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# Identification of Double Seasonal Autoregressive Models: A Bayesian Approach

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## Abstract

In this paper we present a Bayesian methodology to identify the order of double seasonal autoregressive (DSAR) models. Assuming the model errors are normally distributed and using conjugate priors on the model parameters, we derive the joint posterior mass function of the model order in a closed-form. Accordingly, the posterior mass function can be investigated and the best order of DSAR model is chosen as a value with the highest posterior probability. We evaluate the proposed Bayesian methodology using simulation study and real-world hourly internet amount of traffic datasets.

**Key Words:** Bayesian analysis; DSAR models; Posterior distribution; Double seasonality; Hourly internet traffic.

## 1 Introduction

High frequency time series that are observed at small time units may be characterized by exhibiting multiple seasonal patterns. For example, hourly internet traffic data can exhibit intraday and intraweek seasonal patterns. Other examples of high frequency time series contain multiple seasonal patterns include daily hospital admissions, daily usage of water and natural gas, hourly volumes of call center arrivals, hourly traffic on a road, hourly access to computer web sites, and hourly electricity load. The notion of modelling multiple seasonalities is not new and it can be traced back to 1971 when Thompson and Tiao (1971) showed that monthly disconnections of the Wisconsin telephone company have annual and quarterly (double) seasonal patterns. Accordingly, seasonal autoregressive moving average (SARMA) models being widely applied to analyze time series with single seasonal pattern need to be modified and extended to accommodate multiple seasonalities, see for example Box et al. (1994) and Taylor (2003). In addition to SARMA models, other techniques have been extended to fit multiple seasonal time series, which include exponential smoothing methods, neural networks, and innovation state models (Feinberg and Genethliou, 2005).

In particular, the double SARMA (DSARMA) models have been the subject of interest of many researchers and extensively studied and employed in modeling and forecasting double seasonal time series data. Taylor (2003) showed that electricity load in England and Wales features daily (within day) and weekly (within week) seasonal patterns. Taylor et al. (2006) compared the forecast accuracy of six univariate models including DSARMA for electricity demand forecasting in Brazil and in England and Wales. Cruz et al. (2011) empirically compared the predictive accuracy of a set of methods for day-ahead spot price forecasting in the Spanish electricity market.

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Other references may include Taylor (2008a), Taylor (2008b), Caiado (2008), Baek (2010), Mohamed et al. (2011), and Kim (2013) and references therein, among others.

Bayesian analysis of SARMA model for single seasonality has been well established, and different approaches have been developed in literature, including Markov Chain Monte Carlo (MCMC) methods and analytical approximations. Chib and Greenberg (1994) and Marriott et al. (1996) developed a Bayesian analysis of ARMA models using MCMC methods, and Barnett et al. (1996,1997) used MCMC methods to estimate seasonal autoregressive and ARMA models. Ismail (2003) used Gibbs sampling algorithm, as one of MCMC methods, to ease the Bayesian analysis of seasonal autoregressive models. This work was extended by Ismail and Amin (2014) to SARMA models. The main criticism that can be raised against the use of MCMC methods for Bayesian analysis of time series models is that they are highly iterative and computationally intensive. On the other hand, analytical approximations work on approximating the posterior and predictive densities to be standard closed-form distributions that are analytically tractable, see for example Broemeling and Shaarawy (1984), Shaarawy and Ismail (1987), and Shaarawy and Ali (2003).

With respect to the Bayesian analysis of double seasonal autoregressive (DSAR) models, it is immature and few work have been introduced in literature. Recently, Amin and Ismail (2015) used Gibbs sampling algorithm to develop a Bayesian estimation of DSAR models, and Amin (2017) used the analytical approximations to develop a Bayesian inference of DSARMA models. Based on our review for the existing Bayesian analysis of DSAR models, there is no work that considers the problem of DSAR models identification. Since in reality the order of the DSAR models is unknown and needs to be identified or estimated. In order to fill this gap, in this paper we propose a Bayesian method to identify the order of the DSAR models. The main idea of the proposed Bayesian identification method is that we assume the order of the DSAR model is a random variable with a known maximum, and we derive its posterior mass function to select the order as a value with a maximum posterior probability.

The remainder of this paper is organized as follows: Section 2 presents the double seasonal autoregressive (DSAR) models. Section 3 is devoted to summarizing the proposed Bayesian identification method for DSAR models. The simulation study and real application of the proposed Bayesian identification method are given in Section 4. Finally, the conclusions are given in Section 5.

## 2 Double Seasonal Autoregressive (DSAR) Models

A time series $\{y_t\}$ is said to be generated by a double seasonal autoregressive model of order $p$, $P_1$, and $P_2$, denoted by DSAR(p)$(P_1)_{s_1}(P_2)_{s_2}$, if it satisfies

$$\phi_p(B)\Phi_{P_1}(B^{s_1})\Pi_{P_2}(B^{s_2})y_t = \varepsilon_t \tag{1}$$

where $\{\varepsilon_t\}$ is a sequence of independent normal variates with zero mean and variance $\sigma^2$. The backshift operator B is defined as $B^k y_t = y_{t-k}$, $s_1$ and $s_2$ are the seasonal periods. The non seasonal autoregressive polynomial is $\phi(B) = \left(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right)$ with order $p$. In addition, $\Phi_{P_1}(B^{s_1}) = \left(1 - \Phi_1 B^{s_1} - \Phi_2 B^{2s_1} - \cdots - \Phi_{P_1} B^{P_1 s_1}\right)$ with order $P_1$ and $\Pi_{P_2}(B^{s_2}) = \left(1 - \Pi_1 B^{s_2} - \Pi_2 B^{2s_2} - \cdots - \Pi_{P_2} B^{P_2 s_2}\right)$ with order $P_2$ are the seasonal autoregressive polynomials. Finally, the non seasonal and seasonal autoregressive coefficients are $\phi = (\phi_1, \phi_2, \cdots, \phi_p)^T$, $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_{P_1})^T$ and $\Pi = (\Pi_1, \Pi_2, \cdots, \Pi_{P_2})^T$, respectively.

It should be noted that the DSAR model (1) has an extra terms compared with the usual single SAR model. The new term is $\Pi_{P_2}(B^{s_2})$ that accommodates the second seasonal pattern. Accordingly, the model (1) can be written as

$$y_t = \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{j=1}^{P_1} \Phi_j y_{t-js_1} + \sum_{\tau=1}^{P_2} \Pi_\tau y_{t-\tau s_2} - \sum_{i=1}^{p}\sum_{j=1}^{P_1} \phi_i \Phi_j y_{t-i-js_1} - \sum_{i=1}^{p}\sum_{\tau=1}^{P_2} \phi_i \Pi_\tau y_{t-i-\tau s_2}$$
$$- \sum_{j=1}^{P_1}\sum_{\tau=1}^{P_2} \Phi_j \Pi_\tau y_{t-js_1-\tau s_2} + \sum_{i=1}^{p}\sum_{j=1}^{P_1}\sum_{\tau=1}^{P_2} \phi_i \Phi_j \Pi_\tau y_{t-i-js_1-\tau s_2} + \varepsilon_t \tag{2}$$

The matrix form of this model can be written as:

$$y = X\beta + \varepsilon, \tag{3}$$

where $y = (y_1, y_2, \cdots, y_n)^T$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)^T$, $X$ is an $n \times p^*$ design matrix, where $p^* = (1+p)(1+P_1)(1+P_2) - 1$, with the $t^{th}$ row:

$$X_t = \big(y_{t-1}, \dots, y_{t-p}, y_{t-s_1}, y_{t-s_1-1}, \dots, y_{t-s_1-p}, \dots\dots, y_{t-P_1s_1}, y_{t-P_1s_1-1}, \dots, y_{t-P_1s_1-p}, y_{t-s_2},$$
$$y_{t-s_2-1}, \dots, y_{t-s_2-p}, y_{t-s_1-s_2}, y_{t-s_1-s_2-1}, \dots, y_{t-s_1-s_2-p}, \dots\dots, y_{t-P_1s_1-s_2}, y_{t-P_1s_1-s_2-1},$$
$$\dots, y_{t-P_1s_1-s_2-p}, \dots\dots\dots, y_{t-P_2s_2}, y_{t-P_2s_2-1}, \dots, y_{t-P_2s_2-p}, y_{t-s_1-P_2s_2}, y_{t-s_1-P_2s_2-1}, \dots,$$
$$y_{t-s_1-P_2s_2-p}, \dots\dots, y_{t-P_1s_1-P_2s_2}, y_{t-P_1s_1-P_2s_2-1}, \dots, y_{t-P_1s_1-P_2s_2-p}\big),$$

$$\beta = \big(\phi_1, \dots, \phi_p, \Phi_1, -\phi_1\Phi_1, \dots, -\phi_p\Phi_1, \dots\dots, \Phi_{P_1}, -\phi_1\Phi_{P_1}, \dots, -\phi_p\Phi_{P_1}, \Pi_1, -\phi_1\Pi_1, \dots,$$
$$-\phi_p\Pi_1, -\Phi_1\Pi_1, \phi_1\Phi_1\Pi_1, \dots, \phi_p\Phi_1\Pi_1, \dots\dots, -\Phi_{P_1}\Pi_1, \phi_1\Phi_{P_1}\Pi_1, \dots, \phi_p\Phi_{P_1}\Pi_1, \dots\dots\dots,$$
$$\Pi_{P_2}, -\phi_1\Pi_{P_2}, \dots, -\phi_p\Pi_{P_2}, -\Phi_1\Pi_{P_2}, \phi_1\Phi_1\Pi_{P_2}, \dots, \phi_p\Phi_1\Pi_{P_2}, \dots\dots, -\Phi_{P_1}\Pi_{P_2}, \phi_1\Phi_{P_1}\Pi_{P_2},$$
$$\dots, \phi_p\Phi_{P_1}\Pi_{P_2}\big)^T \tag{4}$$

Equation (2) shows that the DSAR model can be written as an autoregressive model of order $(1+p)(1+P_1)(1+P_2) - 1$ with some coefficients that are products of non seasonal and seasonal coefficients. The DSAR model (2) is stationary if the roots of the polynomials $\phi(B)$, $\Phi_{P_1}(B^{s_1})$ and $\Pi_{P_2}(B^{s_2})$ lie outside the unit circle. For more details about the properties of seasonal AR models see Box et al. (1994).

It is worth noting that the design matrix $X$ becomes a function of $p$, $P_1$, and $P_2$ when the DSAR model order is unknown. In this case we can assume that the model order $p$, $P_1$, and $P_2$ is a random variable with known maximum values of $k_1$, $k_2$, and $k_3$ respectively. The prior information about $p$, $P_1$, and $P_2$ can be represented in terms of a prior mass function $\zeta(p, P_1, P_2)$ that can have different forms such as uniform or geometric.

# 3 Proposed Bayesian Identification for DSAR Models

Bayesian analysis of the DSAR models is based on Bayes' theorem that combines the prior distribution of the model parameters with the likelihood function of observed sample to get the posterior distribution.

Regarding the prior specification, we consider the natural conjugate prior. In case of the DSAR model with normally distributed errors, the natural conjugate prior is normal-gamma. So, suppose $\tau = 1/\sigma^2 \sim G(\frac{\nu}{2}, \frac{\lambda}{2})$ and $\beta \sim N_{p^å}(\mu_\beta, \tau^{-1}\Sigma_\beta)$, the joint natural conjugate prior distribution of $\beta$ and $\tau$ can be written as:

$$\zeta_n(\beta,\tau) \propto \tau^{\left(\frac{\nu+p^*}{2}-1\right)} \exp\left\{-\frac{\tau}{2}\left[\lambda + \left(\beta-\mu_\beta\right)^T \Sigma_\beta^{-1}\left(\beta-\mu_\beta\right)\right]\right\}, \tag{5}$$

where $\mu_\beta, \Sigma_\beta, \nu$ and $\lambda$ are hyperparameters need to be estimated, and the products of non seasonal and seasonal coefficients are considered as free coefficients.

In case of no or little information is available about the model parameters $\beta$ and $\tau$ Jeffreys' prior can be employed and it is given as:

$$\zeta_j(\beta,\tau) \propto \tau^{-1}, \tau > 0 \tag{6}$$

Employing a straightforward random variable transformation from $\varepsilon$ to $y$, the conditional likelihood function (on the first $p^*$ initial values) is given by

$$\begin{aligned} L(\beta,\tau,p,P_1,P_2|y) &\propto \tau^{\frac{n-p^*}{2}} \exp\left\{-\frac{\tau}{2}\varepsilon^T\varepsilon\right\}, \\ &\propto \tau^{\frac{n-p^*}{2}} \exp\left\{-\frac{\tau}{2}(y-X\beta)^T(y-X\beta)\right\}, \end{aligned} \tag{7}$$

Based on the likelihood function (7), we update the information about the DSAR model order $p$, P $_1$, and P $_2$ by deriving the posterior probability mass function. To derive this posterior mass function, we need first to obtain the joint posterior of the model parameters $\beta, \tau,\ p$, P $_1$, and P $_2$ and then integrate out the parameters $\beta$ and $\tau$.

For the natural conjugate prior, the joint posterior of the model parameters is obtained as:

$$\begin{aligned} \zeta_n(\beta,\tau,p,P_1,P_2|y) \propto \zeta(p,P_1,P_2)\tau^{\left(\frac{n+\nu}{2}-1\right)} \exp\Big\{-\frac{\tau}{2}\Big[\lambda + \left(\beta-\mu_\beta\right)^T \Sigma_\beta^{-1}\left(\beta-\mu_\beta\right) + \\ (y-X\beta)^T(y-X\beta)]\}. \end{aligned} \tag{8}$$

For Jeffreys' prior, the resulting joint posterior of $\beta$ and $\tau$ is:

$$\zeta_j(\beta,\tau,p,P_1,P_2|y) \propto \zeta(p,P_1,P_2)\tau^{\left(\frac{n-p^*}{2}-1\right)} \exp\left\{-\frac{\tau}{2}(y-X\beta)^T(y-X\beta)\right\}. \tag{9}$$

In the following theorem we use the natural conjugate prior to obtain the joint posterior mass function of the DSAR model order.

**Theorem 1** *Using the joint natural conjugate prior distribution of* $\beta$ *and* $\tau$ *(5) and the* *conditional likelihood function (7), the joint posterior mass function of the DSAR model order* $p$, $P_1$, *and* $P_2$ *is given by*

$$\zeta_n(p, P_1, P_2|y) \propto \zeta(p, P_1, P_2) \left[\frac{|\Sigma_\beta^{-1}|}{|A_n|}\right]^{1/2} \left[y^T y + \lambda + \mu_\beta^T \Sigma_\beta^{-1} \mu_\beta - B_n^T A_n^{-1} B_n\right]^{-\frac{n+\nu}{2}}$$
$$\forall p = 1, \dots, k_1, P_1 = 1, \dots, k_2, P_2 = 1, \dots, k_3. \tag{10}$$

Where $A_n = (X^T X + \Sigma_\beta^{-1})$, $B_n = (X^T y + \Sigma_\beta^{-1} \mu_\beta)$, and $\zeta(p, P_1, P_2)$ is a prior mass function of the model order.

**Proof.** The proof follows by completing the square in the exponent of (??) with respect to $\beta$ and then integrating over $\beta$ and $\tau$ respectively results in the given joint posterior mass function of the DSAR model order.

Based on Theorem 1, we can straightforwardly identify the order of the DSAR model by computing all possible posterior probabilities from the joint posterior mass function (??) and choose the values of $p$, $P_1$, and $P_2$ that have the highest posterior probabilities to be the appropriate DSAR model order for the time series under study. In case of no or little information is available a priori about the model parameters $\beta$ and $\tau$, in the following corollary we obtain the joint posterior mass function of the DSAR model order.

**Corollary 1** *Using Jeffreys' prior of* $\beta$ *and* $\tau$ *(6) and the conditional likelihood function (7), the joint posterior mass function of the DSAR model order* $p$, $P_1$, *and* $P_2$ *is given by*

$$\zeta_j(p, P_1, P_2|y) \propto \zeta(p, P_1, P_2) \frac{\Gamma\left(\frac{n-p^*}{2}\right)}{\pi^{\frac{n-p^{å}}{2}} |X^T X|^{1/2}} \left[y^T y - y^T X (X^T X)^{-1} X^T y\right]^{-\frac{n-p^*}{2}}$$
$$\forall p = 1, \dots, k_1, P_1 = 1, \dots, k_2, P_2 = 1, \dots, k_3. \tag{11}$$

**Proof.** The proof follows directly from Theorem 1 by setting $\lambda = 0$, $\Sigma_\beta^{-1} = 0$, and $\nu = -p^*$.

# 4 Simulation Study and Real Application

In this section we first present a simulation study to evaluate the efficiency of the proposed Bayesian identification method for DSAR models, and then we apply the proposed method to real world time series dataset of internet traffic.

## 4.1 Simulation Study

In this simulation study, we generate 100 time series of size $n$ (from 200 to 500 with an increment of 100 observations) from different DSAR models. Table 0 shows these DSAR models and their corresponding true parameters values. By these DSAR models we try to cover different seasonality patterns with different data type.

| Model | $\phi_1$ | $\phi_2$ | $\Phi_1$ | $\Phi_2$ | $\Pi_1$ | $\Pi_2$ | $\sigma^2$ |
|---|---|---|---|---|---|---|---|
| I.DSAR(1)(1) $_4$(1) $_{28}$ | 0.80 | - | 0.20 | - | 0.30 | - | 1.00 |
| II. DSAR(2)(1) $_4$(1) $_{28}$ | -0.40 | 0.30 | 0.40 | - | 0.40 | - | 1.00 |
| III.DSAR(2)(2) $_4$(1) $_{28}$ | 0.5 | -0.40 | -0.50 | 0.40 | 0.40 | - | 1.00 |
| IV.DSAR(2)(2) $_4$(2) $_{28}$ | 0.4 | 0.40 | 0.60 | -0.50 | 0.30 | 0.50 | 1.00 |

Table 1: Simulation design.

Once the time series datasets are generated from these DSAR models, the Bayesian analysis is performed by assuming Jeffreys' prior for the model parameters $\beta$ and $\tau$, and employing three prior distributions for the model order given as:

$$\zeta_1(p,P_1,P_2)=\frac{1}{k_1}\times\frac{1}{k_2}\times\frac{1}{k_3},\forall p=1,\ldots,k_1,P_1=1,\ldots,k_2,P_2=1,\ldots,k_3(\text{Uniformprior})$$
$$\zeta_2(p,P_1,P_2)=0.5^{p+P_1+P_2},\forall p=1,\ldots,k_1,P_1=1,\ldots,k_2,P_2=1,\ldots,k_3(\text{Geometricprior})$$
$$\zeta_3(p,P_1,P_2)=\frac{k_1-p+1}{k_1+1}\times\frac{k_2-P_1+1}{k_2+1}\times\frac{k_3-P_2+1}{k_3+1},$$
$$\forall p=1,\ldots,k_1,P_1=1,\ldots,k_2,P_2=1,\ldots,k_3(\text{Arithmeticprior}) \quad (12)$$

For each time series, we compute the posterior mass functions of the DSAR model order, $\zeta_1(p,P_1,P_2|y)$, $\zeta_2(p,P_1,P_2|y)$, and $\zeta_3(p,P_1,P_2|y)$ resulting from the employed priors in eq(12) respectively with assuming the maximum values of the model order are $k_1=k_2=k_3=3$, and then we identify the model order as a value with a maximum posterior probability. For all simulated time series, we compute the percentage of correctly identified models by comparing the identified order with the true value of $p$, $P_1$, and $P_2$ used to generate the time series. Results of the simulation study are presented in Table (1).

From the simulation results we can observe some remarks. Firs, the larger sample size the higher percentage of correctly identified models is obtained. Second, the employed priors for the model order result in very similar posteriors, and their impact in the percentage of correctly identified models is small and can be ignored especially for sample size $n\geq 300$. Third, a higher percentage of correctly identified models is obtained for simple DSAR models (Models I and II) compared to complicated DSAR models (Models III and IV). In general, all these simulation results confirm the efficiency of the proposed Bayesian method for identifying the DSAR models.

## 4.2 Application to Real-World Internet Traffic Time Series

The internet traffic time series represents real-world hourly internet amount of traffic (in bits) dataset collected from private Internet service provider (ISP) with centers in 11 European cities. In particular, Cortez et al. (2012) collected this time series dataset from 06:57 hours on 7 June to 11:17 hours on 29 July 2005, which is available online and can be downloaded [†].

[†] http://www3.dsi.uminho.pt/pcortez/series/

| $n$ | $\zeta_1(p,P_1,P_2\|y)$ | $\zeta_2(p,P_1,P_2\|y)$ | $\zeta_3(p,P_1,P_2\|y)$ |
|---|---|---|---|
| | Model I | | |
| 200 | 94 | 97 | 96 |
| 300 | 100 | 100 | 100 |
| 400 | 99 | 100 | 100 |
| 500 | 100 | 100 | 100 |
| | Model II | | |
| 200 | 79 | 72 | 77 |
| 300 | 96 | 94 | 94 |
| 400 | 99 | 99 | 99 |
| 500 | 97 | 99 | 99 |
| 200 | Model III | | |
| 200 | 55 | 52 | 54 |
| 300 | 85 | 84 | 87 |
| 400 | 95 | 94 | 94 |
| 500 | 100 | 100 | 100 |
| | Model IV | | |
| 200 | 21 | 16 | 19 |
| 300 | 64 | 62 | 62 |
| 400 | 91 | 90 | 90 |
| 500 | 96 | 96 | 96 |

Table 2: Simulation results for percentage of correctly identified models.

Figure (1) shows how the daily and weekly seasonal patterns are represented in the internet traffic time series which reveal clearly the double seasonality. Accordingly, it is strongly recommended to consider two seasonal components for this dataset, where the first seasonal component is due to the intraday cycle ($s_1$ = 24) and the second seasonal component is due to the intraweek cycle ($s_2$ =168).

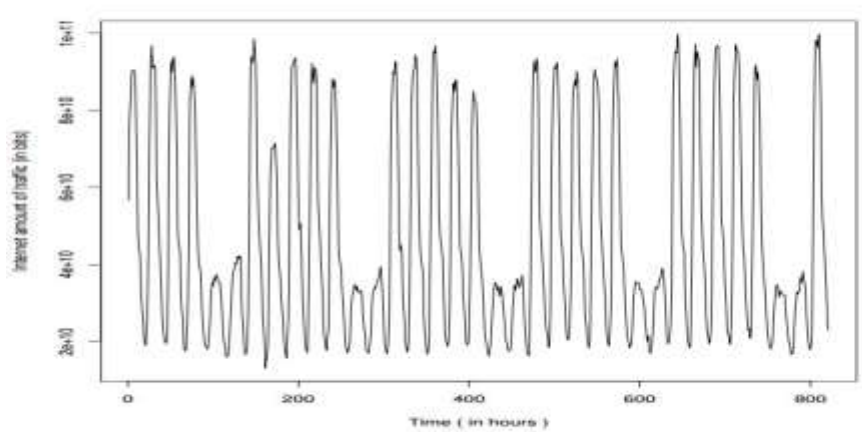


Figure 1: Time plot of the internet traffic time series.

Cortez et al. (2012) tested a collection of DSAR models using different combinations of the

$p$, $P_1$, and $P_2$ values up to a maximum order of 2 in order to identify the best DSAR model that can fit the internet amount of traffic dataset. They decided that DSAR(2)(2) $_{24}$(2) $_{168}$ is the best model that has smallest value of Bayesian information criterion (BIC).

We can apply our proposed Bayesian methodology to identify the best order of DSAR model to fit the internet amount of traffic dataset. We first employ the three prior mass functions of the model order introduced in eqn (12) and then compute the posterior mass functions of the DSAR model order, $\zeta_1(p, P_1, P_2|y)$, $\zeta_2(p, P_1, P_2|y)$, and $\zeta_3(p, P_1, P_2|y)$ with assuming the maximum values of the model order are 3. Results of the posterior probabilities are presented in Table (2). Accordingly, we identify the model order as a value with the maximum posterior probability as highlighted in Table (2). From these results, we can observe that the employed three priors for the model order result in very similar identified DSAR models, which are also similar to that identified by Cortez et al. (2012).

| Model Order | Priors of Model Order | | |
|---|---|---|---|
| (p)(P $_1$)(P $_2$) | $\zeta_1(p, P_1, P_2\|y)$ | $\zeta_2(p, P_1, P_2\|y)$ | $\zeta_3(p, P_1, P_2\|y)$ |
| (1)(1)(1) | 1.3E-109 | 7.0E-109 | 4.0E-109 |
| (2)(1)(1) | 3.6E-03 | 9.9E-03 | 7.5E-03 |
| (3)(1)(1) | 1.1E-03 | 1.5E-03 | 1.1E-03 |
| (1)(2)(1) | 3.6E-112 | 1.0E-111 | 7.5E-112 |
| (2)(2)(1) | 2.1E-06 | 2.8E-06 | 2.8E-06 |
| (3)(2)(1) | 9.4E-08 | 6.5E-08 | 6.5E-08 |
| (1)(3)(1) | 2.0E-113 | 2.8E-113 | 2.1E-113 |
| (2)(3)(1) | 2.3E-13 | 1.6E-13 | 1.6E-13 |
| (3)(3)(1) | 6.2E-16 | 2.1E-16 | 2.1E-16 |
| (1)(1)(2) | 1.7E-107 | 4.8E-107 | 3.6E-107 |
| (2)(1)(2) | 4.4E-01 | **6.1E-01** | **6.1E-01** |
| (3)(1)(2) | 4.0E-02 | 2.7E-02 | 2.8E-02 |
| (1)(2)(2) | 1.5E-113 | 2.0E-113 | 2.0E-113 |
| (2)(2)(2) | 5.4E-06 | 3.7E-06 | 5.0E-06 |
| (3)(2)(2) | 1.3E-11 | 4.5E-12 | 6.0E-12 |
| (1)(3)(2) | 2.4E-116 | 1.7E-116 | 1.7E-116 |
| (2)(3)(2) | 2.9E-18 | 1.0E-18 | 1.4E-18 |
| (3)(3)(2) | 3.4E-23 | 5.9E-24 | 7.9E-24 |
| (1)(1)(3) | 7.0E-104 | 9.6E-104 | 7.2E-104 |
| (2)(1)(3) | **5.1E-01** | 3.5E-01 | 3.5E-01 |
| (3)(1)(3) | 5.7E-05 | 2.0E-05 | 2.0E-05 |
| (1)(2)(3) | 1.2E-112 | 8.0E-113 | 8.1E-113 |
| (2)(2)(3) | 3.1E-11 | 1.1E-11 | 1.4E-11 |
| (3)(2)(3) | 1.3E-19 | 2.2E-20 | 2.9E-20 |
| (1)(3)(3) | 9.2E-115 | 3.2E-115 | 3.2E-115 |
| (2)(3)(3) | 6.3E-26 | 1.1E-26 | 1.4E-26 |
| (3)(3)(3) | 3.9E-37 | 3.3E-38 | 4.5E-38 |
| **Identified** | **DSAR(2)(1)(3)** | **DSAR(2)(1)(2)** | **DSAR(2)(1)(2)** |

Table 3: Posterior probability and identfied models for the real-world time series.

## 5 Conclusions

In this paper we presented a Bayesian methodology to identify the order of DSAR models. We derived the joint posterior mass function of the model order in a closed-form by combining the conjugate priors on the model parameters with the conditional likelihood function, assuming the model errors are normally distributed. Therefore, we can easily inspect the posterior mass function of the DSAR model order and choose a value with the highest posterior probability as the best model order for the time series being analyzed. We conducted a simulation study with different priors for the model order and the results confirmed the efficiency of the proposed Bayesian methodology. In addition, we applied our proposed methodology to real-world hourly internet amount of traffic dataset. Future work may investigate model identification and outliers detection for multivariate double seasonal models.

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# On Expressing Continuous Distributions With Discrete Distributions (A Review Of Literature )

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## Abstract:

In this paper, the motivation of expressing continuous distributions via discrete ones is discussed, discretizing methods are introduced, and the discretized distributions obtained using these methods are also reviewed.

**Key Words:** (Continuous distribution, discretization, survival function, density function, distribution function, failure rate).

## Introduction:

It is sometimes inconvenient or difficult to get samples from a continuous distribution in real life. The observed values are discrete because they are usually measured to only a finite number of decimal places and cannot really constitute all points in a continuum. Even if the measures are taken on a continuous (ratio or interval) scale the observations may be recorded in a way making discrete model more appropriate. Therefore, it is reasonable to consider the observations as coming from a discretized distribution generated from the underlying continuous model.

The reliability (survival) function is sometimes assumed to be a function of a discrete random variable instead of being a function of continuous time random variable. For example, the reliability of an on/off - switching device is a function of the number of operating times, and the reliability of an airplane tire is a function of the number of landings. Similarly, the length of stay in an observation ward is usually measured as number of days, and survival time of leukemia or lung cancer patients may be measured in number of weeks. Many discrete distributions are available to model such mentioned situations, for example, the geometric distribution is a

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discrete alternative of the exponential distribution and the negative binomial distribution is a discrete alternative of the gamma distribution, but it is also well known that these discrete distributions are unsuitable for some situations because of their monotonic hazard rate functions [Jazi et al. (2010)], therefore, research is ongoing to find new discrete distributions suitable to treat different conditions and to fit to various types of lifetime data. In the light of the above contexts, the study of the discretization of continuous distributions is meaningful.

## General approaches of discretization:

Discrete analogues of continuous distributions may be obtained using some methods of discretization. Only four methods will be introduced in this work:

### Approach 1: Discretizing using the survival function:

Suppose that the continuous random variable $X$ has the survival function (sf) $S(x) = \Pr(X \geq x)$, then the random variable $Y = [X]$ (where, $[X]$ is the largest integer less than or equal to $X$) will have the probability mass function (pmf):

$$\mathrm{P}(Y = y) = \mathrm{P}(y \leq X < y+1)$$

$$= \mathrm{P}(X \geq y) - \mathrm{P}(X \geq y+1)$$

$$= S(y) - S(y+1), \quad y = 0, 1, 2, \cdots \qquad (1)$$

The pmf of the random variable $Y$ may be considered as a discrete concentration of the probability density function (pdf) of the continuous random variable $X$. (Roy, 2004).

Using formula (1), it is possible to generate a discrete distribution corresponds to a given continuous distribution. For example, If $X$ is an exponential random variable with the sf $S(x) = \Pr(X \geq x) = e^{-\theta x}$, then the pmf of its discrete version is given by:

$$\mathrm{P}(Y = y) = e^{-\theta y} - e^{-\theta(y+1)} = q^{y} - q^{y+1} = (1-q)\, q^{y}, \qquad y = 0, 1, 2, \cdots$$

where, $q = e^{-\theta}$. This is the geometric distribution.

Discretization of continuous distributions using this method retains the same functional form of the survival function. So, many reliability characteristics remain unchanged [Krishna and Pundir (2009)]. There is enough motivation to use this technique of generating discretized version of continuous distribution with this approach to develop new discrete lifetime models corresponding to the existing continuous one.

Following this approach, Many researchers studied discretization of some known continuous distributions for use as lifetime distributions. Nakagawa and Osaki (1975) proposed a discrete Weibull distribution. Xie et al. (2002) introduced the modified Weibull extension. Bracquemond and Gaudion (2003) introduced the discrete geometric Weibull distribution and investigated the discrete truncated logistic distribution. Roy (2003) derived the PMF of the discrete analogue of the normal distribution and elaborated its application for evaluating the reliability of complex systems as an alternative to simulation methods. Roy (2004) proposed a discrete Rayleigh version as a particular case of the discrete Weibull distribution and used it in approximating the reliability value of a complex system to examine probability integrals arising out of a continuous setting. Krishna and Pundir (2007) derived the discrete Maxwell distribution. The modified Weibull Type I distribution was introduced by Sarhan and Zaindin (2009). Krishna and Pundir (2009) derived the discrete Burr XII distribution and applied it to fit the reliability in series system and a set of real data, they also derived the discrete Pareto distribution as a special case of the discrete Burr XII distribution**.** The discrete extended exponential (telescopic) distribution was introduced by Roknabadi et al. (2009). Jazi et al. (2010) introduced the discrete inverse Weibull distribution. Gómez-Déniz (2010) used the Marshal-Olkin scheme to generalize the SF of the exponential distribution and then got the discrete generalized exponential distribution. Gómez-Déniz and Calderin-Ojeda (2011) introduced the discrete Lindley distribution and used it as an alternative to the Poisson distribution to model automobile claim frequency data. Nooghabi et al. (2011) got the modified Weibull Type II distribution as a special case of the telescopic distribution. Bebbington et al. (2012) explored a discrete analogue of the additive Weibull distribution. Bakouch et al. (2012) proposed another version of the discrete Lindley distribution. Chakraborty and Chakravarty (2012) proposed the discrete gamma distribution and applied it to fit the reliability function of real discrete failure time data and also for modeling probability distribution of count data. Al-Huniti and Al-Dayian (2012) introduced the discrete Burr III distribution and discussed its distributional properties and reliability characteristics along with a graphical description. Nekoukhou et al. (2013) got the discrete generalized exponential distribution, estimated its parameters, and examined its model using a real set of data related to accidents of women working on Shells for five weeks. Khorashadizadeh (2013) proposed the discrete log-logistic distribution as a special case of the discrete Burr XII distribution. Hussain and Ahmad (2014) introduced the discrete inverse Rayleigh distribution. Chakraborty and Chakravarty (2014) derived the discrete Gumbel distribution and applied it to fit three real life count data in relation to maximum flood discharges and annual maximum wind speeds. Almalki and Nadarajah (2014) got the discrete reduced modified Weibull Type II distribution as a special case of the telescopic distribution. Barbiero (2014) proposed the discrete skew Laplace distribution and applied its model to artificial and real data. Gómez-Déniz et al. (2014) generalized the SF of the normal case when $\mu = 0$ and $\sigma>0$ using the Marshal-Olkin scheme and got the discrete half normal distribution which can also be regarded as a particular case from the discrete generalized gamma distribution of Chakraborty (2015a)[see, Chakraborty (2015b)]. Chakraborty (2015a) derived the discrete generalized gamma distribution, discussed the failure rate (fr) behavior for the distribution and demonstrated its application in two real life count data sets. Another version of the discrete Pareto distribution was given by Kozubowski et al. (2015) from which they obtained the discrete truncated Pareto distribution. Chakraborty and Chakravarty (2015) introduced the two-sided power distribution, studied its distributional and reliability properties, and estimated its parameters using ML and moment methods. Alamatsaz et al. (2016) proposed the pmf of the discrete generalized Rayleigh distribution. Chakraborty and Chakravarty (2016) proposed the discrete logistic distribution and applied it to model real life count data.

## **Approach 2: Discretizing using the density function:**

Suppose that $X$ is a continuous random variable with pdf $f(x),\ -\infty < x < \infty$, then, the pmf of the discretized random variable where, $Y = [X]$ is derived as:

$$\mathrm{P}(Y=y)=\frac{f(y)}{\sum_{y=-\infty}^{\infty} f(y)}, \qquad y=0, \pm 1, \pm 2, \cdots \qquad \ldots (2)$$

For example, If $X$ is an exponential random variable with the pdf $f(x)=\theta e^{-\theta x}, \quad x>0,$ then the pmf of its discrete version is given by:

$$P(Y=y)=\frac{f(y)}{\sum\limits_{y=-\infty}^{\infty} f(y)}=\frac{\theta e^{-\theta y}}{\sum\limits_{y=0}^{\infty}\theta e^{-\theta y}}=\frac{e^{-\theta y}}{\sum\limits_{y=0}^{\infty} e^{-\theta y}}, \qquad y=0,1,\ \cdots$$

But, $\sum\limits_{k=0}^{\infty} x^k=\frac{1}{(1-x)}$ , $\Rightarrow \sum\limits_{y=0}^{\infty} e^{-\theta y}=\frac{1}{(1-e^{-\theta})}=(1-e^{-\theta})^{-1}$, then:

$$P(Y=y)=\frac{e^{-\theta y}}{(1-e^{-\theta})^{-1}}=e^{-\theta y}(1-e^{-\theta}), \qquad y=0,1,\ \cdots$$

therefore, $\qquad P(Y=y)=(1-e^{-\theta})\ e^{-\theta y}==(1-q)\,q^{y}, \quad y=0,1,2,\cdots$

This is the geometric distribution where, $q=e^{-\theta}$.

Another example is the discrete analogue of the normal distribution with mean $\mu$ and variance $\sigma^2$ which was derived by Kemp (1997) is given by the following pmf:

$$P(Y=y)=P_y=\frac{\theta^y q^{[y(y-1)/2]}}{\sum\limits_{y=-\infty}^{\infty}\theta^y q^{[y(y-1)/2]}}, \qquad y=0,\pm 1,\pm 2,\cdots$$

where, $\theta=e^{(2C_3\mu-C_2-C_3)}>0$, $0<q=e^{-2C_3}<1$, and the normalizing constant $\frac{1}{\sum\limits_{y=-\infty}^{\infty}\theta^y q^{[y(y-1)/2]}}=e^{\left(-1-C_1-C_3\mu^2\right)}$, where, $C_1$, $C_2$, and $C_3$ are the constants in the Lagrangian $L$ in the Lagrange's method of undetermined multipliers:

$$L\equiv-\sum_{y=-\infty}^{\infty}P_y\ln P_y-C_1\left[\sum_{y=-\infty}^{\infty}P_y-1\right]-C_2\left[\sum_{y=-\infty}^{\infty}y\,P_y-\mu\right]-C_3\left[\sum_{y=-\infty}^{\infty}(y-\mu)^2P_y-\sigma^2\right],$$

and $\qquad \ln P_y+1+C_1+C_2y+C_3(y-\mu)^2=0$

Using formula (2), Siromoney (1964) suggested a discrete general Dirichlet distribution and applied it to model the frequency distribution of the length of wet spells during the period 1932-

1962 in Tambaram in southern India. Lai and Wang (1995) suggested the discrete power function distribution and used it for fitting discrete failure times and a type of mortality data. The discrete analogue of the normal distribution with mean $\mu$ and variance $\sigma^2$ was derived by Kemp (1997) and was applied by Harris et al.(2001) in dynamic analysis of rural establishment count data. Sato et al. (1999) proposed the discrete exponential distribution and applied it to model defect count distribution in semiconductor deposition equipment and defect count distribution per chips. The convolution of the discrete exponential distribution was used by Sato et al. (1999) to give a discrete gamma distribution which was applied again by them to model defect count distribution in semiconductor deposition equipment and defect count distribution per chips. The discrete Gaussian exponential (also called the discrete log normal) distribution was proposed by Bi et al. (2001). The discrete half normal distribution was derived by Kemp (2006) as the maximum entropy distribution on (*0, 1, . . .*) with specified mean and variance. It may be regarded as a special case of the discrete normal distribution of Kemp (1997). A discrete version of the Laplace distribution was derived by Inusah and Kozubowski (2006) and introduced by Andersen et al. (2013a) to estimate Y-STR haplotype frequencies, as described by Andersen et al. (2013b), demonstrating a number of examples using R package. Kozubowski and Inusah (2006) proposed the discrete skew Laplace distribution. Nekoukhou, et al. (2012) introduced a discrete analogue of the generalized exponential distribution and fitted its model to rank frequencies of graphemes in Slovene (a Slavic language). Lekshmi and Sebastian (2014) introduced the discrete generalized Laplace distribution. Nekoukhou et al. (2015) proposed the discrete analogue of the beta-exponential distribution and fitted it again to rank frequencies of graphemes in Slovene.

**Approach 3: Discretizing by shifting the cumulative distribution function:**

If $X$ is a continuous random variable having a cumulative distribution function (cdf) $F_X(x)$, then the pmf of the discrete analogue $Y$ is given by:

$$P(Y=y)=P(y)=F_T\,(y+\delta)-F_T\left[y-(1-\delta)\right],\ y=0,\pm 1,\pm 2,\cdots,\pm i\ ;\quad 0<\delta\leq 1,$$

this pmf retains the form of the original cdf, except for a shift in the location by $\delta$. [Roy and Dasgupta (2001)]. For example, If $X$ is an exponential random variable with the cdf $F(x)=1-e^{-\theta x},\quad x>0,$ then the pmf of its discrete version is given by:

$$
\begin{aligned}
P(Y=y) &= F_T(y+\delta) - F_T[y-(1-\delta)] \\
&= [1-e^{-\theta(y+\delta)}] - \{1-e^{-\theta[y-(1-\delta)]}\} \\
&= e^{-\theta[y-(1-\delta)]} - e^{-\theta(y+\delta)} \\
&= e^{-\theta[y-(1-\delta)]}(1-e^{-\theta}), \quad y=0,1,\cdots
\end{aligned}
$$

For $\delta = 1$, this pmf reduces to the geometric distribution.

A standard form of this pmf is considered to reduce the range of $Y$. For example if $X^*$ is a random variable having a normal distribution with mean $\mu$ and standard deviation $\sigma$, that is, $X^* \sim$ N $(\mu, \sigma^2)$, then, the discrete counterpart of $X^*$ is: $Y^* = \mu + \sigma \,.\, Z^*$,
where, $Z^*$ is the discrete counterpart of $Z \sim$ N $(0, 1)$.

Using this method of discretizing, Perry and Taylor (1985) proposed the discrete analogue of the Ade's distribution and fitted it to $22$ entomological data sets. Roy and Dasgupta (2001) proposed this method in order to evaluate the reliability of complex systems under stress-strength model where available analytic approaches fail to provide a closed solution. According to this method, the cdf of $Y$ (the resulted discrete random variable) retains the form of the cdf of $X$ (the original continuous random variable) and support of $Y$ is determined from a subset of the range of $X$.

**Approach 4: Discretizing using the failure rate:**

Stein and Dattero (1984) proposed this method to obtain a discrete version of the Weibull distribution. According to this method, the fr of $Y$ retains the form of the fr of $X$.

Given the fr of the continuous random variable $X$, $h_X(x)$, then the pmf of $Y$ is:

$$
\begin{aligned}
P(Y=y) &= [1-h_X(0)][1-h_X(1)]\cdots[1-h_X(y-1)]\{1-[1-h_X(y)]\} \\
&= [1-h_X(0)][1-h_X(1)]\cdots[1-h_X(y-1)]h_X(y)
\end{aligned}
$$

$$
= \begin{cases} h_X(0), & y=0 \\ [1-h_X(0)][1-h_X(1)]\cdots[1-h_X(y-1)]h_X(y), & y=1,2,\cdots,d \\ 0, & otherwise \end{cases}
$$

where, $d$ is determined such that $0 \le h_X(x) < 1$, and the sf of $Y$ is given by:

$$
S_Y(y) = [1-h_X(0)][1-h_X(1)]\cdots[1-h_X(y-1)], \quad y=1,2,\cdots
$$

Salvia and Bollinger (1982) and Padgett and Spurrier (1985) gave the following illustrations:

**1.** For constant fr [ $h_X(y) = c$ ], $0 < c \le 1$, $y = 0,1,\ldots$, (the exponential distribution),

$$\begin{aligned} P(Y = y) &= [1 - h_X(0)][1 - h_X(1)]\cdots[1 - h_X(y-1)]h_X(y) \\ &= [(1-c)(1-c)\cdots(1-c)]c = c(1-c)^y, \quad y = 0,1,\ldots \end{aligned}$$

this is the geometric distribution.

**2.** For decreasing fr $[h_X(y) = \frac{c}{y+1}]$, $0 < c \le 1$, $y = 0,1,\ldots$, then:

$$\begin{aligned} P(Y = y) &= [1 - h_X(0)][1 - h_X(1)]\cdots[1 - h_X(y-1)]h_X(y) \\ &= [(1-c)(1-\frac{c}{2})\cdots(1-\frac{c}{y})](\frac{c}{y+1}) \\ &= \frac{c(1-c)(2-c)\cdots(y-c)}{(y+1)!}, \quad y = 0,1,\ldots \end{aligned}$$

**3.** For increasing fr $[h_X(y) = 1 - \frac{c}{y+1}]$, $0 \le c \le 1$, $y = 0,1,\ldots$, then:

$$\begin{aligned} P(Y = y) &= [1 - h_X(0)][1 - h_X(1)]\cdots[1 - h_X(y-1)]h_X(y) \\ &= [c(\frac{c}{2})\cdots(\frac{c}{y})][1 - (\frac{c}{y+1})] \\ &= \frac{(y-c+1)c^y}{(y+1)!}, \quad y = 0,1,\ldots \end{aligned}$$

To insure that $\sum_{y=0}^{c} P(Y = y) = 1$, each probability $P(Y = y)$ has to be multiplied by a positive normalizing constant. The functional form of the fr will not be affected by such a choice. [Chakraborty(2015b)].

Bracquemond and Gaudoin (2003) criticized the assumption of a bounded support and referred such models as not to be very useful practically, asking the question: How can one be sure that the underlying system will fail in less than *c* demands?.

Using this method of discretizing, Stein and Dattero (1984) discretize the Weibull distribution. Padgett and Spurrier (1985) obtained the discrete analogue of the Weibull distribution using the definition of the fr provided by Roy and Gupta (1992). Roy and Gosh (2009) got discrete analogues of the Rayleigh and the Lomax distributions and used the fr of the Lomax distribution to approximate the reliability of complex systems where exact determination of the survival probabilities are analytically intractable.

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# Estimation for Birnbaum-Saunders Distribution in Constant-Stress Accelerated Life Test with Hybrid Censoring

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## Abstract:

This paper introduces the maximum likelihood estimation for the Birnbaum-Saunders distribution based on hybrid censoring in the constant-stress accelerated life test with power law accelerated form. Maximum likelihood estimates (MLEs) of an accelerated life tests model are derived. Prediction of the scale parameter under usual conditions is obtained. Moreover, the reliability function at a certain mission time under the same conditions is also predicted. A Monte Carlo Simulation study is carried out to study the precision of the MLEs for the parameters involved.

**Keywords and Phrases**: Accelerated life testing; Reliability; Constant stress; Birnbaum-Saunders distribution, Maximum likelihood estimation; Inverse power law model; Hybrid censoring

## 1- Introduction

In industrial experiments, most of products and materials with high reliability are difficult to obtain information about their lifetime under normal conditions. It is often very costly and time consuming. To avoid these drawbacks and improve

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the efficiency of experiment, one of the choices may be to resort to Accelerated life testing (ALT) which is often used in order to induce failures of very high reliability devices. In such tests, the units are tested under accelerated conditions (stress) that is extremely more severe than normal use conditions encountered in practice. So, the time necessary to test a sample of such devices under normal conditions very costly, take a long time. So, ALT save time and cost. Stress would induce early failure of the tested units. Over stress may be in the form of temperature, voltage, pressure, load, humidity, vibration,…etc, or some combination of them.

As Nelson [1990] indicated, the stress can be applied in various ways, commonly used methods are constant-stress and step-stress. In a constant stress accelerated test, each unit is run at a prespecified constant stress level which does not vary with time. In use, most products such as semiconductors and microelectronics, capacitors, lamps, …etc, run at a constant stress. This type of stress is widely used and preferred because the stress is constant in most applications, it is much easier to apply and quantify constant stress and models for constant stress are available, widely publicized and empirically verified.

Determining a relationship between stress and scale parameter of the lifetime distribution, extrapolation to the normal use conditions will be carried. This relationship is known as the acceleration model. It is assumed that changing the stress from one level to another affects the value of the parameter only and not the functional form of the lifetime distribution. This is a major assumption of ALT.

Constant stress ALT, was applied by several authors. Among others, are Owen [1997], Aly [1997], Abdel-Ghaly et al. [1998] , Al-Hussaini and Abd-Hamid[2004,2006] andWatkins and John [2008].

Birnbaum and Saunders [1969] proposed a two-parameter failure time distribution for fatigue failure caused under cyclic loading. Fatigue failure based on a model which assumes that failure is due to the development and growth of a dominant crack. This distribution is known as the two-parameter Birnbaum-Saunders ( BS) distribution or as the fatigue life distribution. As Artur and Silvia [2011] indicated that, this distribution is an attractive alternative to the Weibull, Gamma and Log-normal models, since its derivation considers the basic characteristics of the fatigue process.

Statistical analysis for the BS distribution has been limited to complete data by several authors,( for example, see Chang and Tang [1993], Owen [1997], Dupuis and Mills [1998], Xu and Tang [2010]).

little works have been done based on censored data, (for example, see Rieck [1995], Kevin [1999] ,Jeng [2003], Ng et al. [2006],Wang et al[2006], Xu and Tang [2011],Artur and Silvia [2011]) and Attia et al. [2013a,b] , Hamouda [2014,2015]) and Sun and Shi [2016].

The estimates from the censored data are less accurate than those from the complete data. However, this is more than offset by the reduced test time and expense. The most common censoring schemes are type-I and type-II censoring. Consider n units placed on life test, in conventional type-I censoring, the experiment continues up to a pre-specified time . But in conventional type-II censoring , the experiment is terminated after a pre-chosen number of units $r < n$ of failures .The mixture of type-I and type-II censoring schemes is known as hybrid censoring scheme, which was introduced by Epstein (1954) , and it can be described as follows. Consider the following life testing experiment in which n units are placed on test. The lifetimes of the sample units are assumed to be independent and identically (i.i.d) random variables. The test is terminated when a pre-chosen $< n$ , out of n units have failed, or when a prespecified time, $\eta$ , has been reached. In other words, the life test is terminated at a random time $t_{(R_j)} = min\left(t_{(r)}, \eta\right)$ . It is also usually assumed that the failed units are not replaced in the experiment. From now on, we refer to this hybrid censoring scheme as type-I hybrid censoring scheme (Type-I HCS).

In this paper, we discuss the MLEs of constant-stress ALT for two-parameter BS distribution under Type- I HCS. This model is described in details in Section 2, and the value of the scale parameter and the reliability function at a mission time $t_0$ under usual stress are predicted. Section 3 explains the simulation studies for illustrating the theoretical results. Finally, conclusion are included in Section 4.

## 2. Model description and maximum likelihood estimation

Assume the lifetime T is assumed to have a two - parameter BS distribution with the shape and scale parameters α and β respectively. So, the probability density function of T is

$$f(t;\alpha,\beta)=\frac{1}{2\alpha\sqrt{2\pi\beta}}\frac{(t+\beta)}{\left(t^{\frac{3}{2}}\right)}exp\left[\frac{-1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right], t>o\ , \alpha,\beta>o \qquad [1]$$

The scale parameter ($\beta$) is the median of the $B$S distribution which has a wide use in reliability studies. The cumulative distribution function is as follows:

$$F(t;\alpha,\beta)=\Phi\left(\frac{1}{\alpha}\sqrt{\left(\frac{t}{\beta}\right)}-\sqrt{\left(\frac{\beta}{t}\right)}\right), \qquad [2]$$

where Φ (.) denotes the standard normal distribution function. And the reliability function of $BS$ distribution in (1) takes the form:

$$R(t)=1-F(t)=1-\Phi\left(\frac{1}{\alpha}\left(\sqrt{\left(\frac{t}{\beta}\right)}-\sqrt{\left(\frac{\beta}{t}\right)}\right)\right) \qquad [3]$$

The scale parameters $\beta$ and the stress V are related by the inverse power law model, which is described in Mann et al. [1974].

$$\beta_{\mathrm{j}}=CV_j^{-P} \qquad ,C,P>o \qquad j=1,2,..k \qquad [4]$$

Where C is the constant of proportionality, and P is the power of the applied stress, which are the parameters of this model.

Assume that the life testing experiment is conducted under K higher stress $V_j$, j=1,…k and assume that $V_u$ be the normal use conditions such that $V_u<V_1<V_2<\ldots<V_k$. At each $V_j$ there are $n_j$ units put on test. The total number of units in the experiments is $N=\sum_{j=1}^{k} n_j$. When a Type-I HCS is adopted at each stress level, the experiment terminates once the number of pre-chosen number $r_j$ of failures , or pre-determined time $\eta_j$ whichever is reached first.

When $t_{(1j)} \le t_{(2j)} \le \cdots \le t_{(r_j j)}$ denote the observed failure times if $t_{(r_j j)} < \eta_j$ and $t_{(1j)} \le t_{(2j)} \le \cdots \le t_{(d_j j)}$ denote the observed failure times if $t_{(d_j j)} < \eta_j, d_j < r_j$ *and the* $\left((d+1)_j\right)^{th}$ failure does not take place before time point $\eta_j$. So, sampling according to the type-I HCS is terminated at:

$$t_{(R_j)} = \min\left(t_{(r_j)}, \eta_j\right). \qquad [5]$$

Such that : $R_j = r_j$ Case I, otherwise $R_j = d_j$ case II,

Where $d_j$ denotes the number of failures observed before time point $\eta_j$.

The lifetime $t_{ij}$ at stress $V_j$, i=1, .. .$R_j$ and j=1,..k, are assumed to be two-parameter BS distribution. The likelihood function of the experiment is considered to have the following form:

$$L(\alpha, C, P) = \prod_{j=1}^{k} \left[\frac{n_j!}{(n_j - R_j)!}\right] \left[\prod_{i=1}^{R_j} f\left(t_{ij}; \alpha, C, P\right)\right] \left[1 - F\left(t_{(R_j j)}\right)\right]^{n_j - R_j}$$

$$L(\alpha,C,P)=\prod_{j=1}^{K}\frac{n_j!}{(n_j-R_j)!}\left[\prod_{i=1}^{R_j}\frac{1}{2\alpha\sqrt{2\pi(CV_j^{-P})}}\frac{(t_{ij}+CV_j^{-P})}{(t_{ij})^{\frac{2}{3}}}exp\left(\frac{-1}{2\alpha^2}\left(\frac{t_{ij}}{CV_j^{-P}}+\frac{CV_j^{-P}}{t_{ij}}-2\right)\right)\right]\left[1-\Phi\left(\frac{1}{\alpha}\left(\sqrt{\frac{t_{(R_jj)}}{CV_j^{-P}}}-\sqrt{\frac{CV_j^{-P}}{t_{(R_jj)}}}\right)\right)\right]^{n_j-R_j} \quad [6]$$

From a statistical point of view, the method of maximum likelihood yields estimators with good statistical properties. Since these estimators do not always exist in closed from, numerical techniques are used to compute them.

This section discusses the process of obtaining the maximum likelihood estimates of the parameters $\alpha$, C and $\beta$ based on type-I HCS. Both point and interval estimations of the parameters are derived.

Using lnL to denote the natural logarithm of L$(\alpha,C,P)$, then we have

$$lnL=constant-\sum_{j=1}^{k}R_j\left[ln\alpha+\frac{lnC}{2}\right]+\frac{P}{2}\sum_{j=1}^{k}R_j\,lnV_j+\sum_{j=1}^{k}\sum_{i=1}^{R_j}ln\left(t_{ij}+CV_j^{-P}\right)-\frac{2}{3}\sum_{j=1}^{k}\sum_{i=1}^{R_j}ln\left(t_{ij}\right)-\frac{1}{2\alpha^2}\sum_{j=1}^{k}\sum_{i=1}^{R_j}\left(\frac{t_{ij}}{CV_j^{-P}}+\frac{CV_j^{-P}}{t_{ij}}-2\right)+\sum_{j=1}^{k}(n_j-R_{j_j})W(R_jj) \quad [7]$$

where $constant=\sum_{j=1}^{k}R_j\left(\ln n_j!-ln(n_j-R_j)!\right)-kR_j\,ln(2\sqrt{2\pi}\,)$ ,

$$W(R_jj)=\ln\left[1-\Phi\left(\omega_{(R_jj)}\right)\right] \quad , \quad \omega_{(R_jj)}=\frac{1}{\alpha}\left(\sqrt{\frac{t_{(R_jj)}}{CV_j^{-P}}}-\sqrt{\frac{CV_j^{-P}}{t_{(R_jj)}}}\right)$$

MLEs of α, C and P are obtained by getting the partial derivatives of lnL with respect to α, C and P respectively , as follows:

$$\frac{\partial lnL}{\partial\alpha}=-\sum_{j=1}^{k}R_j\quad\left(\frac{1}{\alpha}\right)+\left(\frac{1}{\alpha^3}\right)\sum_{j=1}^{k}\sum_{i=1}^{R_j}\left(\frac{t_{ij}}{CV_j^{-P}}+\frac{CV_j^{-P}}{t_{ij}}-2\right)+\sum_{j=1}^{k}(n_j-R_j)\frac{\partial W(R_jj)}{\partial\alpha} \quad [8]$$

$$\frac{\partial lnL}{\partial C}=-\sum_{j=1}^{k}R_j\left(\frac{1}{2C}\right)+\sum_{j=1}^{k}\sum_{i=1}^{R_j}\frac{V_j^{-P}}{\left(t_{ij}+CV_j^{-P}\right)}-\frac{1}{2\alpha^2}\sum_{j=1}^{k}\sum_{i=1}^{R_j}\left(\frac{V_j^{-P}}{t_{ij}}-\frac{t_{ij}}{C^2V_j^{-P}}\right)+\sum_{j=1}^{k}\left(n_j-R_{j_j}\right)\frac{\partial W(R_jj)}{\partial C} \quad [9]$$

$$\frac{\partial lnL}{\partial P}=\frac{1}{2}\sum_{j=1}^{k}R_j\ln V_J-\sum_{j=1}^{k}\ln V_J\sum_{i=1}^{R_j}\frac{CV_j^{-P}}{\left(t_{ij}+CV_j^{-P}\right)}-\frac{1}{2\alpha^2}\sum_{j=1}^{k}\ln V_j\sum_{i=1}^{R_j}\left(\frac{t_{ij}}{CV_j^{-P}}-\frac{CV_j^{-P}}{t_{ij}}\right)+\sum_{j=1}^{k}\left(n_j-R_j\right)\frac{\partial W(R_jj)}{\partial P} \quad [10]$$

For notational convenience, let us denote the hazard function of the standard normal distribution as: $H(y)=\frac{\emptyset(y)}{1-\Phi(y)}$

$$\frac{\partial W(R_jj)}{\partial \alpha}=H\left(\omega_{(R_jj)}\right)*\frac{1}{\alpha}\left(\omega_{(R_jj)}\right) \quad [11]$$

$$\frac{\partial W(R_jj)}{\partial C}=H\left(\omega_{(R_jj)}\right)*\frac{1}{2\alpha}\left(\sqrt{\frac{t_{(R_jj)}}{C^3V_j^{-P}}}+\sqrt{\frac{V_j^{-P}}{Ct_{(R_jj)}}}\right) \quad [12]$$

$$\frac{\partial W(R_jj)}{\partial P}=H\left(\omega_{(R_jj)}\right)*\frac{-(\ln V_J)}{2\alpha}\left(\sqrt{\frac{t_{(R_jj)}}{CV_j^{-P}}}+\sqrt{\frac{CV_j^{-P}}{t_{(R_jj)}}}\right) \quad [13]$$

To obtain $\hat{\alpha}$, $\hat{C}$ and $\hat{P}$ at which the log likelihood function is maximized, equating the equations (8-10) to zero. Since the closed form solution to thesis equations do not exist. The numerical iteration will be used to solve these equations. To estimate the variance-covariance matrix of the estimated parameters, we use the second derivatives of the logarithm of the likelihood function defined in equation (6).The second derivatives are used to get the information matrix (F), and hence the inverse of F is the asymptotic variance-covariance matrix. The Fisher-information can be expressed in the form:

$$F=\begin{bmatrix}\frac{-\partial^2\ln L}{\partial\alpha^2} & \frac{-\partial^2\ln L}{\partial\alpha\partial C} & \frac{-\partial^2\ln L}{\partial\alpha\partial P}\\ \frac{-\partial^2\ln L}{\partial C\partial\alpha} & \frac{-\partial^2\ln L}{\partial C^2} & \frac{-\partial^2\ln L}{\partial C\partial P}\\ \frac{-\partial^2\ln L}{\partial P d\alpha} & \frac{-\partial^2\ln L}{\partial P\partial C} & \frac{-\partial^2\ln L}{\partial C^2}\end{bmatrix} \quad [14]$$

The inverse of F is the asymptotic variance-covariance matrix of $\left(\hat{\alpha}, \hat{C} \text{ and } \hat{P}\right)$.

$$v = F^{-1} \qquad [15]$$

Then, the second partial derivates are given as follows:

$$\frac{\partial^2 lnL}{\partial \alpha^2} = \sum_{j=1}^{k} R_j \frac{1}{\alpha^2} - \frac{3}{\alpha^4} \sum_{j=1}^{k} \sum_{i=1}^{R_j} \left( \frac{t_{ij}}{CV_j^{-P}} + \frac{CV_j^{-P}}{t_{ij}} - 2 \right) + \sum_{j=1}^{k} (n_j - R_j) \frac{\partial^2 W(R_j j)}{\partial \alpha^2} \qquad [16]$$

$$\frac{\partial^2 lnL}{\partial C^2} = \sum_{j=1}^{k} R_j \frac{1}{2C^2} - \sum_{j=1}^{k} \sum_{i=1}^{R_j} \frac{V_j^{-2P}}{\left(t_{ij} + CV_j^{-P}\right)^2} - \frac{1}{\alpha^2} \sum_{j=1}^{k} \sum_{i=1}^{R_j} \left( \frac{t_{ij}}{C^3 V_j^{-P}} \right) + \sum_{j=1}^{k} (n_j - R_j) \frac{\partial^2 W(R_j j)}{\partial C^2} \qquad [17]$$

$$\frac{\partial^2 lnL}{\partial P^2} = C \sum_{j=1}^{k} (\ln V_j)^2 \sum_{i=1}^{R_j} \left( \frac{t_{ij} V_j^{-P}}{\left(t_{ij} + CV_j^{-P}\right)^2} \right) - \frac{1}{2\alpha^2} \sum_{j=1}^{k} (\ln V_j)^2 \sum_{i=1}^{R_j} \left( \frac{t_{ij}}{CV_j^{-P}} + \frac{CV_j^{-P}}{t_{ij}} \right) + \sum_{j=1}^{k} (n_j - R_j) \frac{\partial^2 W(R_j j)}{\partial P^2} \qquad [18]$$

$$\frac{\partial^2 lnL}{\partial \alpha \partial C} = \frac{1}{\alpha^3} \sum_{j=1}^{k} \sum_{i=1}^{R_j} \left( \frac{V_j^{-P}}{t_{ij}} - \frac{t_{ij}}{C^2 V_j^{-P}} \right) + \sum_{j=1}^{k} (n_j - R_j) \frac{\partial^2 W(R_j j)}{\partial \alpha \partial C} \qquad [19]$$

$$\frac{\partial^2 lnL}{\partial \alpha \partial P} = \frac{1}{\alpha^3} \sum_{j=1}^{k} (\ln V_j) \sum_{i=1}^{R_j} \left( \frac{t_{ij}}{CV_j^{-P}} - \frac{CV_j^{-P}}{t_{ij}} \right) + \sum_{j=1}^{k} (n_j - R_j) \frac{\partial^2 W(R_j j)}{\partial \alpha \partial P} \qquad [20]$$

$$\frac{\partial^2 lnL}{\partial C \partial P} = -\sum_{j=1}^{k} \ln V_j \sum_{i=1}^{R_j} \frac{t_{ij} V_j^{-P}}{\left(t_{ij} + CV_j^{-P}\right)^2} + \frac{1}{2\alpha^2} \sum_{j=1}^{k} (\ln V_j) \sum_{i=1}^{R_j} \left( \frac{V_j^{-P}}{t_{ij}} + \frac{t_{ij}}{C^2 V_j^{-P}} \right) + \sum_{j=1}^{k} (n_j - R_j) \frac{\partial^2 W(R_j j)}{\partial C \partial P} \qquad [21]$$

$$Where \ \frac{\partial^2 W(R_j j)}{\partial \alpha^2} = H^{`}\left(\omega_{(R_j j)}\right) * \frac{1}{\alpha} \left(\omega_{(R_j j)}\right) - H\left(\omega_{(R_j j)}\right) * \frac{1}{\alpha^2} \left(\omega_{(R_j j)}\right) \qquad [22]$$

$$\frac{\partial^2 W(R_j j)}{\partial C^2} = H^{`}\left(\omega_{(R_j j)}\right) \frac{1}{2\alpha} \left( \sqrt{\frac{t_{(R_j j)}}{C^3 V_j^{-P}}} + \sqrt{\frac{V_j^{-P}}{C t_{(R_j j)}}} \right) - H\left(\omega_{(R_j j)}\right) \frac{1}{4\alpha} \left( 3\sqrt{\frac{t_{(R_j j)}}{C^5 V_j^{-P}}} + \sqrt{\frac{V_j^{-P}}{C^3 t_{(R_j j)}}} \right) \qquad [23]$$

$$\frac{\partial^2 W(R_j j)}{\partial P^2} = H^{`}\left(\omega_{(R_j j)}\right) \frac{-(\ln V_J)}{2\alpha} \left( \sqrt{\frac{t_{(R_j j)}}{CV_j^{-P}}} + \sqrt{\frac{CV_j^{-P}}{t_{(R_j j)}}} \right) + H\left(\omega_{(R_j j)}\right) \frac{(\ln V_J)^2}{4\alpha} \left( \sqrt{\frac{t_{(R_j j)}}{CV_j^{-P}}} - \sqrt{\frac{CV_j^{-P}}{t_{(R_j j)}}} \right) \qquad [24]$$

$$\frac{\partial^2 W(R_j j)}{\partial \alpha \partial C} = H^{`}\left(\omega_{(R_j j)}\right) \frac{1}{\alpha} \left(\omega_{(R_j j)}\right) - H\left(\omega_{(R_j j)}\right) \frac{1}{2\alpha} \left( \sqrt{\frac{t_{(R_j j)}}{C^3 V_j^{-P}}} + \sqrt{\frac{V_j^{-P}}{C t_{(R_j j)}}} \right) \qquad [25]$$

$$\frac{\partial^2 W(R_j j)}{\partial \alpha \partial P} = H^{`}\left(\omega_{(R_j j)}\right) \frac{1}{\alpha} \left(\omega_{(R_j j)}\right) + H\left(\omega_{(R_j j)}\right) \frac{(\ln V_J)}{2\alpha} \left( \sqrt{\frac{t_{(R_j j)}}{CV_j^{-P}}} + \sqrt{\frac{CV_j^{-P}}{t_{(R_j j)}}} \right) \qquad [26]$$

$$\frac{\partial^2 W(R_j j)}{\partial C \partial P} = H^{`}\left(\omega_{(R_j j)}\right) \frac{1}{2\alpha} \left( \sqrt{\frac{t_{(R_j j)}}{C^3 V_j^{-P}}} + \sqrt{\frac{V_j^{-P}}{C t_{(R_j j)}}} \right) + H\left(\omega_{(R_j j)}\right) \frac{(\ln V_J)}{4\alpha} \left( \sqrt{\frac{t_{(R_j j)}}{C^3 V_j^{-P}}} - \sqrt{\frac{V_j^{-P}}{C t_{(R_j j)}}} \right) \qquad [27]$$

Getting Fisher-information matrix, it can be said that the MLE $\hat{\alpha}$, $\hat{C}$ and $\hat{P}$ have an asymptotic variance-covariance matrix defined by inverting the Fisher-information matrix.

The approximate confidence intervals of the parameters are derived based on the asymptotic distribution of the MLEs of the elements of the vector of unknown parameters $\Theta = (\alpha, C, P)$. It is known that the asymptotic distribution of the MLEs of $\Theta = (\alpha, C, P)$ is given as follows: $\left(\frac{(\hat{\alpha}-\alpha)}{\sqrt{\mathrm{var}(\hat{\alpha})}}, \frac{(\hat{C}-C)}{\sqrt{\mathrm{var}(\hat{C})}}, \frac{(\hat{P}-P)}{\sqrt{\mathrm{var}(\hat{P})}}\right)$, which can be approximated by a standard normal distribution, see Miller [1981], where $var(\widehat{\Theta})$ is estimated as the asymptotic variance, obtained from[15]. Then, the approximate $100\gamma$ % two sided confidence interval for α, C, P are, respectively, given by:

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\alpha})} \;, \hat{C} \pm Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{C})} \quad and \; \hat{P} \pm Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{P})} \;, \qquad [28]$$

where $Z_{\gamma/2}$ is the $100(\gamma/2)$standard normal percentile.

## 4. Simulation Studies

In this section simulation studies using the MathCAD(14) are conducted to investigate the performances of the maximum likelihood estimates through their estimators, their absolute relative bias (RABias), mean square error (MSE) and the confidence intervals of the MLEs. Using the invariance property of

MLEs, we can estimate the MLE of scale parameter $\beta_j$ through the following equation

$$\hat{\beta}_j = \hat{C} V_j^{-\hat{P}} \qquad , C, P > 0 \quad j = 1,2,3 \qquad [29]$$

The Simulation procedures were described as follows:

Step1. 1000 random samples of sizes 90, 150, 210, 300 and 450 were generated from the BS distribution. Different initial values are selected for all sets of $(\alpha_0, C_0$ and $P_0)$ . There are only three different levels of stress, k=3, the stress values are selected as$(V_1 = .9, V_2 = 1.4, V_3 = 1.9)$. $n_j = \frac{n}{3}$ , $and\ r_j = 60\% n_j\ and\ \eta_1 = 5, \eta_2 = 3, \eta_3 = 2.$

Step2. For all sample sizes, the parameters of the model are estimated under Type-I HCS, for all sets of $(\alpha_0, C_0$ and $P_0)$.

Step3. Equations 8,9 and 10 are solved by using the Numerical method. The estimate of scale parameter $\beta_j$ was obtained from equation [29].

Step4. The estimators , RABias and MSE are tabulated for all sets of $(\alpha_0, C_0$ and $P_0)$ in Tables (1-3) based on Type-I HCS.

Step5. The confidence limit with confidence level $(1 - \gamma) = 0.95$ are tabulated for all sets of parameters in Tables (4-6) based on Type-I HCS.

Step6. Using the invariance property of MLEs, the MLE of the scale parameter $\beta_u$ of BS distribution at usual stress $V_u = .5$, can be estimated by using the following equation:

$$\hat{\beta}_{u} = \hat{C}.5^{-\hat{P}} \qquad ,C,P > 0 \quad j = 1,2,3 \qquad [30]$$

Also, the reliability function at mission time $(t_1 = 1.5, t_2 = 1.8\,, t_3 = 2.1\ and\ t_4 = 2.4)$ are predicted under the same usual condition for all sets of $(\alpha_0, C_0$ and $P_0)$ in Tables [7-9].

## 4. Conclusion

The performances of the estimators are investigated by simulation study and from results of Tables [1-6], it is observed the following :

1- The MLEs for all sets of initial values of parameters have good statistical properties for all sample sizes. As the sample size increases, the MSE of estimators decreases.

2- As the sample size increases, the interval of estimators decreases.

3-As the stress increases, it is evident that the MSE of the estimated scale parameter $\beta_j$ tend to decrease.

4- As the value of $\alpha_0$ decrease, it is evident that the MSE of estimates decrease.

5- It is evident that the MSE of the estimates do not rely on varying the values of $P_0$ and $C_0$ .

From results of Tables [7-9], it is observed the following:

1-As the sample size increases, the reliability function increases.

2- As the values of $C_0$ and $P_0$ increase at the same mission time, the reliability function increases.

3- There is an inverse proportional relationship between the shape parameter $\alpha_0$ and the reliability function.

**Table (1): The Estimates, RABias and MSE of the parameters under Type-I HCS**

| n | Parameters | $r_j = 60\% n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\boldsymbol{\alpha = .25, C = 1.5, P = 1}$ | | | $\boldsymbol{\alpha = .5, C = 1.5, P = 1}$ | | |
| | | Estimators | RABias | MSE | Estimators | RABias | MSE |
| **30** | $\alpha$ | 0.228 | 0.089 | 0.002 | 0.456 | 0.087 | 0.01 |
| | C | 1.49 | 0.006 | 0.011 | 1.49 | 0.005 | 0.045 |
| | P | 1.007 | 0.007 | 0.026 | 1.013 | 0.013 | 0.103 |
| | $\beta_1$ | 1.66 | 0. 004 | 0.019 | 1.667 | 0.003 | 0.077 |
| | $\beta_2$ | 1.06 | 0.009 | 0.003 | 1.059 | 0.014 | 0.012 |
| | $\beta_3$ | 0.788 | 0.009 | 0.003 | 0.783 | 0.007 | 0.014 |
| **90** | $\alpha$ | 0.242 | 0.033 | 0.0007 | 0.483 | 0.033 | 0.003 |
| | C | 1.49 | 0.002 | 0.004 | 1.49 | 0.003 | 0.014 |
| | P | 1.001 | 0.0006 | 0.008 | 0.993 | 0.007 | 0.033 |
| | $\beta_1$ | 1.66 | 0.002 | 0.006 | 1.663 | 0.002 | 0.023 |
| | $\beta_2$ | 1.07 | 0.003 | 0.001 | 1.068 | 0.002 | 0.004 |
| | $\beta_3$ | 0.788 | 0.002 | 0.0004 | 0.792 | 0.003 | 0.005 |
| **150** | $\alpha$ | 0.246 | 0.016 | 0.002 | 0.492 | 0.016 | 0.002 |
| | C | 1.498 | 0.001 | 0.005 | 1.499 | 0.0009 | 0.008 |
| | P | 1.001 | 0.001 | 0.004 | 1.004 | 0.004 | 0.019 |
| | $\beta_1$ | 1.665 | 0.0007 | 0.0006 | 1.667 | 0.0001 | 0.014 |
| | $\beta_2$ | 1.069 | 0.001 | 0.0007 | 1.068 | 0.003 | 0.002 |
| | $\beta_3$ | 0.788 | 0.002 | 0.0007 | 0.787 | 0.003 | 0.003 |
| **210** | $\alpha$ | 0.247 | 0.013 | 0.0003 | 0.494 | 0.012 | 0.001 |
| | C | 1.499 | 0.0009 | 0.002 | 1.499 | 0.0004 | 0.006 |
| | P | 1.001 | 0.0008 | 0.004 | 1.002 | 0.002 | 0.014 |
| | $\beta_1$ | 1.665 | 0.0007 | 0.003 | 1.667 | 0.0003 | 0.010 |
| | $\beta_2$ | 1.071 | 0.001 | 0.0004 | 1.07 | 0.002 | 0.001 |
| | $\beta_3$ | 0.788 | 0.001 | 0.0005 | 0.788 | 0.002 | 0.002 |
| **300** | $\alpha$ | 0.248 | 0.009 | 0.0002 | 0.496 | 0.007 | 0.0008 |
| | C | 1.5 | 0.0002 | 0.001 | 1.5 | 0.0003 | 0.004 |
| | P | 1 | 0.0009 | 0.002 | 1.002 | 0.002 | 0.009 |
| | $\beta_1$ | 1.001 | 0.0004 | 0.001 | 1.667 | 0.0003 | 0.007 |
| | $\beta_2$ | 1.071 | 0.0002 | 0.0003 | 1.07 | 0.001 | 0.001 |
| | $\beta_3$ | 0.789 | 0.00004 | 0.0003 | 0.788 | 0.002 | 0.001 |

**Table (2): The Estimates, RABias and MSE of the parameters under Type-I HCS**

| n | Parameters | $r_j = 60\% n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\boldsymbol{\alpha = .5, C = 1.5, P = 1}$ | | | $\boldsymbol{\alpha = .5, C = 2, P = 1}$ | | |
| | | Estimators | RABias | MSE | Estimators | RABias | MSE |
| **30** | α | 0.456 | 0.087 | 0.01 | 0.453 | 0.094 | 0.009 |
| | C | 1.49 | 0.005 | 0.045 | 1.97 | 0.012 | 0.07 |
| | P | 1.013 | 0.013 | 0.103 | 1.009 | 0.009 | 0.104 |
| | $\beta_1$ | 1.667 | 0.003 | 0.077 | 2.205 | 0.008 | 0.123 |
| | $\beta_2$ | 1.059 | 0.014 | 0.012 | 1.402 | 0.019 | 0.021 |
| | $\beta_3$ | 0.783 | 0.007 | 0.014 | 1.037 | 0.015 | 0.026 |
| **90** | α | 0.483 | 0.033 | 0.003 | 0.487 | 0.026 | 0.003 |
| | C | 1.49 | 0.003 | 0.014 | 1.99 | 0.0003 | 0.024 |
| | P | 0.993 | 0.007 | 0.033 | 1.002 | 0.002 | 0.033 |
| | $\beta_1$ | 1.663 | 0.002 | 0.023 | 2.224 | 0.0009 | 0.123 |
| | $\beta_2$ | 1.068 | 0.002 | 0.004 | 1.426 | 0.002 | 0.04 |
| | $\beta_3$ | 0.792 | 0.003 | 0.005 | 1.052 | 0.0004 | 0.007 |
| **150** | α | 0.492 | 0.016 | 0.002 | 0.491 | 0.018 | 0.002 |
| | C | 1.499 | 0.0009 | 0.008 | 1.99 | 0.001 | 0.015 |
| | P | 1.004 | 0.004 | 0.019 | 1 | 0.00005 | 0.019 |
| | $\beta_1$ | 1.667 | 0.0001 | 0.014 | 2.22 | 0.001 | 0.026 |
| | $\beta_2$ | 1.068 | 0.003 | 0.002 | 1.425 | 0.002 | 0.004 |
| | $\beta_3$ | 0.787 | 0.003 | 0.003 | 1.051 | 0.002 | 0.005 |
| **210** | α | 0.494 | 0.012 | 0.001 | 0.495 | 0.009 | 0.001 |
| | C | 1.499 | 0.0004 | 0.006 | 2.003 | 0.001 | 0.011 |
| | P | 1.002 | 0.002 | 0.014 | 1.003 | 0.003 | 0.014 |
| | $\beta_1$ | 1.667 | 0.0003 | 0.010 | 2.227 | 0.002 | 0.018 |
| | $\beta_2$ | 1.07 | 0.002 | 0.001 | 1.428 | 0.0002 | 0.003 |
| | $\beta_3$ | 0.788 | 0.002 | 0.002 | 1.052 | 0.0004 | 0.003 |
| **300** | α | 0.496 | 0.007 | 0.0008 | 0.496 | 0.008 | 0.0009 |
| | C | 1.5 | 0.0003 | 0.004 | 1.999 | 0.0001 | 0.007 |
| | P | 1.002 | 0.002 | 0.009 | 1.004 | 0.003 | 0.009 |
| | $\beta_1$ | 1.667 | 0.0003 | 0.007 | 2.223 | 0.0003 | 0.013 |
| | $\beta_2$ | 1.07 | 0.001 | 0.001 | 1.426 | 0.002 | 0.002 |
| | $\beta_3$ | 0.788 | 0.002 | 0.001 | 1.05 | 0.002 | 0.002 |

**Table (3): The Estimates, RABias and MSE of the parameters under Type-I HCS**

| n | Parameters | $r_j = 60\% n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\alpha = .5, C = 1.5, P = 1$ | | | $\alpha = .5, C = 1.5, P = 1.5$ | | |
| | | Estimators | RABias | MSE | Estimators | RABias | MSE |
| **30** | $\alpha$ | 0.456 | 0.087 | 0.01 | 0.485 | 0.084 | 0.009 |
| | C | 1.49 | 0.005 | 0.045 | 1.48 | 0.009 | 0.042 |
| | P | 1.013 | 0.013 | 0.103 | 1.49 | 0.004 | 0.098 |
| | $\beta_1$ | 1.667 | 0.003 | 0.077 | 1.75 | 0.007 | 0.079 |
| | $\beta_2$ | 1.059 | 0.014 | 0.012 | 0.89 | 0.012 | 0.008 |
| | $\beta_3$ | 0.783 | 0.007 | 0.014 | 0.57 | 0.004 | 0.007 |
| **90** | $\alpha$ | 0.483 | 0.033 | 0.003 | 0.485 | 0.04 | 0.003 |
| | C | 1.49 | 0.003 | 0.014 | 1.49 | 0.001 | 0.014 |
| | P | 0.993 | 0.007 | 0.033 | 1.5 | 0.004 | 0.034 |
| | $\beta_1$ | 1.663 | 0.002 | 0.023 | 1.75 | 0.0008 | 0.027 |
| | $\beta_2$ | 1.068 | 0.002 | 0.004 | 0.901 | 0.005 | 0.002 |
| | $\beta_3$ | 0.792 | 0.003 | 0.005 | 0.57 | 0.004 | 0.002 |
| **150** | $\alpha$ | 0.492 | 0.016 | 0.002 | 0.491 | 0.019 | 0.002 |
| | C | 1.499 | 0.0009 | 0.008 | 1.498 | 0.001 | 0.008 |
| | P | 1.004 | 0.004 | 0.019 | 1.5 | 0.003 | 0.019 |
| | $\beta_1$ | 1.667 | 0.0001 | 0.014 | 1.757 | 0.0001 | 0.016 |
| | $\beta_2$ | 1.068 | 0.003 | 0.002 | 0.902 | 0.004 | 0.001 |
| | $\beta_3$ | 0.787 | 0.003 | 0.003 | 0.57 | 0.004 | 0.001 |
| **210** | $\alpha$ | 0.494 | 0.012 | 0.001 | 0.495 | 0.011 | 0.001 |
| | C | 1.499 | 0.0004 | 0.006 | 1.495 | 0.003 | 0.005 |
| | P | 1.002 | 0.002 | 0.014 | 1.49 | 0.003 | 0.013 |
| | $\beta_1$ | 1.667 | 0.0003 | 0.010 | 1.75 | 0.003 | 0.011 |
| | $\beta_2$ | 1.07 | 0.002 | 0.001 | 0.904 | 0.002 | 0.001 |
| | $\beta_3$ | 0.788 | 0.002 | 0.002 | 0.573 | 0.0003 | 0.001 |
| **300** | $\alpha$ | 0.496 | 0.007 | 0.0008 | 0.495 | 0.009 | 0.0009 |
| | C | 1.5 | 0.0003 | 0.004 | 1.496 | 0.002 | 0.004 |
| | P | 1.002 | 0.002 | 0.009 | 1.497 | 0.002 | 0.01 |
| | $\beta_1$ | 1.667 | 0.0003 | 0.007 | 1.75 | 0.002 | 0.007 |
| | $\beta_2$ | 1.07 | 0.001 | 0.001 | 0.904 | 0.002 | 0.0009 |
| | $\beta_3$ | 0.788 | 0.002 | 0.001 | 0.573 | 0.0004 | 0.0008 |

**Table (4): Confidence bounds of the parameters at confidence level .95 under Type-I HCS**

| n | Parameters | $r_j = 60\% n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\alpha = .25, C = 1.5, P = 1$ | | | $\alpha = .5, C = 1.5, P = 1$ | | |
| | | Estimators | Variance | (Lower,Upper) | Estimators | Variance | (Lower,Upper) |
| **30** | α | 0.228 | 0.003 | (0.152,0.304) | 0.456 | 0.01 | (0.305,0.608) |
| | C | 1.49 | 0.011 | (1.320,1.680) | 1.49 | 0.045 | (1.167,1.818) |
| | P | 1.007 | 0.026 | (0.740,1.280) | 1.013 | 0.103 | (0.489,1.536) |
| **90** | α | 0.242 | 0.0007 | (0.193,0.291) | 0.483 | 0.003 | (0.385,0.582) |
| | C | 1.49 | 0.004 | (1.39,1.61) | 1.496 | 0.014 | (1.279,1.713) |
| | P | 1.001 | 0.008 | (0.83,1.17) | 0.993 | 0.033 | (0.657,1.328) |
| **150** | α | 0.246 | 0.0004 | (0.207,0.285) | 0.492 | 0.002 | (0.413,0.57) |
| | C | 1.498 | 0.002 | (1.41,1.59) | 1.499 | 0.008 | (1.325,1.67) |
| | P | 1.001 | 0.005 | (0.87,1.14) | 1.004 | 0.019 | (0.738,1.27) |
| **210** | α | 0.247 | 0.0003 | (0.214,0.28) | 0.494 | 0.001 | (0.427,0.561) |
| | C | 1.499 | 0.001 | (1.42,1.57) | 1.499 | 0.006 | (1.351,1.648) |
| | P | 1.001 | 0.004 | (0.886,1.11) | 1.002 | 0.014 | (0.775,1.229) |
| **300** | α | 0.248 | 0.0002 | (0.22,0.276) | 0.496 | 0.0008 | (0.44,0.553) |
| | C | 1.5 | 0.0002 | (1.44,1.563) | 1.5 | 0.004 | (1.374,1.625) |
| | P | 1.001 | 0.0009 | (0.904,1.097) | 1.002 | 0.009 | (0.811,1.193) |

**Table (5): Confidence bounds of the parameters at confidence level .95 under Type-I HCS**

| n | Parameters | $r_j = 60\% n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\alpha = .5, C = 1.5, P = 1$ | | | $\alpha = .5, C = 2, P = 1$ | | |
| | | Estimators | Variance | (Lower,Upper) | Estimators | Variance | (Lower,Upper) |
| **30** | α | 0.456 | 0.01 | (0.305,0.608) | 0.453 | 0.009 | (0.301,0.605) |
| | C | 1.49 | 0.045 | (1.167,1.818) | 1.975 | 0.07 | (1.541,2.41) |
| | P | 1.013 | 0.103 | (0.489,1.536) | 1.009 | 0.104 | (0.486,1.532) |
| **90** | α | 0.483 | 0.003 | (0.385,0.582) | 0.487 | 0.003 | (0.388,0.586) |
| | C | 1.496 | 0.014 | (1.279,1.713) | 1.999 | 0.024 | (1.707,2.292) |
| | P | 0.993 | 0.033 | (0.657,1.328) | 1.002 | 0.033 | (0.664,1.339) |
| **150** | α | 0.492 | 0.002 | (0.413,0.57) | 0.491 | 0.002 | (0.412,0.569) |
| | C | 1.499 | 0.008 | (1.325,1.67) | 1.99 | 0.015 | (1.766,2.227) |
| | P | 1.004 | 0.019 | (0.738,1.27) | 1 | 0.019 | (0.734,1.266) |
| **210** | α | 0.494 | 0.001 | (0.427,0.561) | 0.495 | 0.001 | (0.428,0.592) |
| | C | 1.499 | 0.006 | (1.351,1.648) | 2.003 | 0.011 | (1.804,2.201) |
| | P | 1.002 | 0.014 | (0.775,1.229) | 1.003 | 0.014 | (0.776,1.23) |
| **300** | α | 0.496 | 0.0008 | (0.44,0.553) | 0.496 | 0.0009 | (0.439,0.552) |
| | C | 1.5 | 0.004 | (1.374,1.625) | 1.99 | 0.007 | (1.833,2.166) |
| | P | 1.002 | 0.009 | (0.811,1.193) | 1.004 | 0.009 | (0.813,1.194) |

**Table (6): Confidence bounds of the parameters at confidence level .95 under Type-I HCS**

| n | Parameters | $r_j = 60\% n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\alpha = .5, C = 1.5, P = 1$ | | | $\alpha = .5, C = 1.5, P = 1.5$ | | |
| | | Estimators | Variance | (Lower,Upper) | Estimators | Variance | (Lower,Upper) |
| **30** | α | 0.456 | 0.01 | (0.305,0.608) | 0.485 | 0.009 | (0.305,0.611) |
| | C | 1.49 | 0.045 | (1.167,1.818) | 1.48 | 0.042 | (1.156,1.814) |
| | P | 1.013 | 0.103 | (0.489,1.536) | 1.49 | 0.098 | (0.967,2.022) |
| **90** | α | 0.483 | 0.003 | (0.385,0.582) | 0.485 | 0.003 | (0.386,0.584) |
| | C | 1.496 | 0.014 | (1.279,1.713) | 1.48 | 0.014 | (1.28,1.716) |
| | P | 0.993 | 0.033 | (0.657,1.328) | 1.5 | 0.034 | (1.171,1.844) |
| **150** | α | 0.492 | 0.002 | (0.413,0.57) | 0.491 | 0.002 | (0.412,0.569) |
| | C | 1.499 | 0.008 | (1.325,1.67) | 1.49 | 0.008 | (1.325,1.671) |
| | P | 1.004 | 0.019 | (0.738,1.27) | 1.5 | 0.019 | (1.24,1.771) |
| **210** | α | 0.494 | 0.001 | (0.427,0.561) | 0.495 | 0.001 | (0.428,0.562) |
| | C | 1.499 | 0.006 | (1.351,1.648) | 1.49 | 0.005 | (1.347,1.643) |
| | P | 1.002 | 0.014 | (0.775,1.229) | 1.49 | 0.013 | (1.268,1.722) |
| **300** | α | 0.496 | 0.0008 | (0.44,0.553) | 0.495 | 0.0009 | (0.439,0.551) |
| | C | 1.5 | 0.004 | (1.374,1.625) | 1.49 | 0.004 | (1.372,1.621) |
| | P | 1.002 | 0.009 | (0.811,1.193) | 1.49 | 0.01 | (1.302,1.688) |

Table (7): Estimated shape parameter and the reliability function at $V_u = .5$ with Type-I HCS

| | $r_j = 60\% n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\alpha = .25, C = 1.5, P = 1$ | | | | | $\alpha = .5, C = 1.5, P = 1$ | | | | |
| | **30** | **90** | **150** | **210** | **300** | **30** | **90** | **150** | **210** | **300** |
| $\hat{\beta}_u$ | 3.034 | 3.006 | 3.006 | 3.004 | 3.006 | 3.152 | 3.019 | 3.031 | 3.022 | 3.017 |
| $\hat{R}_u(t_1)$ | .99 | .99 | .99 | .99 | .99 | .89 | .91 | .92 | .92 | .92 |
| $\hat{R}_u(t_2)$ | .96 | .97 | .97 | .97 | .98 | .82 | .83 | .84 | .85 | .85 |
| $\hat{R}_u(t_3)$ | .89 | .92 | .92 | .92 | .92 | .74 | .76 | .76 | .76 | .77 |
| $\hat{R}_u(t_4)$ | .78 | .80 | .81 | .81 | .81 | .65 | .67 | .67 | .67 | .68 |

Table (8): Estimated shape parameter and the reliability function at $V_u = .5$ with Type-I HCS

| | $r_j = 60\%n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\alpha = .5, C = 1.5, P = 1$ | | | | | $\alpha = .5, C = 2, P = 1$ | | | | |
| | **30** | **90** | **150** | **210** | **300** | **30** | **90** | **150** | **210** | **300** |
| $\hat{\beta}_u$ | 3.152 | 3.019 | 3.031 | 3.022 | 3.017 | 4.158 | 4.059 | 4.027 | 4.038 | 4.026 |
| $\hat{R}_u(t_1)$ | .89 | .91 | .92 | .92 | .92 | .96 | .97 | .97 | .97 | .97 |
| $\hat{R}_u(t_2)$ | .82 | .83 | .84 | .85 | .85 | .92 | .94 | .94 | .95 | .95 |
| $\hat{R}_u(t_3)$ | .74 | .76 | .76 | .76 | .77 | .87 | .89 | .90 | .90 | .90 |
| $\hat{R}_u(t_4)$ | .65 | .67 | .67 | .67 | .68 | .81 | .83 | .84 | .84 | .84 |

Table (9): Estimated shape parameter and the reliability function at $V_u = .5$ with Type-I HCS

| | $r_j = 60\%n_j, \eta_1 = 5, \eta_2 = 3, \eta_3 = 2, V_1 = .9, V_2 = 1.4, V_3 = 1.9$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\alpha = .5, C = 1.5, P = 1$ | | | | | $\alpha = .5, C = 1.5, P = 1.5$ | | | | |
| | **30** | **90** | **150** | **210** | **300** | **30** | **90** | **150** | **210** | **300** |
| $\hat{\beta}_u$ | 3.152 | 3.019 | 3.031 | 3.022 | 3.017 | 4.367 | 4.324 | 4.288 | 4.238 | 4.242 |
| $\hat{R}_u(t_1)$ | .89 | .91 | .92 | .92 | .92 | .96 | .98 | .98 | .98 | .98 |
| $\hat{R}_u(t_2)$ | .82 | .83 | .84 | .85 | .85 | .93 | .95 | .95 | .95 | .96 |
| $\hat{R}_u(t_3)$ | .74 | .76 | .76 | .76 | .77 | .88 | .91 | .92 | .92 | .92 |
| $\hat{R}_u(t_4)$ | .65 | .67 | .67 | .67 | .68 | .83 | .86 | .87 | .87 | .87 |

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