R_i$ . The values  $T_{q_i}$  are to be predetermined by:

1. The experimenter, the choice of these times depend on the prior knowledge and ex-

perience of the experimenter about the items on test (Balasooriya and Low (2004)),  
or

2. The  $q_i$ th quantiles of lifetimes distribution

$$P(X \leq T_{q_i}) = q_i \quad \Rightarrow \quad T_{q_i} = F^{-1}(q_i)$$

where  $i = 1, 2, \dots, m$  and  $F^{-1}(\cdot)$  is the inverse of the cdf.

In these situations  $R_i, T_{q_i}$  and  $n$  are fixed and predetermined while  $l_i$  is the number of the survivor items at time  $T_{q_i}$  and  $r = \sum_{i=1}^m l_i$  are random variables. The likelihood function in this case is given by

$$L(\theta) \propto \prod_{i=1}^r f(x_{(i)}; \theta) \prod_{j=1}^m (1 - F(T_{(q_j)}; \theta))^{R_j} \quad (3)$$

where  $x_{(i)}$  is the observed lifetime of the  $i$ th order statistic (Cohen (1963)). Figure (3) describe this scheme of censoring (Balakrishnan and Cramer (2010)).

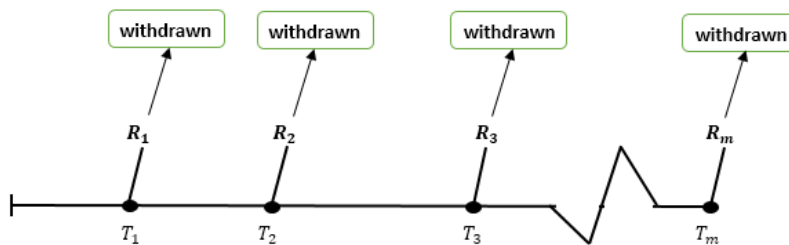


Figure 3: Progressive Type-I Censoring Scheme

Thus, the type-I and complete samples can also be considered as special cases of this censoring. For more details and implementation of this kind of censoring, one may mention Balakrishnan and Cramer (2010), Balakrishnan *et al.* (2011), and Balasooriya and Low (2004).

Krishna and Kumar (2013) tackled in their study GIE distribution under progressive type-II censoring. They obtained the maximum likelihood estimator and associated asymptotic confidence intervals for the unknown parameters, and also discussed reliability characteristics and other distribution properties. Recently, Dey and Dey (2014) discussed Bayes estimates and the associated highest posterior density interval estimates. They have also proved the uniqueness and existence of the maximum likelihood estimators. One may further mention to Singh *et al.* (2013) for random removal case.

This article's goal is to elicit techniques of estimation, maximum likelihood estimators (MLE) and Bayes estimators, concerning a life-test from which the available data are progressively type-I censored. Supposing that the lifetime of the test units are independently GIE distributed, we obtained the MLEs of the unknown parameters of  $GIE(\lambda, \alpha)$  distribution in Section 2. By assuming gamma prior distributions and squared error loss function, we obtained Bayes estimates using Metropolis-Hasting algorithm and the associated highest posterior density interval estimates in Section 3. Finally, a real data set is analyzed for illustrative purposes and also Monte Carlo simulations are carried out to explore the performance of the introduced estimators and comments are obtained depended on this study in Section 4.

## 2 Maximum Likelihood Estimation

In this section, we obtain MLEs for the unknown parameters of the GIE distribution based on progressive type-I censored data. The implementation of progressive type-I censoring scheme can be implement as follows:

- Suppose that a random sample of  $n$  units whose lifetimes follows  $GIE(\lambda, \alpha)$  distribution is put on a life test experiment.
- Prefix  $m$  censoring time points  $T_{q_1}, \dots, T_{q_m}$ , at which fixed number  $R_1, \dots, \dots, R_{m-1}$  of surviving items are randomly removed from the test. The censoring times  $T_{q_i}$  are

selected corresponding to  $P(X \leq T_{q_i}) = q_i$ , where  $X$  follows  $\text{GIE}(\lambda, \alpha)$  distribution.

- The life test is terminates at a pre-specified time  $T_{q_m}$ .

Therefore, we can obtain the progressive type-I censored samples  $\mathbf{x} = (x_{(1)}, x_{(2)}, \dots, x_{(m)})$  that represent the observed lifetimes of the  $n$  units under this scheme of censoring. Then the associated likelihood function of  $\lambda$  and  $\alpha$  given the observed data  $\mathbf{x}$  can be written as

$$L(\lambda, \alpha | \mathbf{x}) \propto (\alpha \lambda)^r \prod_{i=1}^r \left( \frac{\exp\left(\frac{-\lambda}{x_{(i)}}\right)}{x_{(i)}^2} \left[ 1 - \exp\left(\frac{-\lambda}{x_{(i)}}\right) \right]^{\alpha-1} \right) \prod_{j=1}^m \left( \left[ 1 - \exp\left(\frac{-\lambda}{T_{(j)}}\right) \right]^{\alpha} \right)^{R_j} \quad (4)$$

Taking logarithm of  $L(\lambda, \alpha | \mathbf{x})$  to obtain log-likelihood  $\ell$  as

$$\begin{aligned} \ell &\propto r \log \alpha + r \log \lambda + \sum_{i=1}^r \left( -\frac{\lambda}{x_{(i)}} - \log x_{(i)}^2 + (\alpha - 1) \log \left[ 1 - \exp\left(\frac{-\lambda}{x_{(i)}}\right) \right] \right) \\ &+ \alpha \sum_{j=1}^m \left( R_j \log \left[ 1 - \exp\left(\frac{-\lambda}{T_{(j)}}\right) \right] \right) \end{aligned}$$

Differentiating the associated log-likelihood  $\ell$  with respect to  $\lambda$  and  $\alpha$  and equating them to zero, we get:

$$\begin{aligned} \frac{r}{\hat{\lambda}} + \sum_{i=1}^r \frac{\hat{\alpha} - \exp\left(\frac{\hat{\lambda}}{x_{(i)}}\right)}{x_{(i)} \left[ \exp\left(\frac{\hat{\lambda}}{x_{(i)}}\right) - 1 \right]} + \hat{\alpha} \sum_{j=1}^m \frac{R_j}{T_{(j)} \left[ \exp\left(\frac{\hat{\lambda}}{T_{(j)}}\right) - 1 \right]} &= 0, \\ \frac{r}{\hat{\alpha}} + \sum_{i=1}^r \log \left[ 1 - \exp\left(\frac{-\hat{\lambda}}{x_{(i)}}\right) \right] + \sum_{j=1}^m R_j \log \left[ 1 - \exp\left(\frac{-\hat{\lambda}}{T_{(j)}}\right) \right] &= 0, \end{aligned}$$

where  $\hat{\lambda}$  and  $\hat{\alpha}$  are the MLEs of  $\lambda$  and  $\alpha$  respectively. Now, the asymptotic variance-covariance matrix of the MLEs of  $\lambda$  and  $\alpha$  can be obtained by inverting the observed information matrix, and is given by

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} -E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) & -E\left(\frac{\partial^2 \ell}{\partial \lambda \partial \alpha}\right) \\ -E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \lambda}\right) & -E\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right) \end{bmatrix}_{\lambda=\hat{\lambda}, \alpha=\hat{\alpha}}^{-1}$$

where

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \lambda^2} &= -\frac{r}{\lambda^2} - (\alpha - 1) \sum_{i=1}^r \frac{\exp\left(\frac{\lambda}{x_{(i)}}\right)}{x_{(i)}^2 \left[\exp\left(\frac{\lambda}{x_{(i)}}\right) - 1\right]^2} - \alpha \sum_{j=1}^m \frac{R_j \exp\left(\frac{\lambda}{T_{(j)}}\right)}{T_{(j)}^2 \left[\exp\left(\frac{\lambda}{T_{(j)}}\right) - 1\right]^2} \\ \frac{\partial^2 \ell}{\partial \alpha^2} &= -\frac{r}{\alpha^2} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} &= \sum_{i=1}^r \frac{1}{x_{(i)} \left[\exp\left(\frac{\lambda}{x_{(i)}}\right) - 1\right]} + \sum_{j=1}^m \frac{R_j}{T_{(j)} \left[\exp\left(\frac{\lambda}{T_{(j)}}\right) - 1\right]}\end{aligned}$$

According to particular regularity conditions, the two-sided  $100(1 - \gamma)\%$ ,  $0 < \gamma < 1$ , asymptotic confidence intervals for the parameters  $\lambda$  and  $\alpha$  can be obtained as:

$$\begin{aligned}\hat{\lambda} \pm Z_{\frac{\gamma}{2}} \sqrt{\sigma_{11}} \\ \hat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{\sigma_{22}}\end{aligned}$$

where  $\sigma_{11}$  and  $\sigma_{22}$  are the asymptotic variances of the MLEs of  $\lambda$  and  $\alpha$ , respectively, here  $Z_{\frac{\gamma}{2}}$  is the upper  $\frac{\gamma}{2}$ th percentile of the standard normal distribution.

### 3 Bayesian Estimation

In this section, we discuss the Bayesian inference of the unknown parameters of a GIE distribution. For Bayesian parameter estimation we will consider squared error loss function. It can be noted that if the scale parameter  $\lambda$  is known, the shape parameter  $\alpha$  has a conjugate prior, which is a gamma prior. When both the parameters of the model are unknown, a joint conjugate prior for the parameters does not exist. We suggest using independent gamma priors for both  $\lambda$  and  $\alpha$  having pdfs (Dey *et al.* (2016))

$$\begin{aligned}\pi_1(\lambda) &\propto \lambda^{a_1-1} \exp(-b_1\lambda) & \lambda > 0, a_1 > 0, b_1 > 0 \\ \pi_2(\alpha) &\propto \alpha^{a_2-1} \exp(-b_2\alpha) & \alpha > 0, a_2 > 0, b_2 > 0\end{aligned}$$

where the hyper-parameters  $a_1, b_1, a_2, b_2$  are selected to reflect the prior knowledge about

the unknown parameters. The joint prior for  $\lambda$  and  $\alpha$  is given by

$$\begin{aligned}\pi(\lambda, \alpha) &= \pi_1(\lambda)\pi_2(\alpha) \\ \pi(\lambda, \alpha) &\propto \lambda^{a_1-1}\alpha^{a_2-1}\exp(-b_1\lambda - b_2\alpha)\end{aligned}$$

**Hyper-parameter elicitation:** The elicitation of the hyper-parameters will relies on the informative priors. These informative priors will be obtained from the maximum likelihood estimates for  $(\lambda, \alpha)$  by equating the mean and variance of  $(\hat{\lambda}^j, \hat{\alpha}^j)$  with the mean and variance of the considered priors (Gamma priors), where  $j = 1, 2, \dots, k$  and  $k$  is the number of samples available from the generalized inverted exponential distribution. Thus, on equating mean and variance of  $(\hat{\lambda}^j, \hat{\alpha}^j)$  with the mean and variance of gamma priors, we get (Dey *et al.* (2016))

$$\begin{aligned}\frac{1}{k} \sum_{j=1}^k \hat{\lambda}^j &= \frac{a_1}{b_1} & \& \quad \frac{1}{k-1} \sum_{j=1}^k \left( \hat{\lambda}^j - \frac{1}{k} \sum_{j=1}^k \hat{\lambda}^j \right)^2 = \frac{a_1}{b_1^2} \\ \frac{1}{k} \sum_{j=1}^k \hat{\alpha}^j &= \frac{a_2}{b_2} & \& \quad \frac{1}{k-1} \sum_{j=1}^k \left( \hat{\alpha}^j - \frac{1}{k} \sum_{j=1}^k \hat{\alpha}^j \right)^2 = \frac{a_2}{b_2^2}.\end{aligned}$$

Now on solving the above two equations, the estimated hyper-parameters can be written as

$$\begin{aligned}a_1 &= \frac{\left( \frac{1}{k} \sum_{j=1}^k \hat{\lambda}^j \right)^2}{\frac{1}{k-1} \sum_{j=1}^k \left( \hat{\lambda}^j - \frac{1}{k} \sum_{j=1}^k \hat{\lambda}^j \right)^2} & \& \quad b_1 = \frac{\frac{1}{k} \sum_{j=1}^k \hat{\lambda}^j}{\frac{1}{k-1} \sum_{j=1}^k \left( \hat{\lambda}^j - \frac{1}{k} \sum_{j=1}^k \hat{\lambda}^j \right)^2} \\ a_2 &= \frac{\left( \frac{1}{k} \sum_{j=1}^k \hat{\alpha}^j \right)^2}{\frac{1}{k-1} \sum_{j=1}^k \left( \hat{\alpha}^j - \frac{1}{k} \sum_{j=1}^k \hat{\alpha}^j \right)^2} & \& \quad b_2 = \frac{\frac{1}{k} \sum_{j=1}^k \hat{\alpha}^j}{\frac{1}{k-1} \sum_{j=1}^k \left( \hat{\alpha}^j - \frac{1}{k} \sum_{j=1}^k \hat{\alpha}^j \right)^2}.\end{aligned}$$

The corresponding posterior density given the observed data  $\mathbf{x} = (x_{(1)}, x_{(2)}, \dots, x_{(r)})$  can be written as

$$\pi(\lambda, \alpha | \mathbf{x}) = \frac{\pi(\lambda, \alpha) L(\lambda, \alpha)}{\int_0^\infty \int_0^\infty \pi(\lambda, \alpha) L(\lambda, \alpha) d\lambda d\alpha}$$

The posterior density function can be written as

$$\pi(\lambda, \alpha | \mathbf{x}) = K^{-1} \left[ \lambda^{r+a_1-1} \alpha^{r+a_2-1} \exp(-b_1\lambda - b_2\alpha) \prod_{i=1}^r \frac{\exp(\frac{-\lambda}{x_i})}{x_i^2} \left[ 1 - \exp\left(\frac{-\lambda}{x_i}\right) \right]^{\alpha-1} \prod_{j=1}^m \left[ 1 - \exp\left(\frac{-\lambda}{T_j}\right) \right]^{\alpha R_j} \right]$$

where

$$K = \int_0^\infty \int_0^\infty \lambda^{r+a_1-1} \alpha^{r+a_2-1} \exp(-b_1\lambda - b_2\alpha) \prod_{i=1}^r \frac{\exp(\frac{-\lambda}{x_i})}{x_i^2} \left[ 1 - \exp\left(\frac{-\lambda}{x_i}\right) \right]^{\alpha-1} \prod_{j=1}^m \left[ 1 - \exp\left(\frac{-\lambda}{T_j}\right) \right]^{\alpha R_j} d\lambda d\alpha$$

Thus, the posterior density can be rewritten as

$$\pi(\lambda, \alpha | \mathbf{x}) \propto \lambda^{r+a_1-1} \alpha^{r+a_2-1} \exp(-b_1\lambda - b_2\alpha) \prod_{i=1}^r \frac{\exp(\frac{-\lambda}{x_i})}{x_i^2} \left[ 1 - \exp\left(\frac{-\lambda}{x_i}\right) \right]^{\alpha-1} \prod_{j=1}^m \left[ 1 - \exp\left(\frac{-\lambda}{T_j}\right) \right]^{\alpha R_j} \quad (5)$$

**The Bayes Estimator** of any loss function, say  $g(\lambda, \alpha)$  under the squared error, is given by

$$\tilde{g}(\lambda, \alpha) = \int_0^\infty \int_0^\infty g(\lambda, \alpha) \pi(\lambda, \alpha | x) d\lambda d\alpha \quad (6)$$

Unfortunately, Eq. (6) cannot be computed for general  $g(\lambda, \alpha)$ . Therefore, we suggest the most common approximate Bayes estimates of  $\lambda$  and  $\alpha$  Markov Chain Monte Carlo (MCMC).

Markov Chain Monte Carlo (MCMC) is considered to be a computer-driven sampling technique. It permits one to characterize a distribution without knowing all of the distribution mathematical properties by random sampling values out of the distribution (Ravenszwaaij *et al.* (2018)). MCMC is specifically beneficial in Bayesian inference as a result of focusing on posterior distributions that are often hard to work with through analytic examination. In these cases, MCMC allows the user to approximate aspects of

posterior distributions that cannot be directly computed (e.g., random samples from the posterior, posterior means, etc.). To draw samples from a distribution using MCMC:

1. Beginning with an initial guess: just one value that might be plausibly drawn from the distribution.
2. Generating a series of new samples from this initial guess. Every new sample is generated by two steps:
  - Proposal: a proposal for the new sample is generated by adding a small random perturbation to the most recent sample.
  - Acceptance: the new proposal is either accepted as the new sample, or rejected (in which case the old sample retained).

There are many ways of adding random noise to generate proposals, and also diverse techniques to the process of accepting and rejecting, such as: Gibbs-sampling and Metropolis-Hastings algorithm.

### 3.1 Metropolis-Hasting algorithm

Metropolis-Hastings (MH) algorithm is considered to be a beneficial technique for generating random samples from the posterior distribution using a proposal density. To perform the MH algorithm we have to define a proposal distribution  $q(\theta'|\theta)$  and an initial values  $\theta^{(0)}$  of the unknown parameters. For the proposal distribution, we consider a bivariate normal distribution, that is  $q(\theta'|\theta) \equiv N_2(\theta, S_\theta)$ , where  $\theta = (\lambda, \alpha)$  and  $S_\theta$  represent the variance-covariance matrix, we may obtain negative observations that are unacceptable. For the initial values, we guess an appropriate values to  $\lambda$  and  $\alpha$ . Thus, we perform the following steps of MH algorithm to draw sample from the posterior density given by Eq. (5) (Dey and Pradhan (2014)).

**Step 1.** Set initial value of  $\theta$  as  $\theta = \theta^{(0)}$ .

**Step 2.** For  $i = 1, 2, \dots, M$  repeat the following steps:

1. Set  $\theta = \theta^{(i-1)}$ .
2. Generate a new candidate parameter value  $\delta$  from  $N_2(\ln \theta, S_\theta)$ .
3. Set  $\theta' = \exp(\delta)$ .
4. Calculate  $\beta = \min \left( 1, \frac{\pi(\theta'|x)}{\pi(\theta|x)} \right)$ .
5. Update  $\theta^{(i)} = \theta'$  with probability  $\beta$ ; otherwise set  $\theta^{(i)} = \theta$ .

The initial value for  $\theta$  is considered to be the MLE  $\hat{\theta} = (\hat{\lambda}, \hat{\alpha})$  of  $\theta = (\lambda, \alpha)$ . While, the selection of  $S_\theta$  is considered to be the asymptotic variance-covariance matrix  $I^{-1}(\hat{\lambda}, \hat{\alpha})$ , where  $I(\cdot)$  is the Fisher information matrix. It is noticed that, the selection of  $S_\theta$  is an important issue in the MH algorithm where the acceptance rate depends on upon this.

Finally, from the random samples of size  $M$  drawn from the posterior density, some of the initial samples (burn-in) can be discarded, and remaining samples can be further carried out to calculate Bayes estimates. More accurately the Eq. (2) can be estimated as

$$\tilde{g}_{MH}(\lambda, \alpha) = \frac{1}{M - l_B} \sum_{i=l_B}^M g(\lambda_i, \alpha_i) \quad (7)$$

where  $l_B$  represent the number of burn-in samples (Dey *et al.* (2016) ).

### 3.2 Highest posterior density

We suggest utilizing the technique of Chen and Shao (1999) to calculate highest posterior density (HPD) interval estimates for the unknown parameters of the GIE distribution. The technique of Chen and Shao has been broadly utilized for constructing HPD intervals for the unknown parameters of the distribution of interest. In the present study, we will employ the samples drawn using the proposed MH algorithm to construct the interval estimates. More accurately, let us assume that  $\Pi(\theta|\mathbf{x})$  denotes the posterior distribution function of  $\theta$ . Let us further suppose that  $\theta^{(p)}$  be the  $p$ th quantile of  $\theta$ , that is,

$\theta^{(p)} = \inf\{\theta : \Pi(\theta|\mathbf{x}) \geq p\}$ , where  $0 < p < 1$ . Notice that for a given  $\theta^*$ , a simulation consistent estimator of  $\Pi(\theta^*|\mathbf{x})$  can be estimated as

$$\Pi(\theta^*|\mathbf{x}) = \frac{1}{M - l_B} \sum_{i=l_B}^M I_{\theta \leq \theta^*}$$

Here  $I_{\theta \leq \theta^*}$  is the indicator function. Then the corresponding estimate is obtained as

$$\hat{\Pi}(\theta^*|\mathbf{x}) = \begin{cases} 0 & \text{if } \theta^* < \theta_{(l_B)} \\ \sum_{j=l_B}^i \omega_j & \text{if } \theta_{(i)} < \theta^* < \theta_{(i+1)} \\ 1 & \text{if } \theta^* > \theta_{(M)} \end{cases}$$

where  $\omega_j = \frac{1}{M-l_B}$  and  $\theta_{(j)}$  are the ordered values of  $\theta_j$ . Now, for  $i = l_B, \dots, M$ ,  $\theta^{(p)}$  can be approximated by

$$\hat{\theta}^{(p)} = \begin{cases} \theta_{(l_B)} & \text{if } p = 0 \\ \theta_{(i)} & \text{if } \sum_{j=l_B}^{i-1} \omega_j < p < \sum_{j=l_B}^i \omega_j. \end{cases}$$

Now to obtain a  $100(1-p)\%$  HPD credible interval for  $\theta$ , let  $R_j = (\hat{\theta}^{(\frac{j}{M})}, \hat{\theta}^{(\frac{j+(1-p)M}{M})})$  for  $j = l_B, \dots, [pM]$ , here  $[a]$  denotes the largest integer less than or equal to  $a$ . Then choose  $R_{j^*}$  among all the  $R_j$ 's such that it has the smallest width.

## 4 Data Analysis and Simulation Study

The aim of this section is to set a comparison the performance of the different methods of estimation discussed in the previous sections. We analyze a real data set for illustrative purpose; also, a simulation study is employed to check the behavior of the proposed methods as well as to assess the statistical performances of the estimators under progressive type-I censoring scheme. We used *R*-statistical programming language for calculation. Further, we utilize *bbmle* and *HDInterval* packages to compute MLEs and HPD interval in *R*-language.

## 4.1 Real data analysis

The following data set represents the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli. There were 72 observations listed below (Dey *et al.* (2016)):

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57,  
58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76,  
81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146,  
146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

It can be indicated that Dube *et al.* (2015) carried out the goodness of fit test of the GIE distribution to the above data set and noticed that the GIE distribution fits the data set very well than exponential, Rayleigh, gamma, Weibull and inverse Weibull distributions.

From the original data we generate three progressively type-I censored samples with different  $m$  stages and removed items  $R_i$  at the time censoring  $T_{q_i}$  corresponding to the selected quantiles  $q_i$ th quantiles, where  $i = 1, 2, \dots, m$ . These different schemes can be described in Table (1) as following.

Table 1: Different schemes for progressively type-I censored samples

| Scheme | $m$ | $q_i(\%)$            | Censoring time $T_{q_i}$     | Removed items $R_i$  |
|--------|-----|----------------------|------------------------------|----------------------|
| 1      | 3   | (10, 30, 60)         | (33.1, 58, 82.2)             | (5, 5, $R_m$ )       |
| 2      | 4   | (10, 20, 40, 60)     | (33.1, 52.2, 61.4, 82.2)     | (5, 5, 5, $R_m$ )    |
| 3      | 5   | (10, 20, 30, 40, 60) | (33.1, 52.2, 58, 61.4, 82.2) | (5, 5, 5, 5, $R_m$ ) |

where  $R_m = n - (\sum_{j=1}^{m-1} R_j + r)$  and  $r$  is the number of failure items.

In Table (2), we calculate the MLEs of the parameters  $\lambda$  and  $\alpha$  and their associated 95 % asymptotic confidence interval estimates. We also compute Bayes estimates utilizing the MH algorithm under the non-informative prior. Note that the non-informative

prior,  $\pi(\lambda, \alpha) = 1/(\lambda\alpha)$ , becomes the particular case of the proposed prior corresponding to hyper-parameter values  $a_1 = b_1 = a_2 = b_2 = 0$ . It is indicated that, while generating samples from the posterior distribution utilizing the MH algorithm, initial values of  $(\lambda, \alpha)$  are considered as  $(\lambda^{(0)}, \alpha^{(0)}) = (\hat{\lambda}, \hat{\alpha})$ , where  $\hat{\lambda}$  and  $\hat{\alpha}$  are the MLEs of the parameters  $\lambda$  and  $\alpha$  respectively. Thus, we considered the variancecovariance matrix  $S_\theta$  of  $(\ln(\hat{\lambda}), \ln(\hat{\alpha}))$ , that can be easily obtained utilizing the delta method. Finally, we discarded 2000 burn-in samples among the total 10000 samples created from the posterior density, and subsequently obtained Bayes estimates, and HPD interval estimates utilizing the technique of Chen and Shao (1999).

Table 2: Maximum likelihood, Bayesian and associated interval estimates for real data set

| Scheme   | Parameter | Maximum Likelihood Estimation |                                | Bayesian Estimation |                   |
|----------|-----------|-------------------------------|--------------------------------|---------------------|-------------------|
|          |           | MLE                           | Asymptotic confidence interval | MH algorithm        | HPD interval      |
| Complete | $\lambda$ | 3.4212                        | (2.6091, 4.3229)               | 3.2848              | (2.5169, 4.1057)  |
|          | $\alpha$  | 2.5424                        | (1.7406, 3.6682)               | 2.4140              | (1.6011, 3.3446)  |
| 1        | $\lambda$ | 3.2605                        | (2.2246, 4.4819)               | 2.9829              | (1.9200, 4.1107)  |
|          | $\alpha$  | 2.3068                        | (1.2050, 4.4820)               | 2.0147              | (0.84536, 3.3005) |
| 2        | $\lambda$ | 3.3356                        | (2.2541, 4.6157)               | 3.0241              | (1.9172, 4.1937)  |
|          | $\alpha$  | 2.3373                        | (1.1860, 4.6994)               | 2.0232              | (0.8508, 3.4509)  |
| 3        | $\lambda$ | 3.3015                        | (2.2229, 4.5786)               | 2.9697              | (1.8557, 4.0392)  |
|          | $\alpha$  | 2.3431                        | (1.1738, 4.7757)               | 1.9845              | (0.8006, 3.4136)  |

Note that we transformed the original data by divide it on 30 because of non-convergence of MCMC of Bayesian estimates. Also, it can be indicated that complete scheme in the above table can be considered as a special case of progressive type-I censoring when  $R_1 = R_2 = \dots = R_{m-1} = 0$ . In Table (2) all the estimated values of maximum likelihood, Bayesian and associated interval estimates for both the real data sets are presented. From Table (2), it is noticed that MLEs compete well with non-informative Bayes estimates.

The convergence of MCMC estimation for  $\lambda$  and  $\alpha$  can be showed in Figure (4) in case of complete censoring.

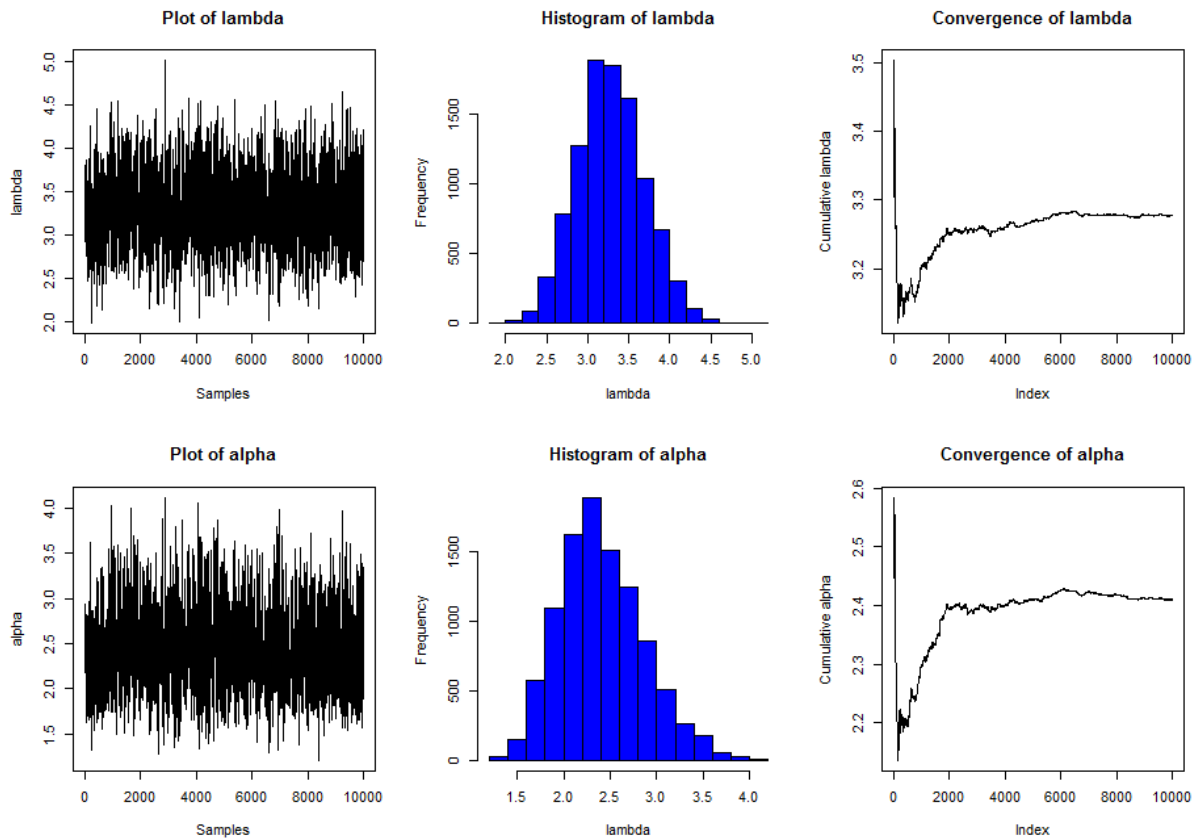


Figure 4: Convergence of MCMC estimation fo $\lambda$  and  $\alpha$

## 4.2 Simulation study

In this section, we employ a Monte Carlo simulation study to compare the performance of proposed methods of estimation. We simulate 1000 generating data from  $GIE(1.5, 2)$  distribution under various progressive type-I censoring schemes for different choice of  $n$ , number of stages  $m = 5$  and different censoring time  $T_{q_i}$  corresponding to the selected quantiles  $q_i$ th quantiles, where  $q_i = (10\%, 20\%, 30\%, 40\%, 60\%)$ . We suppose the following censoring schemes:

**Scheme 1:**  $R_1 = R_2 = \dots = R_4 = 0$  and  $R_5 = n - r$ ,

**Scheme 2:**  $R_1 = 10, R_2 = \dots = R_4 = 0$ ,

**Scheme 3:**  $R_1 = R_2 = R_3 = 0, R_4 = 10$ , and

**Scheme 4:**  $R_1 = R_2 = \dots = R_4 = 5$ .

Based on the generated data, firstly, we calculate maximum likelihood estimates and associated 95% asymptotic confidence interval estimates. Note that the initial guess values are considered to be same as the true parameter values while obtaining maximum likelihood estimates. Further, we calculate Bayes estimates using the MH algorithm under the informative prior (P-I) and the non-informative prior (P-II). Thus:

- For informative priors (P-I), we generate 1000 complete samples each of size 60 from  $GIE(1.5, 2)$  distribution as past samples, and subsequently get the hyper-parameter values as  $a_1 = 22.74, b_1 = 14.20, a_2 = 9.65, b_2 = 4.29$ .
- For non-informative prior (P-II), we assume that hyper-parameter values are  $a_1 = b_1 = a_2 = b_2 = 0$ , thus  $\pi(\lambda, \alpha) = 1/(\lambda\alpha)$ .

These values are then plugged-in to calculate the desired estimates. While utilizing MH algorithm, we take into account the maximum likelihood estimates as initial guess values, and the associated variancecovariance matrix  $S_\theta$  of  $(\ln(\hat{\lambda}), \ln(\hat{\alpha}))$ . At the end, we discarded 2000 burn-in samples among the the overall 10000 samples generated from the posterior density, and subsequently obtained Bayes estimates, and HPD interval estimates utilizing the technique of Chen and Shao (1999).

All the average estimates for both methods are reported in Table 3. Further, the first row represents the average estimates and interval estimates, and in the second row, associated means square errors (MSEs) and average interval lengths (AILs) with coverage percentages (CPs) are reported.

Table 3: Average estimated values, interval estimates, MSEs, AILs and CPs (in %) of GIE distribution with  $\lambda = 1.5$  and  $\alpha = 2$ 

| $n$ | Scheme | Parameter         | MLE            |                              | P-I              |                               | P-II             |                                |
|-----|--------|-------------------|----------------|------------------------------|------------------|-------------------------------|------------------|--------------------------------|
|     |        |                   | Avg MSE        | Asy CI AIL/CP                | Avg MSE          | HPD interval AIL/CP           | Avg MSE          | HPD interval AIL/CP            |
| 25  | 1      | $\lambda$         | 1.643<br>0.021 | (0.741, 2.545)<br>1.804/94.4 | 1.525<br>0.0006  | (0.997, 2.021)<br>1.024/96.2  | 1.254<br>0.061   | (0.263, 2.419)<br>2.156/96.2   |
|     |        | $\alpha$          | 2.554<br>0.306 | (0, 5.186)<br>5.186/93.8     | 2.073<br>0.005   | (1.060, 3.186)<br>2.126/95.9  | 1.756<br>0.060   | (0.139, 4.358)<br>4.219/95.5   |
|     | 2      | $\lambda$         | 1.640<br>0.020 | (0.659, 2.621)<br>1.963/95.1 | 1.522<br>0.0005  | (1.073, 2.052)<br>0.979/98    | 1.103<br>0.158   | (8.4e-17, 2.397)<br>2.397/95.1 |
|     |        | $\alpha$          | 2.806<br>0.650 | (0, 6.324)<br>6.324/94.2     | 2.0856<br>0.007  | (0.9692, 3.3401)<br>2.3709/96 | 1.6157<br>0.148  | (0.002, 5.123)<br>5.121/95.1   |
|     | 3      | $\lambda, \alpha$ |                |                              |                  |                               |                  |                                |
|     | 4      | $\lambda, \alpha$ |                |                              |                  |                               |                  |                                |
| 50  | 1      | $\lambda$         | 1.569<br>0.005 | (0.955, 2.184)<br>1.229/96.6 | 1.523<br>0.0005  | (1.082, 2.032)<br>0.950/97.5  | 1.379<br>0.015   | (0.659, 2.183)<br>1.524/96.2   |
|     |        | $\alpha$          | 2.230<br>0.053 | (0.679, 3.781)<br>3.102/95.6 | 2.075<br>0.006   | (1.138, 3.229)<br>2.091/96.9  | 1.885<br>0.013   | (0.476, 3.921)<br>3.445/95.5   |
|     | 2      | $\lambda$         | 1.572<br>0.005 | (0.936, 2.209)<br>1.272/96.6 | 1.5298<br>0.0009 | (1.0892, 1.9856)<br>0.8964/97 | 1.3793<br>0.0146 | (0.6116, 2.1779)<br>1.5663/96  |
|     |        | $\alpha$          | 2.294<br>0.086 | (0.556, 4.031)<br>3.475/94.9 | 2.061<br>0.004   | (1.067, 3.143)<br>2.075/96.5  | 1.879<br>0.015   | (0.332, 4.013)<br>3.681/95.2   |
|     | 3      | $\lambda$         | 1.584<br>0.007 | (0.942, 2.227)<br>1.284/95.5 | 1.532<br>0.001   | (1.077, 1.996)<br>0.919/96.6  | 1.367<br>0.018   | (0.596, 2.183)<br>1.587/96.3   |
|     |        | $\alpha$          | 2.315<br>0.099 | (0.543, 4.087)<br>3.544/95.5 | 2.052<br>0.003   | (1.172, 3.110)<br>1.938/96.5  | 1.8505<br>0.022  | (0.416, 3.850)<br>3.434/95.6   |
|     | 4      | $\lambda$         | 1.595<br>0.009 | (0.913, 2.277)<br>1.364/95.8 | 1.516<br>0.0002  | (1.018, 1.964)<br>0.946/96.1  | 1.344<br>0.024   | (0.560, 2.264)<br>1.704/96.8   |
|     |        | $\alpha$          | 2.407<br>0.166 | (0.305, 4.510)<br>4.205/94.4 | 2.038<br>0.001   | (0.976, 3.129)<br>2.153/96.3  | 1.837<br>0.027   | (0.273, 4.211)<br>3.938/95.3   |
| 100 | 1      | $\lambda$         | 1.539<br>0.002 | (1.113, 1.966)<br>0.853/96.3 | 1.506<br>4e-05   | (1.118, 1.886)<br>0.768/97.1  | 1.450<br>0.003   | (0.822, 2.000)<br>1.178/95.6   |
|     |        | $\alpha$          | 2.137<br>0.019 | (1.101, 3.172)<br>2.071/96.4 | 2.046<br>0.002   | (1.209, 2.876)<br>1.667/96.6  | 1.972<br>0.001   | (0.793, 3.435)<br>2.642/95.7   |
|     | 2      | $\lambda$         | 1.534<br>0.001 | (1.101, 1.967)<br>0.866/95.8 | 1.520<br>0.0004  | (1.138, 1.931)<br>0.793/97.2  | 1.452<br>0.002   | (0.894, 2.093)<br>1.199/96.9   |
|     |        | $\alpha$          | 2.122<br>0.015 | (1.052, 3.193)<br>2.141/95.5 | 2.065<br>0.004   | (1.234, 3.170)<br>1.936/97    | 1.957<br>0.002   | (0.769, 3.376)<br>2.607/95.7   |
|     | 3      | $\lambda$         | 1.548<br>0.002 | (1.112, 1.984)<br>0.872/95.4 | 1.521<br>0.0005  | (1.111, 1.912)<br>0.801/96.3  | 1.457<br>0.002   | (0.864, 2.083)<br>1.219/96.5   |
|     |        | $\alpha$          | 2.160<br>0.025 | (1.065, 3.254)<br>2.189/95.4 | 2.069<br>0.005   | (1.179, 3.059)<br>1.880/96.4  | 1.964<br>0.001   | (0.674, 3.439)<br>2.765/95.2   |
|     | 4      | $\lambda$         | 1.536<br>0.001 | (1.094, 1.977)<br>0.883/95.8 | 1.518<br>0.0003  | (1.159, 1.911)<br>0.752/97    | 1.435<br>0.004   | (0.884, 2.084)<br>1.200/96.4   |

Note: Asy CI asymptotic confidence interval, Avg average.

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From tabulated values it can be noticed that depended on MSEs, higher values of  $n$  lead to better estimates. It is also noticed that the maximum likelihood estimates compete well with non-informative Bayes estimates, and the performance of the Bayes estimates obtained under informative prior is better than the non-informative Bayes estimates. It can also be noticed that under informative prior the AILs and associated CPs of HPD intervals are better than those of non-informative priors. Furthermore, it is also seen that, in scheme 2 and scheme 3, MSEs, and AILs of associated interval estimates are generally lower when the units are removed at early stages.

## 5 Conclusion

In this paper, we have studied the problem of estimation and prediction for generalized inverted exponential distribution under progressive type-I censoring from classical and Bayesian viewpoints. We derived maximum likelihood estimates and associated asymptotic confidence interval estimates for the unknown parameters of a GIE distribution. Then, we calculated Bayes estimates and the corresponding HPD interval estimates under non-informative and informative priors. We have also discussed as to how to select the values of hyper-parameters based on past samples when informative prior is taken into consideration. Our simulation study indicates that MLEs compete well with non-informative Bayes estimates, and the performance of estimates under informative prior is better than both the non-informative prior and MLEs. Though we have considered squared error loss function under Bayesian set up, yet other loss functions can also be considered. The present work can also be extended to design of optimal progressive censoring sampling plan and other censoring schemes can also be considered.

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