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Bayesian and Non-Bayesian Prediction Based on Generalized Order Statistics from Exponentiated Weibull Poisson Distribution

by

Abd-Elfattah A. M, Amal S. Hassan, and Said G. Nassr

This article, discusses the prediction of a first future observation from exponentiated Weibull Poisson distribution based on generalized order statistics. Bayesian and non-Bayesian approaches have been used to obtain the predictors. Bayesian predictors are examined under different loss functions; such as the symmetric (squared error) and asymmetric (LINEX and general entropy). In addition, Bayesian and non-Bayesian prediction intervals are obtained. A numerical study is performed to compare different Bayesian predictors with the maximum likelihood predictors. Furthermore, all results are specialized to upper record values and type II censoring data.

Keywords: *Generalized Order Statistics; Exponentiated Weibull Poisson Distribution; Upper Record Values; Type II Censored Data; Bayesian Prediction.*

1. Introduction

The concept of generalized order statistics (GOS) was introduced by Kamps (1995) as an unified approach which contains many models of ordered random variables with different interpretations. Moreover, many other models of ordered random variables, e.g. ordinary order statistics, ordinary record values, progressively type II censored data and sequential order statistics are all particular cases of the GOS.

The random variables $X(1, n, m, k), X(2, n, m, k), \dots, X(r, n, m, k)$ are called r GOS if their joint probability density function is as follows (see, Kamps 1995))

$$f_{X(1, n, m, k), X(2, n, m, k), \dots, X(r, n, m, k)}(x_{(1)}, x_{(2)}, \dots, x_{(r)}) \\ = C_{r-1} \left(\prod_{i=1}^{r-1} [\bar{F}(x_{(i)})]^{m_i} f(x_{(i)}) \right) [\bar{F}(x_{(r)})]^{\gamma_{r-1}} f(x_{(r)}); \quad (1.1) \\ \text{for } F^{-1}(0) < x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(r)} < F^{-1}(1)$$

where, $C_{r-1} = \left(\prod_{j=1}^{r-1} \gamma_j \right)$, $r = 1, \dots, n$, $\gamma_j = k + n - j + M_j$, $M_j = \sum_{i=j}^{r-1} m_i$, $\gamma_n = k$ and $\bar{F}(x) = 1 - F(x)$.

For suitable choices of the parameters, GOS reduces to the well-known ordered random variable, as follows:

- i. If $m_i = 0 \forall i = 1, \dots, r, k = 1$, and $\gamma_r = n - r + 1$ then (1.1) reduces to type II censored data.
- ii. If $m_i = -1, \forall i = 1, 2, \dots, r - 1, k = 1$ and $\gamma_r = 1$ then (1.1) reduces to upper record values.

There is some research regarding Bayesian and non-Bayesian estimation based on GOS, for example see Ahsanallah (2000), Habibullah and Ahsanullah (2000), Jaheen (2005), Aboeleneen (2010), Abu El Fotouh and Nassar (2011), Abd Ellah (2012), Moghadam, *et al.* (2012), Abushal and AL-Zaydi (2012), Ateya (2013), Abd-Elfattah, *et al.* ((2013), (2015)). Recently there has been a considerable amount of interest in the predication problems, for more details, see Kim and Han (2014), Sharaf El-Deen, *et al.* (2014), Kotb (2015).

Mahmoudi and Sepahdar (2012) introduced a new compounding distribution with decreasing, increasing, bathtub shaped and unimodel failure rate called the exponentiated Weibull Poisson (EWP) distribution. This distribution contains several lifetime sub-models such as; generalized exponentiated Poisson, complementary Weibull Poisson, complementary exponential Poisson, exponential Rayleigh Poisson and Rayleigh Poisson distributions. The probability density function (pdf) of EWP with set of parameters $\phi \equiv (\alpha, \lambda, \theta, \beta)$ takes the following form

$$f(x; \phi) = \frac{\alpha \lambda \theta \beta^\lambda x^{\lambda-1} e^{-(\beta x)^\lambda} \left(1 - e^{-(\beta x)^\lambda}\right)^{\alpha-1} \exp \left[\theta \left(1 - e^{-(\beta x)^\lambda}\right)^\alpha \right]}{(e^\theta - 1)}; \quad x > 0, \alpha, \lambda, \theta, \beta > 0 \quad (1.2)$$

where, θ and β are the scale parameters and α and λ are the shape parameters of the distribution. The corresponding distribution function is given by

$$F(x; \phi) = \frac{e^{\theta \left(1 - e^{-(\beta x)^\lambda}\right)^\alpha} - 1}{(e^\theta - 1)}; \quad x > 0, \alpha, \lambda, \theta, \beta > 0 \quad (1.3)$$

This article is concerned with the maximum likelihood and Bayesian predictors for the first future observation of the shape parameters for exponentiated Weibull Poisson based on the first r GOS. This work was done under the assumption of squared error (symmetric), LINEX and general entropy (asymmetric) loss functions. Numerical study is

carried out to compare Bayes predictors based on the three loss functions with the maximum likelihood predictors. The results are specialized to upper record values and type II censored data.

This article can be organized as follows. Section 2 obtains the maximum likelihood estimators of the unknown shape parameters for EWP distribution based on r GOS. In the same section, the maximum likelihood (ML) and Bayesian predictors (point and interval) for the first future observation based on GOS are obtained. In Section 3, based on upper record values, the ML estimators of the EWP distribution are derived. Also, the ML and Bayesian predictors (point and interval) for the first future observation are obtained. Section 4 concerns with the ML and Bayesian predictors (point and interval) for the first future observation for EWP under type II censored data. In Section 5, a numerical computation is developed to illustrate theoretical results. Finally, conclusions are presented in Section 6. Tables are displayed in the appendix.

2. Bayesian and Non-Bayesian Predictors Based on r GOS

In this section, the ML and Bayesian predictors of a first observation from EWP distribution is obtained under r GOS. In order to obtain the ML and Bayesian predictor, firstly the ML estimators of the two unknown shape parameters from EWP distribution must be obtained.

2.1 Maximum likelihood Estimator Based on r GOS

Suppose that, $X(1, n, m, k), X(2, n, m, k), \dots, X(r, n, m, k)$, $k > 0$ and $m \in R$ be the first r GOS from a sample of size n drawn from the EWP population with pdf (1.2), then the log-likelihood function denoted by L can be obtained from (1.2), (1.3) and (1.1), and written as

$$L = \ln C_{r-1} + r \ln(\alpha) + r \ln(\lambda) + r \ln(\theta) + r \lambda \ln(\beta) - r \ln(e^\theta - 1) - \sum_{i=1}^r (\beta x_{(i)})^\lambda + (\lambda - 1) \sum_{i=1}^r \ln(x_{(i)}) + (\alpha - 1) \sum_{i=1}^r \ln(E_i) + \theta \sum_{i=1}^r (E_i)^\alpha + \sum_{i=1}^{r-1} m_i \ln\left(\frac{e^\theta - e^{\theta(E_i)^\alpha}}{e^\theta - 1}\right) + (\gamma_r - 1) \ln\left(\frac{e^\theta - e^{\theta(E_r)^\alpha}}{e^\theta - 1}\right), \quad (2.1)$$

where, $E_i = \left(1 - e^{-(\beta x_{(i)})^\lambda}\right)$ and $E_r = \left(1 - e^{-(\beta x_{(r)})^\lambda}\right)$.

The ML estimators $\hat{\alpha}_1$ and $\hat{\lambda}_1$ for α and λ can be obtained by setting $\frac{\partial L}{\partial \alpha}$ and $\frac{\partial L}{\partial \lambda}$ equal zero. Consequently, the system of equations are reduced into two nonlinear equations as follows

$$\frac{r}{\hat{\alpha}_1} + \sum_{i=1}^r \ln E_i + \theta \sum_{i=1}^r (E_i)^{\hat{\alpha}_1} \ln E_i + \theta \sum_{i=1}^{r-1} m_i W_i \ln E_i + \theta(\gamma_r - 1)W_r = 0, \quad (2.2)$$

and

$$\begin{aligned} \frac{r}{\hat{\lambda}_1} + r \ln \beta + \sum_{i=1}^r \ln x_{(i)} - \sum_{i=1}^r T_i + (\hat{\alpha}_1 - 1) \sum_{i=1}^r A_i + \theta \hat{\alpha}_1 \sum_{i=1}^r A_i (E_i)^{\hat{\alpha}_1} \\ + \hat{\alpha}_1 \theta \sum_{i=1}^{r-1} m_i W_1 i + \hat{\alpha}_1 \theta (\gamma_r - 1) W_1 r = 0, \end{aligned} \quad (2.3)$$

where, $E_{1i} = [e^{\theta(E_i)^\alpha} - e^\theta]$, $E_{1r} = [e^{\theta(E_r)^\alpha} - e^\theta]$, $T_i = (\beta x_{(i)})^\lambda \ln(\beta x_{(i)})$,

$T_r = (\beta x_{(r)})^\lambda \ln(\beta x_{(r)})$, $A_i = e^{-(\beta x_{(i)})^\lambda} T_i (E_i)^{-1}$, $A_r = e^{-(\beta x_{(r)})^\lambda} T_r (E_r)^{-1}$,

$W_i = e^{\theta(E_i)^\alpha} (E_i)^\alpha (E_{1i})^{-1}$, $W_r = e^{\theta(E_r)^\alpha} (E_r)^\alpha (E_{1r})^{-1}$, $W_1 r = W_r A_r$,

$W_1 i = W_i A_i$, $A_i = e^{-(\beta x_{(i)})^\lambda} T_i (E_i)^{-1}$, and $A_r = e^{-(\beta x_{(r)})^\lambda} T_r (E_r)^{-1}$.

Equations (2.2) and (2.3) cannot be solved analytically; statistical software can be used to solve these equations numerically.

2.2 Maximum Likelihood Predictor Based on r GOS

Suppose that the first r GOS $X(1, n, m, k), \dots, X(r, n, m, k)$ have been observed and we want to predict the future GOS $X(r+1, n, m, k), \dots, X(n, n, m, k)$. Let $Y_s = X_s$, $s = r+1, r+2, \dots, n$, $m_i = m$, $\forall i = 1, \dots, r$. The conditional density function of the s^{th} future GOS given that the first r GOS is given, see Kamps (1995), by

$$f(y_s | \phi; \underline{x}) = \begin{cases} \frac{c_{s-1}}{c_{r-1} (s-r-1)!} [h_m(F(y_s)) - h_m(F(x_r))]^{s-r-1} \frac{[1-F(y_s)]^{\gamma_{s-1}} f(y_s)}{[1-F(x_r)]^{\gamma_{r+1}}}, & m \neq -1 \\ \frac{k^{s-r}}{(s-r-1)!} [h_m(F(y_s)) - h_m(F(x_r))]^{s-r-1} \frac{[1-F(y_s)]^{k-1} f(y_s)}{[1-F(x_r)]^k}, & m = -1 \end{cases} \quad (2.4)$$

$$\text{where, } h_m[x] = \begin{cases} \frac{-(1-x)^{m+1}}{m+1}, & m \neq -1 \\ -\ln(1-x), & m = -1 \end{cases} \quad (2.5)$$

$C_{\ell-1} = \prod_{j=1}^{\ell} \gamma_j$, $\ell = s, r$, and $\gamma_j = k + (n-j)(m+1)$, $\forall j \in \{1, 2, \dots, n-1\}$.

Therefore, the marginal density function of Y_s will be obtained for the two different cases; for $m \neq -1$ and for $m = -1$. Substituting (1.3) in (2.5), one can obtain,

$$h_m[F(x_r)] = \begin{cases} \frac{-1}{m+1} (e^\theta - e^{\theta(E_r)^\alpha})^{m+1} (e^\theta - 1)^{-(m+1)}, & m \neq -1 \\ -[\ln(e^\theta - e^{\theta(E_r)^\alpha}) - \ln(e^\theta - 1)] & , \quad m = -1 \end{cases}. \quad (2.6)$$

Then, the conditional density function of the s^{th} future GOS given the first r from EWP distribution could be obtained from (2.4) and (2.6). Therefore,

$$f(y_s|\phi;\underline{x}) = \begin{cases} \frac{c_{s-1} \alpha \lambda \theta \beta^\lambda y_s^{\lambda-1}}{c_{r-1} (s-r-1)!} (e^\theta - 1)^{\gamma_{r+1} - \gamma_s - (m+1)(s-r-1)} e^{\theta(E_s)^\alpha - (\beta y_s)^\lambda} \frac{[e^\theta - e^{\theta(E_s)^\alpha}]^{\gamma_s - 1}}{[e^\theta - e^{\theta(E_r)^\alpha}]^{\gamma_{r+1}}} \\ (E_s)^{\alpha-1} \left(\frac{1}{m+1}\right)^{s-r-1} \left[(e^\theta - e^{\theta(E_r)^\alpha})^{m+1} - (e^\theta - e^{\theta(E_s)^\alpha})^{m+1}\right]^{s-r-1}, & m \neq -1 \\ \frac{k^{s-r} \alpha \lambda \theta \beta^\lambda y_s^{\lambda-1} e^{-(\beta y_s)^\lambda} (E_s)^{\alpha-1} e^{\theta(E_s)^\alpha}}{(s-r-1)!} \frac{[e^\theta - e^{\theta(E_s)^\alpha}]^{k-1}}{[e^\theta - e^{\theta(E_r)^\alpha}]^k} \\ [\ln(e^\theta - e^{\theta(E_r)^\alpha}) - \ln(e^\theta - e^{\theta(E_s)^\alpha})]^{s-r-1}, & m = -1 \end{cases} \quad (2.7)$$

By expanding $\left[(e^\theta - e^{\theta(E_r)^\alpha})^{m+1} - (e^\theta - e^{\theta(E_s)^\alpha})^{m+1}\right]^{s-r-1}$ and $[\ln(e^\theta - e^{\theta(E_r)^\alpha}) - \ln(e^\theta - e^{\theta(E_s)^\alpha})]^{s-r-1}$, using their binomial series expansions, (s and r are positive integers and $s \geq r$), the density function (2.7) may be written as

$$f(y_s|\phi;\underline{x}) = \begin{cases} f_1(y_s|\phi;\underline{x}), & m \neq -1 \\ f_2(y_s|\phi;\underline{x}), & m = -1 \end{cases}, \quad (2.8)$$

where, $f_1(y_s|\phi;\underline{x}) = A \alpha \lambda \theta \beta^\lambda (e^\theta - 1)^{\gamma^*} y_s^{\lambda-1} e^{-(\beta y_s)^\lambda} e^{\theta(E_s)^\alpha} (E_s)^{\alpha-1}$

$$\sum_{j=0}^{s-r-1} a_j [e^\theta - e^{\theta(E_s)^\alpha}]^{V_j^* - 1} [e^\theta - e^{\theta(E_r)^\alpha}]^{V_j},$$

$$f_2(y_s|\phi;\underline{x}) = \frac{k^{s-r} \alpha \lambda \theta \beta^\lambda}{(s-r-1)!} y_s^{\lambda-1} e^{-(\beta y_s)^\lambda} e^{\theta(E_s)^\alpha} (E_s)^{\alpha-1} [e^\theta - e^{\theta(E_s)^\alpha}]^{k-1}$$

$$[e^\theta - e^{\theta(E_r)^\alpha}]^{-k} \sum_{j=0}^{s-r-1} a_j [\ln(e^\theta - e^{\theta(E_s)^\alpha})]^j [\ln(e^\theta - e^{\theta(E_r)^\alpha})]^{s-r-j-1},$$

$$A = \frac{c_{s-1}}{c_{r-1} (s-r-1)!} \left(\frac{1}{m+1}\right)^{s-r-1}, a_j = \binom{s-r-1}{j} (-1)^j, V_j^* = j(m+1) + \gamma_s,$$

$$E_s = (1 - e^{-(\beta y_s)^\lambda}), \gamma^* = \gamma_{r+1} - \gamma_s - (m+1)(s-r-1) \text{ and } V_j = (m+1)(s-r-j-1) - \gamma_{r+1}.$$

The one sample maximum likelihood prediction estimator of the future observation Y_s , $s = r + 1, r + 2, \dots, n$ based on the first r GOS can be evaluated after replacing the shape parameters α and λ by their ML estimators $\hat{\alpha}_1$ and $\hat{\lambda}_1$ which can be obtained from (2.2) and (2.3), as follows

$$\hat{y}_{s(ML)} = E(y_s | \hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) = \int_{x_r}^{\infty} y_s f(y_s | \hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s. \quad (2.9)$$

Substituting (2.8) in (2.9), one obtains

$$\hat{y}_{s(ML)} = \begin{cases} \int_{x_r}^{\infty} y_s f_1(y_s | \hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s, & m \neq -1 \\ \int_{x_r}^{\infty} y_s f_2(y_s | \hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s, & m = -1 \end{cases}, \quad (2.10)$$

where, $f_1(y_s | \hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) = A \hat{\alpha}_1 \hat{\lambda}_1 \theta \beta^{\hat{\lambda}_1} (e^{\theta} - 1)^{r^*} y_s^{\hat{\lambda}_1 - 1} e^{-(\beta y_s)^{\hat{\lambda}_1}} e^{\theta(1 - e^{-(\beta y_s)^{\hat{\lambda}_1}})^{\hat{\alpha}_1}}$

$$(1 - e^{-(\beta y_s)^{\hat{\lambda}_1}})^{\hat{\alpha}_1 - 1} \sum_{j=0}^{s-r-1} a_j \left[e^{\theta} - e^{\theta(1 - e^{-(\beta y_s)^{\hat{\lambda}_1}})^{\hat{\alpha}_1}} \right]^{V_j^* - 1} \left[e^{\theta} - e^{\theta(1 - e^{-(\beta x_r)^{\hat{\lambda}_1}})^{\hat{\alpha}_1}} \right]^{V_j},$$

$$f_2(y_s | \hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) = \frac{k^{s-r}}{(s-r-1)!} \hat{\alpha}_1 \hat{\lambda}_1 \theta \beta^{\hat{\lambda}_1} y_s^{\hat{\lambda}_1 - 1} e^{-(\beta y_s)^{\hat{\lambda}_1}} e^{\theta(1 - e^{-(\beta y_s)^{\hat{\lambda}_1}})^{\hat{\alpha}_1}} \\ \left[1 - e^{-(\beta y_s)^{\hat{\lambda}_1}} \right]^{\hat{\alpha}_1 - 1} \left[e^{\theta} - e^{\theta(1 - e^{-(\beta y_s)^{\hat{\lambda}_1}})^{\hat{\alpha}_1}} \right]^{\hat{\alpha}_1 - 1} \left[e^{\theta} - e^{\theta(1 - e^{-(\beta x_r)^{\hat{\lambda}_1}})^{\hat{\alpha}_1}} \right]^{-k} \\ \sum_{j=0}^{s-r-1} a_j \left[\ln \left(e^{\theta} - e^{\theta(1 - e^{-(\beta y_s)^{\hat{\lambda}_1}})^{\hat{\alpha}_1}} \right) \right]^j \left[\ln \left(e^{\theta} - e^{\theta(1 - e^{-(\beta x_r)^{\hat{\lambda}_1}})^{\hat{\alpha}_1}} \right) \right]^{s-r-j-1}.$$

A $(1 - \tau)$ 100 % maximum likelihood predictive bounds (MLPB) for the future observation Y_s , $s = r + 1, r + 2, \dots, n$ given the first r based on GOS, such that

$P[L(\underline{x}) \leq Y_s \leq U(\underline{x})] = 1 - \tau$, are as follows

$$P[Y_s > L(\underline{x}) | \underline{x}] = \int_{L(\underline{x})}^{\infty} f(y_s | \hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s = 1 - \frac{\tau}{2}, \quad (2.11)$$

and

$$P[Y_s > U(\underline{x}) | \underline{x}] = \int_{U(\underline{x})}^{\infty} f(y_s | \hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s = \frac{\tau}{2}. \quad (2.12)$$

Substituting (2.8) in (2.11) and (2.12) the MLPB for the future observations Y_s , $s = r + 1, r + 2, \dots, n$ given the first r based on GOS are given by

$$P[Y_s > L(\underline{x})|\underline{x}] = \begin{cases} \int_{L(\underline{x})}^{\infty} f_1(y_s|\hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s, & m \neq -1 \\ \int_{L(\underline{x})}^{\infty} f_2(y_s|\hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s, & m = -1 \end{cases}, \quad (2.13)$$

and

$$P[Y_s > U(\underline{x})|\underline{x}] = \begin{cases} \int_{U(\underline{x})}^{\infty} f_1(y_s|\hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s, & m \neq -1 \\ \int_{U(\underline{x})}^{\infty} f_2(y_s|\hat{\alpha}_1, \hat{\lambda}_1; \underline{x}) dy_s, & m = -1 \end{cases}. \quad (2.14)$$

2.3 Bayesian Predictors Based on r GOS

Suppose that $X(1, n, m, k), X(2, n, m, k), \dots, X(r, n, m, k)$, $k > 0$ and $m \in R$ be the first r GOS from the EWP population with pdf (1.2), then the likelihood function can be obtained from (1.2), (1.3) and (1.1), and written as

$$L[\alpha, \lambda, \theta, \beta; \underline{x}] = C_{r-1} \alpha^r \lambda^r \theta^r \beta^{r\lambda} (e^\theta - 1)^{-(r+\gamma_{r-1})} J^*[\alpha, \lambda], \quad (2.15)$$

where, $C_{r-1} = (\prod_{j=1}^{r-1} \gamma_j)$, $r = 1, \dots, n$, $\gamma_j = k + n - j + \sum_{i=1}^{r-1} m_i$, $\gamma_n = k$ and

$$J^*[\alpha, \lambda] = [\prod_{i=1}^r x_{(i)}^{\lambda-1}] [\prod_{i=1}^r (E_i)^{\alpha-1}] \exp \left[\theta \sum_{i=1}^r (E_i)^\alpha - \sum_{i=1}^r (\beta x_{(i)})^\lambda \right] \left[\prod_{i=1}^{r-1} \left(\frac{e^{\theta} - e^{\theta(E_i)^\alpha}}{(e^\theta - 1)} \right)^{m_i} \right] [e^\theta - e^{\theta(E_r)^\alpha}]^{\gamma_{r-1}}$$

To develop the Bayesian estimator of α and λ , following Singh, *et al.* (2005), the joint non-informative prior density of α and λ is given by

$$g(\alpha, \lambda) \propto \frac{1}{\alpha \lambda}; \quad \alpha, \lambda > 0. \quad (2.16)$$

where, α and λ are positive constants. Combining (2.15) and (2.16) to obtain the posterior density of α and λ given the data, is as follows

$$\pi(\alpha, \lambda|\underline{x}) = \frac{C_{r-1} \alpha^{r-1} \lambda^{r-1} \theta^r \beta^{r\lambda} (e^\theta - 1)^{-(r+\gamma_{r-1})} J^*[\alpha, \lambda]}{J_1^*}, \quad \alpha, \lambda > 0, \quad (2.17)$$

where, $J_1^* = C_{r-1} \int_0^\infty \int_0^\infty \alpha^{r-1} \lambda^{r-1} \theta^r \beta^{r\lambda} (e^\theta - 1)^{-(r+\gamma_{r-1})} J^*[\alpha, \lambda] d\alpha d\lambda$.

The one sample Bayesian prediction density function (BPDF) of the future observation, Y_s , $s = r + 1, r + 2, \dots, n$ could be obtained by using the following equation

$$h(y_s|\underline{x}) = \int_0^\infty \int_0^\infty f(y_s|\phi; \underline{x}) \pi(\alpha, \lambda|\underline{x}) d\alpha d\lambda \quad (2.18)$$

Therefore, the BPDF of the future Y_s , $s = r + 1, r + 2, \dots, n$ given the first r GOS is given by substituting (2.17), (2.8) in (2.18) to be

$$h(y_s | \underline{x}) = \begin{cases} h_1(y_s | \underline{x}) = \int_0^\infty \int_0^\infty f_1(y_s | \phi; \underline{x}) \pi(\alpha, \lambda | \underline{x}) d\alpha d\lambda, & m \neq -1 \\ h_2(y_s | \underline{x}) = \int_0^\infty \int_0^\infty f_2(y_s | \phi; \underline{x}) \pi(\alpha, \lambda | \underline{x}) d\alpha d\lambda, & m = -1 \end{cases}, \quad (2.19)$$

$$\text{where, } h_1(y_s | \underline{x}) = C_{r-1} A \int_0^\infty \int_0^\infty \frac{\alpha^r \lambda^r \theta^{(r+1)} \beta^{\lambda(r+1)}}{J_1^*} (e^\theta - 1)^{\gamma^{**}} y_s^{\lambda-1} (E_s)^{\alpha-1}$$

$$\exp \left[\theta (E_s)^\alpha - (\beta y_s)^\lambda + \theta \sum_{i=1}^r (E_i)^\alpha - \sum_{i=1}^r (\beta x_{(i)})^\lambda \right]$$

$$\sum_{j=0}^{s-r-1} a_j [e^\theta - e^{\theta(E_s)^\alpha}]^{V_j^*-1} [e^\theta - e^{\theta(E_r)^\alpha}]^{V_{1j}} \left[\prod_{i=1}^r (x_{(i)})^{\lambda-1} \right]$$

$$\left[\prod_{i=1}^r (E_i)^{\alpha-1} \right] \left[\prod_{i=1}^{r-1} \left(\frac{e^\theta - e^{\theta(E_i)^\alpha}}{e^\theta - 1} \right)^{m_i} \right] d\alpha d\lambda,$$

$$h_2(y_s | \underline{x}) = \frac{C_{r-1} k^{s-r}}{(s-r-1)!} \int_0^\infty \int_0^\infty \frac{\alpha^r \lambda^r \theta^{(r+1)} \beta^{\lambda(r+1)}}{J_1^*} (e^\theta - 1)^{-(r+\gamma_r-1)} y_s^{\lambda-1}$$

$$(E_s)^{\alpha-1} \exp \left[\theta (E_s)^\alpha - (\beta y_s)^\lambda + \theta \sum_{i=1}^r (E_i)^\alpha - \sum_{i=1}^r (\beta x_{(i)})^\lambda \right]$$

$$[e^\theta - e^{\theta(E_s)^\alpha}]^{k-1} [e^\theta - e^{\theta(E_r)^\alpha}]^{-k+\gamma_r-1} \sum_{j=0}^{s-r-1} a_j [\ln(e^\theta - e^{\theta(E_s)^\alpha})]^j$$

$$[\ln(e^\theta - e^{\theta(E_r)^\alpha})]^{s-r-j-1} \left[\prod_{i=1}^r (x_{(i)})^{\lambda-1} \right] \left[\prod_{i=1}^r (E_i)^{\alpha-1} \right]$$

$$\left[\prod_{i=1}^{r-1} \left(\frac{e^\theta - e^{\theta(E_i)^\alpha}}{e^\theta - 1} \right)^{m_i} \right] d\alpha d\lambda,$$

where, $\gamma^{**} = \gamma_{r+1} - \gamma_s - (m+1)(s-r-1) - (r+\gamma_r-1)$, and

$$V_{1j} = (m+1)(s-r-j-1) - \gamma_{r+1} + \gamma_r - 1.$$

Therefore, based on the first r GOS the Bayesian predictive estimator (BPE) of the future Y_s , $s = r+1, r+2, \dots, n$ under squared error loss function, denoted by $\tilde{y}_{s(BPESL)}$, can be obtained as posterior mean as follows

$$\tilde{y}_{s(BPESL)} = \begin{cases} \int_{x_r}^\infty y_s h_1(y_s | \underline{x}) dy_s, & m \neq -1 \\ \int_{x_r}^\infty y_s h_2(y_s | \underline{x}) dy_s, & m = -1 \end{cases}, \quad (2.20)$$

Alternatively, under the LINEX loss function the BPE of the future Y_s , $s = r+1, r+2, \dots, n$; denoted by $\tilde{\tilde{y}}_{s(BPELL)}$; is given by

$$\tilde{\tilde{y}}_{s(BPELL)} = \begin{cases} -\frac{1}{a} \log[E(e^{-ay_s} | \underline{x})] = -\frac{1}{a} \log \left[\int_{x_r}^\infty e^{-ay_s} h_1(y_s | \underline{x}) dy_s \right], & m \neq -1 \\ -\frac{1}{a} \log[E(e^{-ay_s} | \underline{x})] = -\frac{1}{a} \log \left[\int_{x_r}^\infty e^{-ay_s} h_2(y_s | \underline{x}) dy_s \right], & m = -1 \end{cases}, \quad (2.21)$$

Moreover, the BPE of the future Y_s , $s = r + 1, r + 2, \dots, n$ under general entropy; denoted by $\hat{y}_{s(BPEGL)}$; can be obtained as follows

$$\hat{y}_{s(BPEGL)} = \begin{cases} [E((y_s)^{-v})]^{-\frac{1}{v}} = \left[\int_{x_r}^{\infty} (y_s)^{-v} h_1(y_s | \underline{x}) dy_s \right]^{-\frac{1}{v}}, & m \neq -1 \\ [E((y_s)^{-v})]^{-\frac{1}{v}} = \left[\int_{x_r}^{\infty} (y_s)^{-v} h_2(y_s | \underline{x}) dy_s \right]^{-\frac{1}{v}}, & m = -1 \end{cases}, \quad (2.22)$$

The integrals (2.20) to (2.22) are very hard to obtain, therefore a numerical procedure is applied to obtain the Bayesian predictive estimator of the future observations Y_s .

A $(1 - \tau)$ 100 % BPE for the future observations Y_s , $s = r + 1, r + 2, \dots, n$ given the future r such that $P[L(\underline{x}) \leq Y_s \leq U(\underline{x})] = 1 - \tau$, is as follows

$$P[Y_s > L(\underline{x}) | \underline{x}] = \int_{L(\underline{x})}^{\infty} h(y_s | \underline{x}) dy_s = 1 - \frac{\tau}{2}, \quad (2.23)$$

and

$$P[Y_s > U(\underline{x}) | \underline{x}] = \int_{U(\underline{x})}^{\infty} h(y_s | \underline{x}) dy_s = \frac{\tau}{2}. \quad (2.24)$$

Substituting (2.19) in (2.23) and (2.24), one can obtained

$$P[Y_s > L(\underline{x}) | \underline{x}] = \begin{cases} \int_{L(\underline{x})}^{\infty} h_1(y_s | \underline{x}) dy_s, & m \neq -1 \\ \int_{L(\underline{x})}^{\infty} h_2(y_s | \underline{x}) dy_s, & m = -1 \end{cases}, \quad (2.25)$$

and

$$P[Y_s > U(\underline{x}) | \underline{x}] = \begin{cases} \int_{U(\underline{x})}^{\infty} h_1(y_s | \underline{x}) dy_s, & m \neq -1 \\ \int_{U(\underline{x})}^{\infty} h_2(y_s | \underline{x}) dy_s, & m = -1 \end{cases}. \quad (2.26)$$

The special cases of generalized order statistics, namely, upper record values and type II censored data will be shown in the next sections.

3. Bayesian and Non-Bayesian Predictors Based on r upper record values

In the current section, the ML and Bayesian predictors of a first observation from EWP distribution is obtained under upper record values. Firstly the ML estimators of the two unknown shape parameters from EWP distribution must be obtained.

3.1. Maximum likelihood estimators based on upper record values

The log-likelihood of the first r upper record values of size n drawn from EWP distribution is special case of generalized order statistics, in which, if put $m_i = -1, \forall i = 1, 2, \dots, r-1, k = 1$ and $\gamma_r = 1$ in (2.1). Therefore, the log-likelihood denoted by L_1 takes the following form

$$L_1 \propto r \ln \alpha + r \ln \lambda + r \ln \theta + r \lambda \ln \beta - r \ln(e^\theta - 1) + (\lambda - 1) \sum_{i=1}^r \ln x_{(i)} - \sum_{i=1}^r (\beta x_{(i)})^\lambda + (\alpha - 1) \sum_{i=1}^r \ln E_i + \theta \sum_{i=1}^r (E_i)^\alpha - \sum_{i=1}^r \ln \left(\frac{e^\theta - e^{\theta(E_i)^\alpha}}{e^\theta - 1} \right). \quad (3.1)$$

ML estimators $\hat{\alpha}_2$ and $\hat{\lambda}_2$ of α and λ can be obtained from (3.1) by solving the following two non-linear equations as follows:

$$\frac{\partial L_1}{\partial \alpha} = \frac{r}{\alpha} + \sum_{i=1}^r \ln E_i + \theta \sum_{i=1}^r (E_i)^{\hat{\alpha}_2} \ln E_i - \theta \sum_{i=1}^r W_i \ln E_i = 0, \quad (3.2)$$

and,

$$\frac{\partial L_1}{\partial \lambda} = \frac{r}{\lambda_2} + r \ln \beta + \sum_{i=1}^r \ln x_{(i)} - \sum_{i=1}^r T_i + (\hat{\alpha}_2 - 1) \sum_{i=1}^r A_i + \hat{\alpha}_2 \theta \sum_{i=1}^r A_i (E_i)^{\hat{\alpha}_2} - \hat{\alpha}_2 \theta \sum_{i=1}^r W_i = 0. \quad (3.3)$$

Equations (3.2) and (3.3) have no explicit solutions, therefore an iterative technique is applied for solving this nonlinear equations to obtain $\hat{\alpha}_2$ and $\hat{\lambda}_2$.

3.2 Maximum likelihood predictors based on upper record values

This subsection discusses as special case from the first r GOS, the ML predictors for the future observation Y_s from EWP distribution when scale parameters θ and β are known based on upper record values.

Assuming that shape parameters α and λ are unknown, the ML predictors of the future upper record values Y_s , is obtained by substituting $k = 1$ and $m = -1$ in (2.10) as follows,

$$\hat{y}_{s1(ML)} = \int_{x_r}^{\infty} y_s f_2^*(y_s | \hat{\alpha}_2, \hat{\lambda}_2; \underline{x}) dy_s, \quad (3.4)$$

where, $f_2^*(y_s | \hat{\alpha}_2, \hat{\lambda}_2; \underline{x}) = \frac{1}{(s-r-1)!} \hat{\alpha}_2 \hat{\lambda}_2 \theta \beta^{\hat{\lambda}_2} y_s^{(\hat{\lambda}_2-1)} e^{-(\beta y_s)^{\hat{\lambda}_2}}$

$$\sum_{j=0}^{s-r-1} a_j \left[\ln \left(e^\theta - e^{\theta(1-e^{-(\beta y_s)^{\hat{\lambda}_2}})^{\hat{\alpha}_2}} \right) \right]^j \left[\ln \left(e^\theta - e^{\theta(1-e^{-(\beta x_r)^{\hat{\lambda}_2}})^{\hat{\alpha}_2}} \right) \right]^{s-r-j-1} e^{\theta(1-e^{-(\beta y_s)^{\hat{\lambda}_2}})^{\hat{\alpha}_2}} \left[1 - e^{-(\beta y_s)^{\hat{\lambda}_2}} \right]^{\hat{\alpha}_2-1} \left[e^\theta - e^{\theta(1-e^{-(\beta x_r)^{\hat{\lambda}_2}})^{\hat{\alpha}_2}} \right]^{-1}.$$

Furthermore, the ML predictive bounds for the future Y_s based on upper record values can be obtained by setting $m = -1$ and $k = 1$ in (2.13) and (2.14)

$$P[Y_s > L(\underline{x})|\underline{x}] = \int_{L(\underline{x})}^{\infty} f_2^*(y_s|\hat{\alpha}_2, \hat{\lambda}_2; \underline{x}) dy_s, \quad (3.5)$$

and

$$P[Y_s > U(\underline{x})|\underline{x}] = \int_{U(\underline{x})}^{\infty} f_2^*(y_s|\hat{\alpha}_2, \hat{\lambda}_2; \underline{x}) dy_s. \quad (3.6)$$

3.3 Bayesian predictors based on upper record values

The Bayesian prediction density function for the future observation, Y_s based on upper record values denoted by $h_2^*(y_s|\underline{x})$ can be obtained by setting $m = -1$, $k = 1$ and $\gamma_r = 1$ in (2.19) as follows

$$h_2^*(y_s|\underline{x}) = \int_0^{\infty} \int_0^{\infty} f_2(y_s|\phi; \underline{x}) \pi(\alpha, \lambda|\underline{x}) d\alpha d\lambda,$$

$$\begin{aligned} h_2^*(y_s|\underline{x}) &= \frac{C_{r-1}}{(s-r-1)!} \int_0^{\infty} \int_0^{\infty} \frac{\alpha^r \lambda^r \theta^{(r+1)} \beta^{\lambda(r+1)}}{J_2^*} (e^{\theta} - 1)^{-r} y_s^{\lambda-1} (E_s)^{\alpha-1} \\ &\quad \exp \left[\theta (E_s)^{\alpha} - (\beta y_s)^{\lambda} + \theta \sum_{i=1}^r (E_i)^{\alpha} - \sum_{i=1}^r (\beta x_{(i)})^{\lambda} \right] \left[\prod_{i=1}^r (x_{(i)})^{\lambda-1} \right] \\ &\quad \sum_{j=0}^{s-r-1} a_j [\ln(e^{\theta} - e^{\theta(E_s)^{\alpha}})]^j [\ln(e^{\theta} - e^{\theta(E_r)^{\alpha}})]^{s-r-j-1} \left[\prod_{i=1}^r (E_i)^{\alpha-1} \right] \\ &\quad \left[\prod_{i=1}^r (e^{\theta} - e^{\theta(E_i)^{\alpha}})^{-1} \right] [e^{\theta} - e^{\theta(E_r)^{\alpha}}]^{-1} d\alpha d\lambda, \end{aligned} \quad (3.7)$$

where, $J_2^* = C_{r-1} \int_0^{\infty} \int_0^{\infty} \alpha^{r-1} \lambda^{r-1} \theta^r \beta^{\lambda} (e^{\theta} - 1)^{-r} Q_1^*[\alpha, \lambda] d\alpha d\lambda$, and

$$\begin{aligned} Q_1^*[\alpha, \lambda] &= \left[\prod_{i=1}^r x_{(i)}^{\lambda-1} \right] \left[\prod_{i=1}^r (E_i)^{\alpha-1} \right] \exp \left[\theta \sum_{i=1}^r (E_i)^{\alpha} - \sum_{i=1}^r (\beta x_{(i)})^{\lambda} \right] \\ &\quad \left[\prod_{i=1}^r (e^{\theta} - e^{\theta(E_i)^{\alpha}})^{-1} \right]. \end{aligned}$$

Therefore, the BPE of the future observation Y_s based on upper record values under square error, LINEX and general entropy loss functions; denoted by $\tilde{y}_{s1(BPESL)}$, $\tilde{\tilde{y}}_{s1(BPELL)}$ and $\hat{\tilde{y}}_{s1(BPEGL)}$ respectively; can be calculated through the following equations as follows

$$\tilde{y}_{s1(BPESL)} = \int_{x_r}^{\infty} y_s h_2^*(y_s|\underline{x}) dy_s, \quad (3.8)$$

$$\tilde{\tilde{y}}_{s1(BPELL)} = -\frac{1}{a} \log[E(e^{-ay_s}|\underline{x})] = -\frac{1}{a} \log \left[\int_{x_r}^{\infty} e^{-ay_s} h_2^*(y_s|\underline{x}) dy_s \right], \quad (3.9)$$

and

$$\hat{\tilde{y}}_{s1(BPEGL)} = [E(y_s^{-v})]^{-\frac{1}{v}} = \left[\int_{x_r}^{\infty} (y_s)^{-v} h_2^*(y_s|\underline{x}) dy_s \right]^{-\frac{1}{v}}. \quad (3.10)$$

The integrals involved in (3.8), (3.9) and (3.10) can't be solved analytically, so it's necessary to apply a numerical technique and computer facilities to evaluate the Bayesian predictive estimator of the future observation Y_s .

The Bayesian predictive bounds (BPB) of the future Y_s based on upper record values can be obtained through the following,

$$P[Y_s > L(\underline{x})|\underline{x}] = \int_{L(\underline{x})}^{\infty} h_2^*(y_s|\underline{x}) dy_s = 1 - \frac{\tau}{2}, \quad (3.11)$$

and

$$P[Y_s > U(\underline{x})|\underline{x}] = \int_{U(\underline{x})}^{\infty} h_2^*(y_s|\underline{x}) dy_s = \frac{\tau}{2}. \quad (3.12)$$

4. Bayesian and Non-Bayesian Predictors Based on type II censored data

This section deals with the ML and Bayesian predictors of a first observation from EWP distribution under type II censored data. In the following subsections, the ML estimators of the two unknown shape parameters are derived. Also, the maximum likelihood and Bayesian predictors are obtained.

4.1 Maximum likelihood estimators based on type II censored data

The log-likelihood function based on type II censored from r sample of size n can be obtained as a special case from the log-likelihood function of GOS in (2.1) by putting; $m_i = 0 \forall i = 1, \dots, r-1$, $k = 1$ and $\gamma_r = n - r + 1$. Therefore, the log-likelihood function, denoted by L_2 , can be written as

$$L_2 \propto r \ln \alpha + r \ln \lambda + r \ln \theta + r \lambda \ln \beta - r \ln(e^\theta - 1) + (\lambda - 1) \sum_{i=1}^r \ln x_{(i)} - \sum_{i=1}^r (\beta x_{(i)})^\lambda + (\alpha - 1) \sum_{i=1}^r \ln E_i + \theta \sum_{i=1}^r (E_i)^\alpha + (n - r) \ln \left(\frac{e^\theta - e^{\theta(E_r)^\alpha}}{e^\theta - 1} \right). \quad (4.1)$$

ML estimators $\hat{\alpha}_3$ and $\hat{\lambda}_3$ of α and λ can be obtained from (4.1) by solving the following two equations as follows:

$$\frac{\partial L_2}{\partial \alpha} = \frac{r}{\hat{\alpha}_3} + \sum_{i=1}^r \ln E_i + \theta \sum_{i=1}^r (E_i)^{\hat{\alpha}_3} \ln E_i + \theta(n - r)W_r \ln E_r = 0, \quad (4.2)$$

and

$$\begin{aligned} \frac{\partial L_2}{\partial \lambda} &= \frac{r}{\hat{\lambda}_3} + r \ln \beta + \sum_{i=1}^r \ln x_{(i)} - \sum_{i=1}^r T_i + (\hat{\alpha}_3 - 1) \sum_{i=1}^r A_i + \theta \hat{\alpha}_3 \sum_{i=1}^r A_i (E_i)^{\hat{\alpha}_3} \\ &+ \hat{\alpha}_3 \theta (n - r) W_1 = 0 \end{aligned} \quad (4.3)$$

To obtain the ML estimator $\hat{\alpha}_3$ and $\hat{\lambda}_3$, an iterative method is used to solve numerically (4.2) and (4.3).

4.2 Maximum likelihood predictors based on type II censored data

ML prediction technique is considered based on type II censored data when scale parameters θ and β are known and shape parameters α and λ are unknown. The maximum likelihood predictive estimator Y_s is obtained by setting $m = 0, k = 1$ and $\gamma_r = n - r + 1$ in (2.10) when $m \neq -1$ as follow

$$\hat{y}_{s2(ML)} = \int_{x_r}^{\infty} y_s f_1^*(y_s | \hat{\alpha}_3, \hat{\lambda}_3; \underline{x}) dy_s, \quad (4.4)$$

where,

$$f_1^*(y_s | \hat{\alpha}_3, \hat{\lambda}_3; \underline{x}) = A_1 \hat{\alpha}_3 \hat{\lambda}_3 \theta \beta^{\hat{\lambda}_3} y_s^{\hat{\lambda}_3-1} e^{-(\beta y_s) \hat{\lambda}_3} e^{\theta(1-e^{-(\beta y_s) \hat{\lambda}_3})^{\hat{\lambda}_3}} \\ \sum_{j=0}^{s-r-1} a_j \left[e^{\theta} - e^{\theta(1-e^{-(\beta y_s) \hat{\lambda}_3})^{\hat{\lambda}_3}} \right]^{(j+n-s)} \left[e^{\theta} - e^{\theta(1-e^{-(\beta x_r) \hat{\lambda}_3})^{\hat{\lambda}_3}} \right]^{(s-n-j-1)} \left[1 - e^{-(\beta y_s) \hat{\lambda}_3} \right]^{\hat{\lambda}_3-1},$$

where, $A_1 = \frac{C_{s-1}}{C_{r-1} (s-r-1)!}$.

The maximum likelihood predictive bounds for the future Y_s based on type II censored data can be obtained by setting $m = 0, k = 1$ and $\gamma_r = n - r + 1$ in (2.13) and (2.14) when $m \neq -1$ as follows

$$P[Y_s > L(\underline{x}) | \underline{x}] = \int_{L(\underline{x})}^{\infty} f_1^*(y_s | \hat{\alpha}_3, \hat{\lambda}_3; \underline{x}) dy_s = 1 - \frac{\tau}{2}, \quad (4.5)$$

and

$$P[Y_s > U(\underline{x}) | \underline{x}] = \int_{U(\underline{x})}^{\infty} f_1^*(y_s | \hat{\alpha}_3, \hat{\lambda}_3; \underline{x}) dy_s = \frac{\tau}{2}. \quad (4.6)$$

4.3 Bayesian predictors based on type II censored data

The Bayesian prediction density function for the future observation, Y_s based on the type II censored data can be obtained by setting $m = 0, k = 1$ and $\gamma_r = n - r + 1$ in (2.19), then the BPDF when $m \neq -1$ is as follows

$$h_1^*(y_s | \underline{x}) = \int_0^{\infty} \int_0^{\infty} f_1(y_s | \phi; \underline{x}) \pi(\alpha, \lambda | \underline{x}) d\alpha d\lambda, \\ = \frac{C_{s-1}}{(s-r-1)!} \int_0^{\infty} \int_0^{\infty} \frac{\alpha^r \lambda^r \theta^{(r+1)} \beta^{\lambda(r+1)}}{J_3^*} (e^{\theta} - 1)^{-n} y_s^{\lambda-1} (E_s)^{\alpha-1} \left[\prod_{i=1}^r (x_{(i)})^{\lambda-1} \right] \\ \exp \left[\theta (E_s)^{\alpha} - (\beta y_s)^{\lambda} + \theta \sum_{i=1}^r (E_i)^{\alpha} - \sum_{i=1}^r (\beta x_{(i)})^{\lambda} \right] \left[\prod_{i=1}^r (E_i)^{\alpha-1} \right] \\ \sum_{j=0}^{s-r-1} a_j \left[e^{\theta} - e^{\theta (E_s)^{\alpha}} \right]^{n+j-s} \left[e^{\theta} - e^{\theta (E_r)^{\alpha}} \right]^{(s-r-j-1)} d\alpha d\lambda, \quad (4.7)$$

where, $J_3^* = C_{r-1} \int_0^{\infty} \int_0^{\infty} \alpha^{r-1} \lambda^{r-1} \theta^r \beta^{r\lambda} (e^{\theta} - 1)^{-(n)} Q_2^*[\alpha, \lambda] d\alpha d\lambda$, and

$$Q_2^*[\alpha, \lambda] = \exp \left[\theta \sum_{i=1}^r (E_i)^\alpha - \sum_{i=1}^r (\beta x_{(i)})^\lambda \right] [e^\theta - e^{\theta(E_r)^\alpha}]^{n-r} \left[\prod_{i=1}^r x_{(i)}^{\lambda-1} \right] \left[\prod_{i=1}^r (E_i)^{\alpha-1} \right].$$

Therefore, based on type II censored data the BPE of the future observation Y_s under squared error loss function, denoted by $\tilde{y}_{s2(BPESL)}$, can be obtained as follows

$$\tilde{y}_{s2(BPESL)} = \int_{x_r}^{\infty} y_s h_1^*(y_s | \underline{x}) dy_s, \quad m \neq -1, \quad (4.8)$$

Based on, the LINEX loss function the BPE of the future observation Y_s ; denoted by $\tilde{\tilde{y}}_{s2(BPELL)}$; is given by

$$\tilde{\tilde{y}}_{s2(BPELL)} = -\frac{1}{a} \log [E(e^{-ay_s} | \underline{x})] = -\frac{1}{a} \log \left[\int_{x_r}^{\infty} e^{-ay_s} h_1^*(y_s | \underline{x}) dy_s \right], m \neq -1 \quad (4.9)$$

Furthermore, the BPE of the future observation Y_s under general entropy; denoted by $\hat{\tilde{y}}_{s2(BPEGL)}$; can be obtained as follows

$$\hat{\tilde{y}}_{s2(BPEGL)} = [E((y_s)^{-v})]^{-\frac{1}{v}} = \left[\int_{x_r}^{\infty} (y_s)^{-v} h_1^*(y_s | \underline{x}) dy_s \right]^{-\frac{1}{v}}, \quad m \neq -1, \quad (4.10)$$

As mentioned earlier, the integrals involved in (4.8), (4.9) and (4.10) are not solvable analytically, therefore, a numerical technique and computer facilities are needed to evaluate the BPE of the future observation Y_s .

Furthermore, the BPB for the future Y_s based on type II censored data is given as follows

$$p[Y_s > L(\underline{x}) | \underline{x}] = \int_{L(\underline{x})}^{\infty} h_1^*(y_s | \underline{x}) dy_s = 1 - \frac{\tau}{2}, \quad (4.11)$$

and

$$p[Y_s > U(\underline{x}) | \underline{x}] = \int_{U(\underline{x})}^{\infty} h_1^*(y_s | \underline{x}) dy_s = \frac{\tau}{2}. \quad (4.12)$$

5. Numerical Illustration

Numerical study has been performed, to investigate the properties of the maximum likelihood and Bayesian predictors for the first future observation Y_s from EWP distribution based on GOS. The investigated properties are length of ML and BPB of the first future observation Y_s . The numerical study can be summarized through the following steps:

Step 1: 1000 random samples of sizes 20, 50, 100, 150 and 200 are generated from EWP distribution and the numbers of the surviving unites (r) are selected as 15, 35, 70, 125 and 145 for different samples under type II censored data.

Step 2: Also, generated random samples from EWP distribution and selected ($r = 5, 8, 10$) record and considered this records as the observed upper records, and predicted the first future record values.

Step 3: Select different sets of the unknown parameters values as case I $\equiv (\alpha = 0.75, \lambda = 0.5)$, case II $\equiv (\alpha = 0.75, \lambda = 0.75)$, case III $\equiv (\alpha = 0.75, \lambda = 1.5)$, case IV $\equiv (\alpha = 0.5, \lambda = 0.8)$, case V $\equiv (\alpha = 0.75, \lambda = 0.8)$, and case VI $\equiv (\alpha = 1.25, \lambda = 0.8)$, while parameters (θ, β) are known under upper record values.

Step 4: Also, select different sets of the unknown parameters values as case I $\equiv (\alpha = 0.75, \lambda = 0.5)$, case II $\equiv (\alpha = 0.75, \lambda = 0.75)$, case III $\equiv (\alpha = 0.5, \lambda = 0.8)$, and case IV $\equiv (\alpha = 0.75, \lambda = 0.8)$, while parameters (θ, β) are known under type II censored data.

Step 5: Iteration technique is used for solving the two nonlinear equations for α and λ to obtain ML predictors under type II censored data and upper record values based on one sample.

Step 6: The integral (3.8) to (3.10) are calculated numerically to obtain the Bayes predictor of the first future observation; $\tilde{Y}_{s1(BPESL)}$, $\tilde{\tilde{Y}}_{s1(BPELL)}$ and $\hat{\tilde{Y}}_{s1(BPEGL)}$, under symmetric and asymmetric loss functions for upper record values.

Step 7: Based on type II censored data, the Bayes predictor of the first future observation, $\tilde{Y}_{s2(BPESL)}$, $\tilde{\tilde{Y}}_{s2(BPELL)}$ and $\hat{\tilde{Y}}_{s2(BPEGL)}$, are calculated by numerically solving (4.8), (4.9) and (4.10).

Step 8: Bayes predictor of the first future observation are tabulated based on upper record values and type II censored data.

Step 9: The two sided confidence limit with confidence level $\tau = 0.95$ of the first future observation Y_s under ML and Bayesian prediction are obtained based on type II censored data and upper record valued for one sample prediction.

Numerical results are reported in Tables (1) to (5) and represented through some Figures from [(1) to (6)], from these tables and figures, the following conclusions can be observed on the properties of predictors from the first future observation based on upper record values, and type II censored samples:

1. For all methods of estimation, it is clear that length of confidence interval of the first future observation decreases as sample size increases. (see, Tables 1 to 5 and see for example Figures 1 to 4.

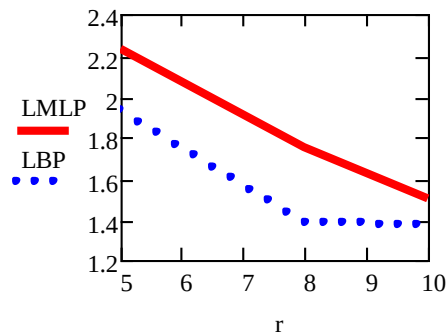


Figure 1: The length of ML and BPB of the first future observation based on upper record values for $(\alpha = 0.75)$ ($\lambda = 0.5, 0.75, 1.5$).

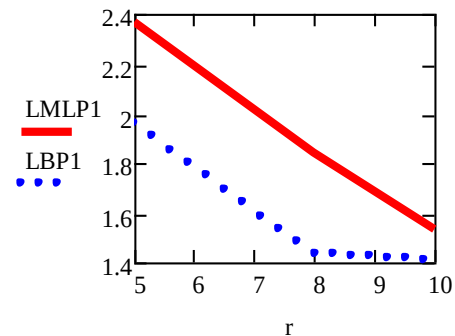


Figure 2: The length of ML and BPB of the first future observation based on upper record values for $(\alpha = 0.5, 0.75, 1.25)$ ($\lambda = 0.8$).

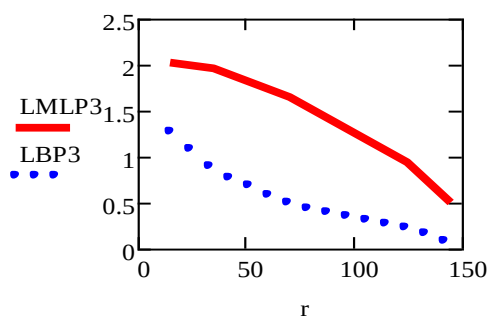


Figure 3: The length of ML and BPB of the first future observation based on type II censored data for $\alpha = 0.75$, $\lambda = 0.75$.

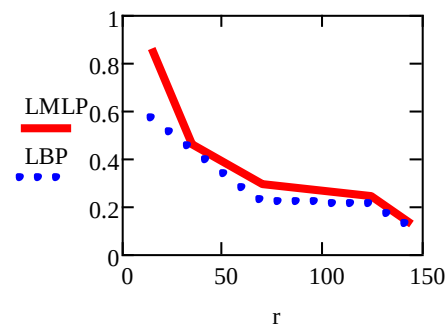


Figure 4: The length of ML and BPB of the first future observation based on type II censored data for $\alpha = 0.75$, $\lambda = 0.25$

2. For fixed value of α , and as the value of shape parameter λ increases, the length of the first future observation based on maximum likelihood and Bayesian prediction methods decreases. (see, Table 1 and Figure 1).
3. Also, For fixed value of λ , and as the value of shape parameter α increases, the length of the first future observation based on maximum likelihood and Bayesian prediction methods decreases. (see, Table 1 and Figure 2).
4. Bayes predictors of the first future observation Y_s under asymmetric (LINEX, general entropy) is over-estimates for $(a < 0, v < 0)$, and when $(a > 0, v > 0)$ the Bayes predictors of the first future observation Y_s is under-estimates. As anticipated, the Bayes predictors under asymmetric loss function are the same as the Bayes predictors relative to SE loss function (for a close to 0, and $v = -1$). This is one of the useful properties of working with the asymmetric loss functions. (see, Tables 1 to 5 and see for example Figures 5 and 6).

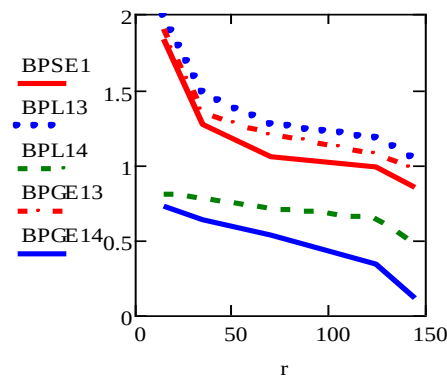


Figure 5: The Bayesian predictors of the first future observation based on type II censored data for $\alpha = 0.75$, $\lambda = 0.25$.

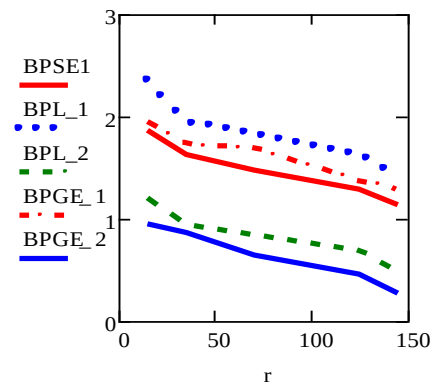


Figure 6: The Bayesian predictors of the first future observation based on type II censored data $\alpha = 0.75$, $\lambda = 0.8$.

5. Bayesian estimators under the GE loss function have the smallest MSEs as compared with the Bayesian estimators under the LINEX and SE loss functions based on upper record values and type II censored data. (see, Tables 1 to 5 and Figures 5 and 6).
6. The length of the Bayesian prediction bounds are shorter than the ML prediction bounds based on upper record values and type II censored data. (see, Tables 1 to 5 and Figures 1 to 6).
7. It has been noticed from Tables 1 to 5 that both of ML and Bayesian predictions of the first future observation Y_s are very close to the value of actual observations when the sample size increase .
8. Results perform better when r gets larger. (see, Tables 2 to 5 and Figures 1 to 6).
9. In most cases, Bayes predictors of the first future observation Y_s under general entropy based on type II censored data has the smallest Bayes predictors based on upper record values. (see, Tables 1 to 5 and Figures 5 and 6).

6. Conclusions

This study deals with the first future observation Y_s from EWP distribution based on r GOS. The ML and Bayesian predictors of the first future observation Y_s are obtained. In addition, the interval predictors of the first future observation Y_s are obtained for large samples. In particular, the ML and Bayesian predictors of the first future observation Y_s based on upper record values and type II censored samples are derived as special case from GOS.

Numerical study revealed that the length of confidence interval of the first future observation decreases as sample size increases. The Bayesian predictors under the GE

loss function have the smallest MSEs as compared with the Bayesian predictors under the LINEX and SE loss functions based on upper record values and type II censored data. The length of the Bayesian prediction bounds are shorter than the maximum likelihood prediction bounds based on upper record values and type II censored data.

In most cases, the Bayesian predictors of the first future observation Y_s under general entropy based on type II censored data are the smallest Bayesian predictors based on upper record values.

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Appendix

Table (1): ML and Bayes predictive bounds for the first future observation based on upper record values at = 95% .

r	Parameters (α, λ)	ML prediction			Bayesian Prediction			MSE under SE loss function	MSE under LINEX loss function		MSE under GE loss function	
		Lower	Upper	Length	Lower	Upper	Length		$a = -2$	$a = 2$	$a = -2$	$a = 2$
5	(0.75, 0.5)	0.028	2.262	2.234	0.168	2.099	1.931	1.894	2.217	1.535	2.015	0.957
8		0.653	2.403	1.750	0.999	2.378	1.379	1.276	1.438	1.099	1.259	0.864
10		1.184	2.685	1.501	1.245	2.620	1.375	0.709	0.993	0.684	0.796	0.649
5	(0.75, 0.75)	0.278	2.366	2.088	0.516	2.423	1.907	1.356	1.694	1.282	1.385	0.855
8		0.502	2.180	1.678	0.778	2.202	1.424	0.921	1.208	0.881	1.177	0.836
10		1.432	2.734	1.302	1.586	2.709	1.123	0.664	0.836	0.507	0.739	0.476
5	(0.75, 1.5)	0.221	2.046	1.825	0.586	2.203	1.617	0.704	0.812	0.687	0.784	0.648
8		0.671	2.262	1.591	0.905	2.073	1.168	0.536	0.748	0.446	0.628	0.359
10		1.302	2.352	1.050	1.645	2.486	0.841	0.302	0.548	0.268	0.408	0.164
5	(0.5, 0.8)	0.085	2.453	2.368	0.094	2.053	1.959	2.076	2.320	1.665	2.163	1.264
8		0.438	2.281	1.843	0.859	2.293	1.434	1.702	2.062	1.548	1.947	1.111
10		1.033	2.568	1.535	1.184	2.589	1.405	0.944	1.390	0.855	1.276	0.743
5	(0.75, 0.8)	0.245	2.411	2.166	0.438	1.842	1.404	1.628	2.190	1.254	1.974	1.043
8		0.473	2.216	1.743	0.685	2.217	1.532	1.055	1.643	0.906	1.457	0.894
10		1.328	2.765	1.437	1.284	2.607	1.323	0.869	1.084	0.694	0.953	0.539
5	(1.25, 0.8)	0.274	2.104	1.830	0.548	1.771	1.223	1.225	1.749	0.929	1.478	0.843
8		0.638	2.261	1.623	0.734	2.260	1.526	0.852	1.064	0.734	0.946	0.702
10		1.438	2.561	1.123	0.985	2.082	1.097	0.695	0.839	0.599	0.793	0.445

Table (2): ML and Bayes predictive bounds for the first future observation based on type II censoring at $\tau = 95\%$ for the set $(\alpha = 0.75, \lambda = 0.25)$.

Case (n, r)	$x_{(r+1)}$	ML prediction			Bayesian Prediction			MSE under SE loss function	MSE under LINEX loss function		MSE under GE loss function	
		Lower	Upper	Length	Lower	Upper	Length		$a = -2$	$a = 2$	$a = -2$	$a = 2$
(20, 15)	$x_{(16)}$	1.431	2.290	0.859	1.405	1.972	0.567	1.830	1.986	0.806	1.903	0.728
	$x_{(17)}$	1.606	2.410	0.804	1.418	1.959	0.541	1.805	1.965	0.787	1.932	0.726
	$x_{(18)}$	1.654	2.443	0.789	1.456	1.971	0.515	1.690	1.902	0.785	1.921	0.720
(50, 35)	$x_{(36)}$	1.575	2.027	0.456	1.213	1.644	0.431	1.269	1.465	0.776	1.343	0.630
	$x_{(37)}$	1.571	1.995	0.424	1.222	1.625	0.403	1.247	1.432	0.764	1.326	0.620
	$x_{(38)}$	1.509	1.902	0.393	1.206	1.582	0.376	1.245	1.403	0.699	1.302	0.609
(100, 70)	$x_{(71)}$	2.497	2.786	0.289	2.156	2.371	0.215	1.056	1.264	0.716	1.203	0.530
	$x_{(72)}$	2.490	2.759	0.269	2.120	2.331	0.211	1.048	1.243	0.712	1.193	0.530
	$x_{(73)}$	2.396	2.660	0.264	2.143	2.352	0.209	1.008	1.204	0.659	1.189	0.481
(150, 125)	$x_{(126)}$	1.574	1.812	0.238	1.482	1.689	0.207	0.988	1.173	0.649	1.074	0.343
	$x_{(127)}$	1.564	1.793	0.229	1.476	1.681	0.205	0.965	1.154	0.641	1.064	0.334
	$x_{(128)}$	1.524	1.744	0.220	1.452	1.652	0.200	0.862	1.435	0.620	1.053	0.320
(200, 145)	$x_{(146)}$	1.649	1.770	0.121	1.345	1.454	0.109	0.849	1.043	0.473	0.965	0.118
	$x_{(147)}$	1.629	1.747	0.118	1.582	1.690	0.108	0.825	1.024	0.394	0.953	0.112
	$x_{(148)}$	1.612	1.728	0.116	1.350	1.453	0.103	0.806	1.025	0.243	0.932	0.015

Table (3): ML and Bayes predictive bounds for the first future observation based on type II censoring at $\tau = 95\%$ for the set $(\alpha = 0.75, \lambda = 0.75)$.

Case (n, r)	$x_{(r+1)}$	ML prediction			Bayesian Prediction			MSE under SE loss function	MSE under LINEX loss function		MSE under GE loss function	
		Lower	Upper	Length	Lower	Upper	Length		$a = -2$	$a = 2$	$a = -2$	$a = 2$
(20, 15)	$x_{(16)}$	0.662	2.692	2.030	0.651	1.915	1.264	1.940	2.317	1.342	2.274	0.938
	$x_{(17)}$	0.627	2.805	2.178	0.735	1.838	1.103	1.731	2.044	1.229	2.024	0.928
	$x_{(18)}$	0.649	2.756	2.107	0.982	1.949	0.967	1.375	1.655	1.114	1.534	0.890
(50, 35)	$x_{(36)}$	0.864	2.819	1.955	0.999	1.862	0.863	1.586	1.945	1.127	1.845	0.856
	$x_{(37)}$	0.899	2.728	1.829	1.037	1.758	0.721	1.291	1.543	1.115	1.532	0.781
	$x_{(38)}$	0.926	2.479	1.553	1.164	1.756	0.592	1.132	1.282	1.035	1.137	0.763
(100, 70)	$x_{(71)}$	1.257	2.912	1.655	1.287	1.767	0.480	1.233	1.438	0.984	1.354	0.662
	$x_{(72)}$	1.345	2.943	1.598	1.359	1.749	0.390	1.025	1.138	0.964	1.097	0.638
	$x_{(73)}$	1.355	2.840	1.485	1.363	1.683	0.320	0.976	1.035	0.932	1.024	0.590
(150, 125)	$x_{(126)}$	1.809	2.739	0.930	1.826	2.046	0.220	0.865	0.995	0.648	0.964	0.586
	$x_{(127)}$	1.847	2.754	0.907	1.897	2.067	0.170	0.755	0.954	0.634	0.932	0.545
	$x_{(128)}$	1.903	2.738	0.835	1.935	2.075	0.140	0.713	0.903	0.615	0.896	0.534
(200, 145)	$x_{(146)}$	2.203	2.709	0.506	2.214	2.280	0.066	0.538	0.753	0.611	0.732	0.513
	$x_{(147)}$	2.053	2.543	0.490	2.134	2.180	0.046	0.437	0.745	0.608	0.721	0.275
	$x_{(148)}$	2.045	2.526	0.481	2.144	2.181	0.037	0.283	0.648	0.604	0.621	0.218

Table (4): ML and Bayes predictive bounds for the first future observation based on type II censoring at $\tau = 95\%$ for the set $(\alpha = 0.5, \lambda = 0.8)$.

Case (n, r)	$x_{(r+1)}$	ML prediction			Bayesian Prediction			MSE under SE loss function	MSE under LINEX loss function		MSE under GE loss function	
		Lower	Upper	Length	Lower	Upper	Length		$a = -2$	$a = 2$	$a = -2$	$a = 2$
(20, 15)	$x_{(16)}$	1.937	2.914	0.977	1.334	1.707	0.373	1.496	1.728	1.158	1.616	0.863
	$x_{(17)}$	1.816	2.755	0.939	1.267	1.614	0.347	1.406	1.645	1.011	1.602	0.849
	$x_{(18)}$	1.955	2.879	0.924	1.232	1.555	0.323	1.404	1.664	1.007	1.589	0.804
(50, 35)	$x_{(36)}$	1.258	1.806	0.548	1.734	1.959	0.225	1.394	1.544	0.872	1.453	0.793
	$x_{(37)}$	1.231	1.734	0.503	1.846	2.065	0.219	1.391	1.531	0.747	1.437	0.734
	$x_{(38)}$	1.227	1.694	0.467	1.835	2.036	0.201	1.381	1.512	0.705	1.423	0.695
(100, 70)	$x_{(71)}$	2.090	2.875	0.285	1.102	1.269	0.167	1.379	1.464	0.669	1.384	0.437
	$x_{(72)}$	2.013	2.283	0.270	1.230	1.396	0.166	1.371	1.436	0.665	1.398	0.425
	$x_{(73)}$	1.899	2.154	0.255	1.218	1.379	0.161	1.368	1.432	0.661	1.376	0.417
(150, 125)	$x_{(126)}$	1.009	1.213	0.204	1.203	1.360	0.157	1.155	1.274	0.564	1.264	0.397
	$x_{(127)}$	0.999	1.195	0.196	1.214	1.369	0.155	1.145	1.245	0.539	1.256	0.376
	$x_{(128)}$	0.996	1.185	0.189	1.224	1.378	0.154	1.126	1.223	0.533	1.246	0.375
(200, 145)	$x_{(146)}$	1.310	1.421	0.111	1.298	1.424	0.126	0.962	1.198	0.398	0.902	0.163
	$x_{(147)}$	1.304	1.413	0.109	1.287	1.407	0.120	0.953	1.184	0.365	0.843	0.154
	$x_{(148)}$	1.293	1.399	0.106	1.275	1.389	0.114	0.935	1.184	0.335	0.823	0.135

Table (5): ML and Bayes predictive bounds for the first future observation based on type II censoring at $\tau = 95\%$ for the set $(\alpha = 0.75, \lambda = 0.8)$.

Case (n, r)	$x_{(r+1)}$	ML prediction			Bayesian Prediction			MSE under SE loss function	MSE under LINEX loss function		MSE under GE loss function	
		Lower	Upper	Length	Lower	Upper	Length		$a = -2$	$a = 2$	$a = -2$	$a = 2$
(20, 15)	$x_{(16)}$	2.204	2.936	0.732	1.776	1.973	0.197	1.857	2.362	1.199	1.946	0.954
	$x_{(17)}$	2.212	2.916	0.704	1.770	1.953	0.183	1.836	2.238	1.136	1.924	0.943
	$x_{(18)}$	2.416	2.973	0.557	1.764	1.937	0.173	1.824	2.213	1.121	1.905	0.924
(50, 35)	$x_{(36)}$	1.468	1.909	0.441	1.512	1.661	0.149	1.635	1.957	0.952	1.734	0.863
	$x_{(37)}$	1.424	1.831	0.407	1.445	1.592	0.147	1.624	1.935	0.924	1.723	0.848
	$x_{(38)}$	1.421	1.800	0.379	1.370	1.512	0.142	1.623	1.924	0.913	1.711	0.832
(100, 70)	$x_{(71)}$	1.247	1.430	0.183	1.297	1.377	0.080	1.469	1.834	0.843	1.687	0.648
	$x_{(72)}$	1.241	1.417	0.176	1.276	1.354	0.078	1.457	1.823	0.832	1.638	0.636
	$x_{(73)}$	1.212	1.383	0.171	1.255	1.331	0.076	1.425	1.803	0.813	1.624	0.632
(150, 125)	$x_{(126)}$	2.148	2.400	0.252	1.535	1.607	0.072	1.295	1.624	0.698	1.375	0.458
	$x_{(127)}$	2.111	2.351	0.240	1.513	1.583	0.070	1.278	1.622	0.684	1.345	0.453
	$x_{(128)}$	2.100	2.331	0.231	1.482	1.551	0.069	1.267	1.611	0.664	1.322	0.413
(200, 145)	$x_{(146)}$	1.432	1.526	0.094	1.239	1.285	0.046	1.138	1.427	0.496	1.296	0.286
	$x_{(147)}$	1.429	1.521	0.092	1.258	1.302	0.044	1.143	1.423	0.486	1.276	0.275

Bayesian Estimation of $R = P[Y < X]$ for Burr Type XII Distribution Using Median Ranked Set Sampling

By

¹Amal S. Hassan, ²Assar, S. M and ³Yahya, M

Abstract

This article deals with the estimation of $R = P[Y < X]$ when X and Y are two independent Burr type XII distributions having the same scale parameter. Bayesian technique is used to estimate R based on median ranked set sampling, ranked set sampling and simple random sampling. Bayesian estimators of R are obtained under squared error and linear exponential loss functions. The Bayesian estimator of R cannot be obtained in explicit form, and therefore it has been implemented using a numerical technique. Analysis of a simulated data have been presented to compare the performances of the Bayes estimates based on median ranked set sampling with the corresponding Bayes estimates based on simple random sampling and ranked set sampling.

Keywords: *Burr XII distribution, ranked set sampling, median ranked set sampling, Bayesian estimation.*

1. Introduction

The concept of ranked set sampling (RSS) was first proposed by McIntyre (1952) as a useful sampling technique for improving the accuracy of estimating mean pasture yield. He found that the estimator based on RSS is more efficient than the corresponding estimator based on simple random sampling (SRS). Takahasi and Wakimoto (1968) provided the necessary mathematical theory of RSS. Dell and Clutter (1972) showed that, regardless of ranking errors, the RSS estimator of a population mean is unbiased and at least as precise as the SRS estimator with the same number of quantifications.

RSS procedure involves randomly drawing n sets of n units each from the population. It is assumed that the units in each set can be ranked visually. From the first set of n units, the smallest ranked unit is measured; the second smallest ranked unit is measured from the second set of n units. The procedure is continued until the largest unit from the n^{th} sample is selected for measurement. Thus, a total of n -measured units, which represent one cycle, are obtained. This process may be repeated r times until a sample number of nr units are obtained. These nr units form the RSS data.

Muttlak (1997) suggested the median ranked set sampling (MRSS) as a modification of ranked set sampling, which reduce the errors in ranking and increase the efficiency over RSS with perfect ranking for some probability distribution functions. The MRSS procedure can be summarized as follows. Select n random samples of size n units from the target population and select the median element of each ordered set if n is odd. However if n is

^{1,2} Institute of Statistical Studies and Research, Cairo University, Egypt.

³ Modern Academy for Management and Computer Science, Egypt.

Cairo University-Institute of Statistical Studies and Research

even, it selects the $(n/2)^{th}$ smallest observation from each of the first $(n/2)$ ordered sets and the second smallest $(n + 2/2)^{th}$ observation from the second $(n/2)$ sets. This selection procedure yield a MRSS of size n , and may be repeated r times to give a MRSS of size nr .

The two-parameter Burr type XII distribution was first introduced in the literature by Burr (1942), and has been gained special attention in the last two decades due to its importance in practical situations. This distribution is widely used in some areas such as business, engineering, quality control and reliability. The cumulative distribution function (cdf) of the Burr XII distribution denoted by Burr (c, b) is given as

$$F(x; c, b) = 1 - (1 + x^c)^{-b}, x > 0, c > 0, b > 0,$$

where c and b are shape parameters. The corresponding probability density function (pdf) is given by

$$f(x; c, b) = bcx^{c-1}(1 + x^c)^{-(b+1)}, x > 0, c > 0, b > 0.$$

In the context of reliability, the stress strength model describes the life of a component which has a random strength X and is subjected to random stress Y . The component fails at the instant that the stress applied to it exceeds strength and the component will function satisfactorily whenever $(Y < X)$. Thus, $P(Y < X)$ stress-strength model is a measure of the reliability of the component. This model first considered by Birnbaum (1956) and the formal term stress strength model appeared in the title of Church and Harris (1970), Kotz et al. (2003) collected the theoretical and practical results of the theory and applications of the stress strength relationships during the last decades. Estimation problem of $R = P(Y < X)$ based on SRS technique have been studied by several authors (see, for example, Awad and Charraf (1986), McCool (1991), Surles and Padgett (1998), Raqab and Kundu (2005), Kundu and Gupta (2006), Raqab et al. (2008), Kundu and Raqab (2009), Rezaei et al. (2010) and Ali et al. (2012), Al-Mutairi et al. (2015)).

In the literature, there were few works for the estimating problems of $P(Y < X)$ based on RSS. Sengupta and Mukhuti (2008) considered the estimation of R using RSS data, they proved that the RSS provides an unbiased estimator with smaller variance as compared with SRS of same sizes, even when the rankings are imperfect. Muttlak et al. (2010) considered the estimation of R when X and Y are independently distributed exponential random variables with different scale parameters using RSS data. Hussian (2014) studied the estimation of R when X and Y are two generalized inverted exponential random variables with different parameters using SRS and RSS techniques. Hassan et al. (2014) proposed the estimation problem of $P(Y < X)$ for Burr type XII distribution based on RSS. They showed that the estimators based on RSS are more efficient than the estimators based on SRS data. Hassan et al. (2015) introduced the estimation of $P(Y < X)$ for Burr XII distribution under

several modifications of RSS using maximum likelihood method; they proposed MRSS, extreme ranked set sampling and percentile ranked set sampling. They concluded that the maximum likelihood estimates (MLEs) based on RSS and its modifications are more efficient than the estimators based on SRS data.

To our knowledge, in the literature, there were no studies that have been performed about stress strength model based on RSS and MRSS using Bayesian approach. Therefore, this article aims to estimate R based on SRS, RSS and MRSS techniques. Bayesian estimators of R using non-informative priors under symmetric and asymmetric loss functions will be considered. Numerical study will be performed to compare the different estimates.

2. Bayesian Estimator for R Based on SRS

In this section, Bayesian estimator of R using non-informative priors will be obtained under the assumptions that the shape parameter c is known and the shape parameters b and a have independent non-informative priors with the following probability density functions:

$$\pi(b) \propto \frac{1}{b}; \quad b > 0, \quad (1)$$

and

$$\pi(a) \propto \frac{1}{a}; \quad a > 0. \quad (2)$$

Let $\underline{X} = \{X_1, \dots, X_p\}$ be a simple random sample from Burr (c, b) and $\underline{Y} = \{Y_1, \dots, Y_q\}$ be a simple random sample from Burr (c, a) , therefore the likelihood function for the observed sample is given by

$$L(\underline{x}, \underline{y} | b, a) \propto \prod_{i=1}^p b (1 + x_i^c)^{-b} \prod_{j=1}^q a (1 + y_j^c)^{-a}. \quad (3)$$

The posterior density functions of b and a can be obtained by combining the priors density functions in (1) and (2) with the likelihood function in (3) as follows

$$\pi(b, a | \underline{x}, \underline{y}) \propto b^{p-1} a^{q-1} e^{-b\lambda_1} e^{-a\lambda_2}, \quad (4)$$

where $\lambda_1 = \sum_{i=1}^p \ln(1 + x_i^c)$ and $\lambda_2 = \sum_{j=1}^q \ln(1 + y_j^c)$.

Bayesian estimator of R can be obtained using the following traditional transformation technique as follows: Let

$$r_1 = \frac{a}{a+b} \text{ and } A_1 = a + b, \text{ then } a = r_1 A_1 \text{ and } b = A_1(1 - r_1), \quad 0 < r_1 < 1, \quad A_1 > 0, \quad (5)$$

then, the posterior density function (4) will be written as follows

$$\pi(r_1, A_1 | \underline{x}, \underline{y}) \propto A_1^{p+q-1} (1 - r_1)^{p-1} r_1^{q-1} e^{-A_1[(1-r_1)\lambda_1 + r_1\lambda_2]},$$

integrate out of A_1

$$\pi(r_1|\underline{x}, \underline{y}) = \Psi \frac{(1-r_1)^{p-1} r_1^{q-1}}{[(1-r_1)\lambda_1 + r_1\lambda_2]^{p+q}}, \quad (6)$$

here Ψ is a constant and it can be obtained by integrate out of r_1 as follows

$$\Psi^{-1} = \int_0^1 \frac{(1-r_1)^{p-1} r_1^{q-1}}{[\lambda_2]^{p+q} \left[\frac{(1-r_1)\lambda_1}{r_1\lambda_2} + 1 \right]^{p+q}} dr_1.$$

$$\text{Let } \eta_1 = \frac{(1-r_1)}{r_1}, r_1 = \frac{1}{1+\eta_1} \text{ and } 1-r_1 = \frac{\eta_1}{1+\eta_1}, \quad (7)$$

therefore,

$$\Psi^{-1} = \int_0^\infty \frac{(\eta_1)^{p-1}}{[\lambda_2]^{p+q} \left[\eta_1 \left(\frac{\lambda_1}{\lambda_2} \right) + 1 \right]^{p+q}} d\eta_1, \quad (8)$$

Let $\eta_2 = \eta_1 \left(\frac{\lambda_1}{\lambda_2} \right)$, then (8) will be as follows

$$\Psi^{-1} = \frac{1}{(\lambda_1)^p (\lambda_2)^q} \int_0^\infty \frac{(\eta_2)^{p-1}}{[\eta_2 + 1]^{p+q}} d\eta_2 = \frac{B(p, q)}{(\lambda_1)^p (\lambda_2)^q},$$

where $B(p, q)$ is beta function. So, the posterior probability density function of R denoted by $\pi(r_1|\underline{x}, \underline{y})$, for $0 < r < 1$, takes the following form

$$\pi(r_1|\underline{x}, \underline{y}) = \frac{(\lambda_1)^p (\lambda_2)^q}{B(p, q)} \frac{(1-r_1)^{p-1} r_1^{q-1}}{[(1-r_1)\lambda_1 + r_1\lambda_2]^{p+q}}. \quad (9)$$

Using (9), Bayesian estimator of R under squared error (SE) loss function, denoted by $\hat{R}_{SE}$ is given by

$$\hat{R}_{SE} = \frac{\lambda_1^p \lambda_2^q}{B(p, q)} \int_0^1 \frac{(1-r_1)^{p-1} r_1^q}{[(1-r_1)\lambda_1 + r_1\lambda_2]^{p+q}} dr_1. \quad (10)$$

Under linear exponential (LINEX) loss function, Bayesian estimator of R , denoted by $\hat{R}_{LIN}$, can be obtained as follows

$$\begin{aligned} \hat{R}_{LIN} &= \frac{-1}{\vartheta} \ln[E(e^{-\vartheta r_1})], \\ \hat{R}_{LIN} &= \frac{-1}{\vartheta} \ln \left\{ \frac{\lambda_1^p \lambda_2^q}{B(p, q)} \int_0^1 \frac{e^{-\vartheta r_1} (1-r_1)^{p-1} r_1^{q-1}}{[(1-r_1)\lambda_1 + r_1\lambda_2]^{p+q}} dr_1 \right\}. \end{aligned} \quad (11)$$

Clearly, the computations of $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ in (10) and (11) are complicated. So numerical technique will be used to evaluate $\hat{R}_{SE}$ and $\hat{R}_{LIN}$.

3. Bayesian Estimator for R Based on RSS

In this section, Bayesian estimator of R under SE and LINEX loss functions based on RSS data using non-informative Jeffrey priors will be obtained. Let $\{\dot{X}_{i(i)s}, i = 1, 2, \dots, n; s = 1, 2, \dots, r\}$ be a ranked set sample with sample size $p = nr$, where n is the set size and r is the number of cycles from Burr (c, b) . Then the pdf of $\dot{X}_{i(i)s}$ is given by:

$$f_i(\dot{x}_{i(i)s}) = \frac{n!}{(i-1)!(n-i)!} b c \dot{x}_{i(i)s}^{c-1} (1 + \dot{x}_{i(i)s}^c)^{-[b(n-i+1)+1]} (1 - (1 + \dot{x}_{i(i)s}^c)^{-b})^{i-1}, \quad \dot{x}_{i(i)s} > 0.$$

Let $\underline{X} = \{\dot{X}_{1(1)s}, \dot{X}_{2(2)s}, \dots, \dot{X}_{n(n)s}, s = 1, 2, \dots, r\}$ be a ranked set sample from Burr (c, b) and the prior density of b as in (1). Then the likelihood function is as follow

$$L(\underline{\dot{X}}|b) \propto b^p e^{-b \sum_{s=1}^r \sum_{i=1}^n (n-i+1) \ln(1 + \dot{x}_{i(i)s}^c)} \prod_{s=1}^r \prod_{i=1}^n \left(1 - (1 + \dot{x}_{i(i)s}^c)^{-b}\right)^{i-1},$$

By using binomial expansion, then $L(\underline{\dot{X}}|b)$ can be written as

$$L(\underline{\dot{X}}|b) \propto b^p e^{-b \sum_{s=1}^r \sum_{i=1}^n (n-i+1) \ln(1 + \dot{x}_{i(i)s}^c)} \prod_{s=1}^r \prod_{i=1}^n \left[\sum_{L=0}^{i-1} (-1)^L \binom{i-1}{L} (1 + \dot{x}_{i(i)s}^c)^{-bL} \right],$$

Then,

$$L(\underline{\dot{X}}|b) \propto \sum_{L_1^1=0}^0 \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s}(i) \right) \right] b^p e^{-b \sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^s}(i) \right)}, \quad (12)$$

where $T_{L_i^s}(i) = (n + L_i^s - i + 1) \ln(1 + \dot{x}_{i(i)s}^c)$ and $\Phi_{L_i^s}(i) = (-1)^{L_i^s} \binom{i-1}{L_i^s}$.

Combine the prior density (1) and the likelihood function (12), to obtain the posterior density function of b , denoted by $\pi_{RSS}(b|\underline{\dot{X}})$, as follows

$$\pi_{RSS}(b|\underline{\dot{X}}) \propto \sum_{L_1^1=0}^0 \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s}(i) \right) \right] b^{p-1} e^{-b \sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^s}(i) \right)}. \quad (13)$$

By similar way, let $\{\dot{Y}_{j(j)s}, j = 1, \dots, m; s = 1, \dots, r\}$ be the ranked set sample of size $q = mr$ from Burr (c, a) . Then the pdf of $\dot{Y}_{j(j)s}$ is given by:

$$f_j(\dot{y}_{j(j)s}) = \frac{m!}{(j-1)!(m-j)!} a c \dot{y}_{j(j)s}^{c-1} (1 + \dot{y}_{j(j)s}^c)^{-[a(m-j+1)+1]} (1 - (1 + \dot{y}_{j(j)s}^c)^{-a})^{j-1}, \dot{y}_{j(j)s} > 0,$$

Let $\underline{\dot{Y}} = \{\dot{Y}_{1(1)s}, \dot{Y}_{2(2)s}, \dots, \dot{Y}_{m(m)s}; s = 1, 2, \dots, r\}$ be a ranked set sample from Burr (c, a) , where m is the set size and r is the number of cycles. The likelihood function will be obtained as follows

$$L(\underline{\dot{Y}}|a) \propto \sum_{\tau_1^1=0}^0 \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots \sum_{\tau_1^r=0}^0 \dots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s}(j) \right) \right] a^q e^{-a \left(\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s}(j) \right)}, \quad (14)$$

where $T_{\tau_j^s}(j) = (m + \tau_j^s - j + 1) \ln(1 + \dot{y}_{j(j)s}^c)$ and $\Phi_{\tau_j^s}(j) = (-1)^{\tau_j^s} \binom{j-1}{\tau_j^s}$.

Combine the prior density (2) and the likelihood function (14) to obtain the posterior density function of a , denoted by $\pi_{RSS}(a|\underline{y})$, as follows

$$\pi_{RSS}(a|\underline{y}) \propto \sum_{\tau_1^0=0}^0 \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \sum_{\tau_1^r=0}^0 \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] a^{q-1} e^{-a \left(\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right)}. \quad (15)$$

Assuming that b and a are independent, from (13) and (15), the joint bivariate posterior density function of b and a will be

$$\pi_{RSS}(b, a|\underline{x}, \underline{y}) \propto \sum_{L_1^1=0}^0 \cdots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \cdots \sum_{L_n^2=0}^{n-1} \cdots \sum_{L_1^r=0}^0 \cdots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \sum_{\tau_1^r=0}^0 \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] b^{p-1} a^{q-1} e^{-b \sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} - a \sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)}}.$$

Bayesian estimator of R can be obtained using the standard transformation technique defined in (5) as follows

$$\pi_{RSS}(A_1, r_1|\underline{x}, \underline{y}) \propto \sum_{L_1^1=0}^0 \cdots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \cdots \sum_{L_n^2=0}^{n-1} \cdots \sum_{L_1^r=0}^0 \cdots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \sum_{\tau_1^r=0}^0 \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] \times (1 - r_1)^{p-1} r_1^{q-1} A_1^{p+q-1} e^{-A_1 \left((1-r_1) \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right] + r_1 \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right] \right)}.$$

Integrate out of A_1 , the posterior density function of R using non-informative prior for $0 < r_1 < 1$ is given by

$$\pi_{RSS}(r_1|\underline{x}, \underline{y}) = \Psi \sum_{L_1^1=0}^0 \cdots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \cdots \sum_{L_n^2=0}^{n-1} \cdots \sum_{L_1^r=0}^0 \cdots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \sum_{\tau_1^r=0}^0 \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] \times (1 - r_1)^{p-1} r_1^{q-1} \left[(1 - r_1) \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right] + r_1 \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right] \right]^{-(p+q)},$$

where Ψ is a constant and it can be obtained as follows

$$\Psi^{-1} = \sum_{L_1^1=0}^0 \cdots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \cdots \sum_{L_n^2=0}^{n-1} \cdots \sum_{L_1^r=0}^0 \cdots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \sum_{\tau_1^r=0}^0 \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] \int_0^1 \frac{(1-r_1)^{p-1} (r_1)^{q-1}}{\left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right]^{p+q} \left[\frac{(1-r_1) \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right] + r_1 \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right]}{r_1 \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right]} + 1 \right]^{p+q}} dr_1. \quad (16)$$

Using transformation technique defined in (7), then (16) takes the following form

$$\Psi^{-1} = \sum_{L_1^1=0}^0 \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots \sum_{\tau_1^r=0}^0 \dots$$

$$\sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] \int_0^\infty \frac{(\eta_1)^{p-1}}{\left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right]^{p+q} \left[\eta_1 \frac{\left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right]}{\left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right]} + 1 \right]^{p+q}} d\eta_1. \quad (17)$$

Let, $\eta_3 = \eta_1 \left(\frac{\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)}}{\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)}} \right)$, then (17) takes the following form

$$\Psi^{-1} = \sum_{L_1^1=0}^0 \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots$$

$$\times \dots \sum_{\tau_1^r=0}^0 \dots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] \frac{B(p,q)}{\left(\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right)^q \left(\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right)^p},$$

Therefore, the posterior density function of R based on RSS, denoted by $\pi_{RSS}(r_1 | \underline{\dot{x}}, \underline{\dot{y}})$,

for $0 < r < 1$ is given by

$$\pi_{RSS}(r_1 | \underline{\dot{x}}, \underline{\dot{y}}) =$$

$$\sum_{L_1^1=0}^0 \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots$$

$$\dots \sum_{\tau_1^r=0}^0 \dots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] (1-r_1)^{p-1} r_1^{q-1}$$

$$\times \left[(1-r_1) \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right] + r_1 \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right] \right]^{-(p+q)} \left[\sum_{L_1^1=0}^0 \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots \sum_{\tau_1^r=0}^0 \dots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \right]$$

$$\times \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] B(p, q) \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right]^{-p} \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right]^{-q} \right]^{-1}. \quad (18)$$

Using (18), Bayesian estimator of R , under SE loss function, denoted by $\hat{R}_{SE}$, is obtained as follows

$$\hat{R}_{SE} = \int_0^1 r_1 \pi_{RSS}(r_1 | \underline{\dot{x}}, \underline{\dot{y}}) dr_1,$$

$$\hat{R}_{SE} = \int_0^1 \sum_{L_1^1=0}^0 \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots$$

$$\dots \sum_{\tau_1^r=0}^0 \dots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] (1-r_1)^{p-1} r_1^q$$

$$\times \left[(1-r_1) \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right] + r_1 \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right] \right]^{-(p+q)} \left[\sum_{L_1^1=0}^0 \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^0 \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots \sum_{\tau_1^r=0}^0 \dots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s(i)} \right) \right] \right]$$

$$\times \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s(j)} \right) \right] B(p, q) \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s(i)} \right]^{-p} \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s(j)} \right]^{-q} \right]^{-1} dr_1. \quad (19)$$

Bayesian estimator of R using non-informative prior under LINEX loss function denoted by $\hat{R}_{LIN}$, is given as follows

$$\begin{aligned}\hat{R}_{LIN} &= \frac{-1}{\vartheta} \ln \left\{ \int_0^1 e^{-\vartheta r_1} \pi_{RSS} (r_1 | \underline{x}, \underline{y}) dr_1 \right\}, \\ \hat{R}_{LIN} &= \frac{-1}{\vartheta} \ln \left\{ \int_0^1 \sum_{L_1=0}^0 \dots \sum_{L_n=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1=0}^0 \dots \sum_{\tau_m=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots \right. \\ &\quad \dots \sum_{\tau_1^r=0}^0 \dots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s}(i) \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s}(j) \right) \right] e^{-\vartheta r_1 + \ln((1-r_1)^{p-1} r_1^{q-1})} \\ &\quad \times \left[(1-r_1) \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s}(i) \right] + r_1 \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s}(j) \right] \right]^{-(p+q)} \left[\sum_{L_1=0}^0 \dots \sum_{L_n=0}^{n-1} \sum_{L_1^2=0}^0 \dots \sum_{L_n^2=0}^{n-1} \dots \right. \\ &\quad \dots \sum_{L_1^r=0}^0 \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1=0}^0 \dots \sum_{\tau_m=0}^{m-1} \sum_{\tau_1^2=0}^0 \dots \sum_{\tau_m^2=0}^{m-1} \dots \sum_{\tau_1^r=0}^0 \dots \sum_{\tau_m^r=0}^{m-1} B(p, q) \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^s}(i) \right) \right] \\ &\quad \left. \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^s}(j) \right) \right] \left[\sum_{s=1}^r \sum_{i=1}^n T_{L_i^s}(i) \right]^{-p} \left[\sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^s}(j) \right]^{-q} \right]^{-1} dr_1 \right\}. \quad (20)\end{aligned}$$

Bayesian estimators $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ of R under SE and LINEX loss functions using Jeffrey priors based on ranked set sample data for the simple case of a single cycle can be obtained by substituting $(r = 1)$ in (19) and (20) respectively as follows

$$\begin{aligned}\hat{R}_{SE} &= \left[B(n, m) \sum_{L_1=0}^0 \dots \sum_{L_n=0}^{n-1} \sum_{\tau_1=0}^0 \dots \sum_{\tau_m=0}^{m-1} \left(\prod_{i=1}^n \Phi_{L_i}(i) \right) \left(\prod_{j=1}^m \Phi_{\tau_j}(j) \right) \left(\sum_{i=1}^n T_{L_i}(i) \right)^{-n} \right. \\ &\quad \times \left(\sum_{j=1}^m T_{\tau_j}(j) \right)^{-m} \left. \right]^{-1} \int_0^1 \sum_{L_1=0}^0 \dots \sum_{L_n=0}^{n-1} \sum_{\tau_1=0}^0 \dots \sum_{\tau_m=0}^{m-1} \left(\prod_{i=1}^n \Phi_{L_i}(i) \right) \left(\prod_{j=1}^m \Phi_{\tau_j}(j) \right) \\ &\quad \times (1-r_1)^{n-1} r_1^m \left[(1-r_1) \left[\sum_{i=1}^n T_{L_i}(i) \right] + r_1 \left[\sum_{j=1}^m T_{\tau_j}(j) \right] \right]^{-(n+m)} dr_1. \quad (21)\end{aligned}$$

and,

$$\begin{aligned}\hat{R}_{LIN} &= \frac{-1}{\vartheta} \ln \left\{ \int_0^1 \left[\sum_{L_1=0}^0 \dots \sum_{L_n=0}^{n-1} \sum_{\tau_1=0}^0 \dots \sum_{\tau_m=0}^{m-1} \left(\prod_{i=1}^n \Phi_{L_i}(i) \right) \left(\prod_{j=1}^m \Phi_{\tau_j}(j) \right) B(n, m) \left(\sum_{i=1}^n T_{L_i}(i) \right)^{-n} \right. \right. \\ &\quad \left. \left(\sum_{j=1}^m T_{\tau_j}(j) \right)^{-m} \right]^{-1} \sum_{L_1=0}^0 \dots \sum_{L_n=0}^{n-1} \sum_{\tau_1=0}^0 \dots \sum_{\tau_m=0}^{m-1} \left(\prod_{i=1}^n \Phi_{L_i}(i) \right) \left(\prod_{j=1}^m \Phi_{\tau_j}(j) \right) e^{-\vartheta r + \ln((1-r_1)^{n-1} r_1^{m-1})} \\ &\quad \times \left[(1-r_1) \left[\sum_{i=1}^n T_{L_i}(i) \right] + r_1 \left[\sum_{j=1}^m T_{\tau_j}(j) \right] \right]^{-(n+m)} dr_1 \left. \right\}. \quad (22)\end{aligned}$$

Clearly, it is not easy to obtain a closed form solution of (21) and (22). Therefore, an iterative technique must be applied to solve these equations numerically to obtain an estimate of R .

4. Bayesian Estimator for R Based on MRSS

In this section, Bayesian estimator of $R = P(Y < X)$ based on MRSS technique will be obtained in case of even and odd set sizes under SE as a symmetric loss function and LINEX as asymmetric loss function using non-informative priors.

4.1 Bayesian estimator for R with odd set size

In this subsection, Bayesian estimator for R based on MRSS with odd set size will be derived. Let $\{X_{i(g)s}^*; i = 1, 2, \dots, n, s = 1, 2, \dots, r\}$ is a MRSS from Burr (c, b) with sample size

$p = nr$, where n is the set size, r is the number of cycles and $g = \frac{n+1}{2}$. Then the pdf of $X_{i(g)s}^*$ will be as follows:

$$f_g(x_{i(g)s}^*) = \frac{n!}{[(g-1)!]^2} b c x_{i(g)s}^{*c-1} [1 + x_{i(g)s}^{*c}]^{-(bg+1)} \left[1 - (1 + x_{i(g)s}^{*c})^{-b}\right]^{g-1}, x_{i(g)s}^* > 0.$$

Let $\underline{X}^* = \{X_{1(g)s}^*, X_{2(g)s}^*, \dots, X_{n(g)s}^*; s = 1, 2, \dots, r\}$ where $g = (n+1)/2$ be a median ranked set sample data from Burr (c, b) and the prior density of b as in (1). The likelihood function denoted by $L(\underline{x}^*|b)$ will be as follows

$$L(\underline{x}^*|b) \propto b^p e^{-bg \sum_{s=1}^r \sum_{i=1}^n \ln(1+x_{i(g)s}^{*c})} \prod_{s=1}^r \prod_{i=1}^n \left[1 - (1 + x_{i(g)s}^{*c})^{-b}\right]^{g-1},$$

by using binomial expansion,

$$\begin{aligned} L(\underline{x}^*|b) &\propto b^p e^{-bg \sum_{s=1}^r \sum_{i=1}^n \ln(1+x_{i(g)s}^{*c})} \prod_{s=1}^r \prod_{i=1}^n \left[\sum_{L^*=0}^{g-1} (-1)^{L^*} \binom{g-1}{L^*} (1 + x_{i(g)s}^{*c})^{-bL^*}\right], \\ L(\underline{x}^*|b) &\propto b^p \sum_{L_1^*=0}^{g-1} \sum_{L_2^*=0}^{g-1} \dots \sum_{L_n^*=0}^{g-1} \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \dots \sum_{L_n^{2*}=0}^{g-1} \dots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \dots \sum_{L_n^{r*}=0}^{g-1} \dots \\ &\times \left[\prod_{s=1}^r \prod_{i=1}^n (-1)^{L_i^{s*}} \binom{g-1}{L_i^{s*}}\right] e^{-b \sum_{s=1}^r \sum_{i=1}^n (g+L_i^{s*}) \ln(1+x_{i(g)s}^{*c})}. \end{aligned} \quad (23)$$

Combine the prior density (1) and the likelihood function in (23) to obtain the posterior density of b denoted by $\pi_{MRSS}(b|\underline{x}^*)$ as follows

$$\begin{aligned} \pi_{MRSS}(b|\underline{x}^*) &\propto b^{p-1} \sum_{L_1^*=0}^{g-1} \sum_{L_2^*=0}^{g-1} \dots \sum_{L_n^*=0}^{g-1} \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \dots \sum_{L_n^{2*}=0}^{g-1} \dots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \dots \\ &\sum_{L_n^{r*}=0}^{g-1} \left[\prod_{s=1}^r \prod_{i=1}^n \Phi_{L_i^{s*}}(i)\right] e^{-b \sum_{s=1}^r \sum_{i=1}^n T_{L_i^{s*}}(i)}. \end{aligned} \quad (24)$$

where $\Phi_{L_i^{s*}}(i) = (-1)^{L_i^{s*}} \binom{g-1}{L_i^{s*}}$ and $T_{L_i^{s*}}(i) = (g + L_i^{s*}) \ln(1 + x_{i(g)s}^{*c})$.

Similarly, Let $\{Y_{j(h)s}^*; j = 1, 2, \dots, m, s = 1, 2, \dots, r\}$ is a MRSS from Burr (c, a) with sample size $q = mr$, where m is the set size, r is the number of cycles. Then the pdf of $Y_{j(h)s}^*$ will be as follows:

$$f_h(y_{j(h)s}^*) = \frac{m!}{[(h-1)!]^2} a c y_{j(h)s}^{*c-1} [1 + y_{j(h)s}^{*c}]^{-(ah+1)} \left[1 - (1 + y_{j(h)s}^{*c})^{-a}\right]^{h-1}, y_{j(h)s}^* > 0.$$

Let $\underline{Y}^* = \{Y_{1(h)s}^*, Y_{2(h)s}^*, \dots, Y_{m(h)s}^*; s = 1, 2, \dots, r\}$ be a median ranked set sample from Burr (c, a) and the prior density of a as in (2). The likelihood function is as follows

$$L(\underline{y}^*|a) \propto a^q e^{-ah \sum_{s=1}^r \sum_{j=1}^m \ln(1+y_{j(h)s}^{*c})} \prod_{s=1}^r \prod_{j=1}^m \left[1 - (1 + y_{j(h)s}^{*c})^{-a}\right]^{h-1},$$

by using binomial expansion,

$$\begin{aligned} L(\underline{y}^*|a) &\propto a^q e^{-ah \sum_{s=1}^r \sum_{j=1}^m \ln(1+y_{j(h)s}^{*c})} \prod_{s=1}^r \prod_{j=1}^m \left[\sum_{\tau^*=0}^{h-1} (-1)^{\tau^*} \binom{h-1}{\tau^*} (1 + y_{j(h)s}^{*c})^{-a\tau^*}\right], \\ L(\underline{y}^*|a) &\propto a^q \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \dots \sum_{\tau_m^*=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \dots \sum_{\tau_m^{2*}=0}^{h-1} \dots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \dots \\ &\sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^r \prod_{j=1}^m \Phi_{\tau_j^{s*}}(j)\right] e^{-a \sum_{s=1}^r \sum_{j=1}^m T_{\tau_j^{s*}}(j)}, \end{aligned} \quad (25)$$

where $T_{\tau_j^{s*}}(j) = (h + \tau_j^{s*}) \ln(1 + y_{j(h)s}^{*c})$ and $\Phi_{\tau_j^{s*}}(j) = (-1)^{\tau_j^{s*}} \binom{h-1}{\tau_j^{s*}}$.

Combine the prior density (2) and the likelihood function (25) to obtain the posterior density of a denoted by $\pi_{MRSS}(a | \underline{y}^*)$ as follows

$$\pi_{MRSS} \left(a \left| \underline{y}^* \right. \right) \propto a^{q-1} \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \cdots \sum_{\tau_m^*=0}^{h-1} \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \cdots \sum_{\tau_m^*=0}^{h-1} \cdots \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \cdots \sum_{\tau_m^*=0}^{h-1} \left[\prod_{s=1}^r \prod_{j=1}^m \Phi_{\tau_j^s}(j) \right] e^{-a \sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^s}(j) \right)}, \quad (26)$$

Assume that b and a are independent, from posterior densities in (24) and (26), the joint bivariate posterior density of b and a based on MRSS data will be

$$\begin{aligned} \pi_{MRSS} \left(b, a | \underline{x}^*, \underline{y}^* \right) &\propto \sum_{L_1^*=0}^{g-1} \sum_{L_2^*=0}^{g-1} \cdots \sum_{L_n^*=0}^{g-1} \sum_{L_1^*=0}^{g-1} \sum_{L_2^*=0}^{g-1} \cdots \sum_{L_n^*=0}^{g-1} \cdots \sum_{L_1^*=0}^{g-1} \sum_{L_r^*=0}^{g-1} \cdots \\ &\sum_{L_n^*=0}^{g-1} \cdots \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \cdots \sum_{\tau_m^*=0}^{h-1} \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \cdots \sum_{\tau_m^*=0}^{h-1} \cdots \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \cdots \sum_{\tau_m^*=0}^{h-1} \left[\prod_{s=1}^r \prod_{i=1}^n \Phi_{L_i^*}(i) \right] \\ &\left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^*}(j) \right) \right] b^{p-1} e^{-b \sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^*}(i) \right)} a^{q-1} e^{-a \sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^*}(j) \right)}. \end{aligned} \quad (27)$$

By applying the transformation technique (5) in (27) the joint posterior density function of b and a based on MRSS in case of odd set size will be as follows

$$\begin{aligned} \pi_{MRSS} \left(A_1, r_1 | \underline{x}^*, \underline{y}^* \right) &\propto \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \\ &\sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^{s*}}(i) \right) \right] \\ &\left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{L_j^{s*}}(j) \right) \right] r_1^{q-1} (1-r_1)^{p-1} A_1^{p+q-1} e^{-A_1 [(1-r_1) \sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right) + r_1 \sum_{s=1}^r \sum_{j=1}^m \left(T_{L_j^{s*}}(j) \right)]}. \end{aligned}$$

Integrate out of A_1 , the posterior density function of r_1 based on MRSS in case of odd set size for $0 < r_1 < 1$ will be as follows

$$\begin{aligned} \pi_{MRSS} \left(r_1 | \underline{x}^*, \underline{y}^* \right) &= \Psi^* \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \\ &\quad \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^{s*}}(i) \right) \right] \\ &\quad \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^{s*}}(j) \right) \right] r_1^{q-1} (1-r_1)^{p-1} \left[(1-r_1) \sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right) + r_1 \sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right) \right]^{-(p+q)}, \end{aligned} \quad (28)$$

where Ψ^* is a constant and is obtained as follows

$$\begin{aligned} \Psi^{*-1} &= \int_0^1 \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \\ &\sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^{s*}}(i) \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^{s*}}(j) \right) \right] \\ &\times \frac{r_1^{-p-1} (1-r_1)^{p-1}}{\left(\sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right) \right)^{p+q} \left[\frac{(1-r_1) \sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right)}{r_1 \left(\sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right) \right)} + 1 \right]^{(p+q)}} dr_1, \end{aligned} \quad (29)$$

Using transformation (7), defined in previous section, then (29) takes the following form

$$\begin{aligned} \Psi^{*-1} &= \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \\ &\sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^{s*}}(i) \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^{s*}}(j) \right) \right] \\ &\times \int_0^\infty \frac{(\eta_1)^{p-1}}{\left(\sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right) \right)^{p+q} \left[\frac{\sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right)}{\sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right)} + 1 \right]^{(p+q)}} d\eta_1, \end{aligned} \quad (30)$$

Let, $\eta_4 = \eta_1 \left(\frac{\sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right)}{\sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right)} \right)$, then (30) will be written as follows

$$\begin{aligned} \Psi^{*-1} &= \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \\ &\sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^{s*}}(i) \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^{s*}}(j) \right) \right] \\ &\times \frac{1}{\left(\sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right) \right)^p \left(\sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right) \right)^q} \int_0^\infty \frac{(\eta_4)^{p-1}}{[\eta_4 + 1]^{(p+q)}} d\eta_4, \end{aligned}$$

$$\begin{aligned} \Psi^{*-1} &= \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \\ &\sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^{s*}}(i) \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^{s*}}(j) \right) \right] \\ &\times \frac{B(p, q)}{\left(\sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right) \right)^p \left(\sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right) \right)^q}. \end{aligned}$$

Therefore, the posterior density function of R based on MRSS in case of odd set size, for $0 < r < 1$ is given by

$$\begin{aligned} \pi_{MRSS} \left(r_1 | \underline{x}^*, \underline{y}^* \right) &= \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \\ &\sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^{s*}}(i) \right) \right] \\ &\left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^{s*}}(j) \right) \right] r_1^{q-1} (1 - r_1)^{p-1} \left[(1 - r_1) \sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right) \right. \\ &\left. + r_1 \sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right) \right]^{-(p+q)} \left[\sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \right. \\ &\sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} B(p, q) \\ &\left. \left[\prod_{s=1}^r \prod_{i=1}^n \left(\Phi_{L_i^{s*}}(i) \right) \right] \left[\prod_{s=1}^r \prod_{j=1}^m \left(\Phi_{\tau_j^{s*}}(j) \right) \right] \left(\sum_{s=1}^r \sum_{i=1}^n \left(T_{L_i^{s*}}(i) \right) \right)^{-p} \left(\sum_{s=1}^r \sum_{j=1}^m \left(T_{\tau_j^{s*}}(j) \right) \right)^{-q} \right]^{-1}. \end{aligned} \quad (31)$$

Using (31) Bayesian estimator of R based on MRSS for odd set size using non-informative prior under SE and LINEX loss functions respectively, can be obtained as follows

$$R_{SE}^* = \int_0^1 r_1 \pi_{MRSS}(r_1 | \underline{x}^*, \underline{y}^*) dr_1, \quad (32)$$

and

$$R_{LIN}^* = \frac{-1}{\vartheta} \ln \int_0^1 e^{-\vartheta r_1} \pi_{MRSS}(r_1 | \underline{x}^*, \underline{y}^*) dr_1. \quad (33)$$

respectively.

Bayesian estimators R_{SE}^* and R_{LIN}^* of R in case of simple case when ($r = 1$) based on MRSS data for odd set size under SE and LINEX loss functions using non-informative prior can be obtained by substituting ($r = 1$) in (32) and (33) as follows

$$\begin{aligned} R_{SE}^* &= \int_0^1 \left[B(n, m) \sum_{L_1^*=0}^{g-1} \sum_{L_2^*=0}^{g-1} \dots \sum_{L_n^*=0}^{g-1} \left[\prod_{i=1}^n (-1)^{L_i^*} \binom{g-1}{L_i^*} \right] \left[\sum_{i=1}^n (g + L_i^*) \ln(1 + x_{i(g)}^{*c}) \right]^{-n} \right. \\ &\times \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \dots \sum_{\tau_m^*=0}^{h-1} \left[\prod_{j=1}^m (-1)^{\tau_j^*} \binom{h-1}{\tau_j^*} \right] \left[\sum_{j=1}^m (h + \tau_j^*) \ln(1 + y_{j(h)}^{*c}) \right]^{-m} \left. \right]^{-1} \sum_{L_1^*=0}^{g-1} \sum_{L_2^*=0}^{g-1} \dots \\ &\times \sum_{L_n^*=0}^{g-1} \left[\prod_{i=1}^n (-1)^{L_i^*} \binom{g-1}{L_i^*} \right] \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \dots \sum_{\tau_m^*=0}^{h-1} \left[\prod_{j=1}^m (-1)^{\tau_j^*} \binom{h-1}{\tau_j^*} \right] r_1^m (1 - r_1)^{n-1} \\ &\times \left[(1 - r_1) \sum_{i=1}^n (g + L_i^*) \ln(1 + x_{i(g)}^{*c}) + r_1 \sum_{j=1}^m (h + \tau_j^*) \ln(1 + y_{j(h)}^{*c}) \right]^{-n+m} dr_1. \end{aligned} \quad (34)$$

and

$$\begin{aligned} R_{LIN}^* &= \frac{-1}{\vartheta} \ln \left\{ \left[B(n, m) \sum_{L_1^*=0}^{g-1} \sum_{L_2^*=0}^{g-1} \dots \sum_{L_n^*=0}^{g-1} \left[\prod_{i=1}^n (-1)^{L_i^*} \binom{g-1}{L_i^*} \right] \left[\sum_{i=1}^n (g + L_i^*) \ln(1 + x_{i(g)}^{*c}) \right]^{-n} \right. \right. \\ &\sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \dots \sum_{\tau_m^*=0}^{h-1} \left[\prod_{j=1}^m (-1)^{\tau_j^*} \binom{h-1}{\tau_j^*} \right] \left[\sum_{j=1}^m (h + \tau_j^*) \ln(1 + y_{j(h)}^{*c}) \right]^{-m} \left. \right]^{-1} \int_0^1 \sum_{L_1^*=0}^{g-1} \sum_{L_2^*=0}^{g-1} \dots \\ &\sum_{L_n^*=0}^{g-1} \left[\prod_{i=1}^n (-1)^{L_i^*} \binom{g-1}{L_i^*} \right] \sum_{\tau_1^*=0}^{h-1} \sum_{\tau_2^*=0}^{h-1} \dots \sum_{\tau_m^*=0}^{h-1} \left[\prod_{j=1}^m (-1)^{\tau_j^*} \binom{h-1}{\tau_j^*} \right] e^{-\vartheta r_1 \ln[(1-r_1)^{n-1} r_1^{m-1}]} \\ &\times \left. \left[(1 - r_1) \left[\sum_{i=1}^n (g + L_i^*) \ln(1 + x_{i(g)}^{*c}) \right] + r_1 \left[\sum_{j=1}^m (h + \tau_j^*) \ln(1 + y_{j(h)}^{*c}) \right] \right]^{-(n+m)} dr_1 \right\}. \end{aligned} \quad (35)$$

As it seems, it is not easy to obtain a closed form solution to (34) and (35). Therefore, an iterative technique must be applied to solve these equations numerically.

4.2 Bayesian estimator for R with even set size

The main aim of this subsection is to obtain Bayes estimators of R based on MRSS in case of even set size using non-informative Jeffery priors under SE and LINEX loss functions.

Let $\{X_{1(u)s}^{**}, X_{2(u)s}^{**}, \dots, X_{u(u)s}^{**}\} \cup \{X_{u+1(u+1)s}^{**}, \dots, X_{n(u+1)s}^{**}\}$ be a MRSS from Burr (c, b), where $u = n/2$, then the pdfs of $X_{i(u)s}^{**}$ and $X_{i(u+1)s}^{**}$ are given as follows:

$$f_u(x_{i(u)s}^{**}) = D_5 b c x_{i(u)s}^{**c-1} [1 + x_{i(u)s}^{**c}]^{-(b(u+1)+1)} \left[1 - (1 + x_{i(u)s}^{**c})^{-b} \right]^{u-1}, x_{i(u)s}^{**} > 0,$$

and

$$f_{u+1}(x_{i(u+1)s}^{**}) = D_5 b c x_{i(u+1)s}^{**c-1} [1 + x_{i(u+1)s}^{**c}]^{-(bu+1)} \left[1 - (1 + x_{i(u+1)s}^{**c})^{-b} \right]^u, x_{i(u+1)s}^{**} > 0.$$

Let the set $\underline{X}^{**} = \{X_{1(u)s}^{**}, \dots, X_{u(u)s}^{**}, X_{u+1(u+1)s}^{**}, \dots, X_{n(u+1)s}^{**}; s = 1, 2, \dots, r\}$ be a MRSS from Burr (c, b), where $u = n/2$. The likelihood function denoted by $L(\underline{x}^{**} | b)$ with even set size will be as follows

$$L(\underline{x}^{**}|b) \propto \prod_{s=1}^r \left[\prod_{i=1}^u b[1 + x_{i(u)s}^{**c}]^{-b(u+1)} \left[1 - (1 + x_{i(u)s}^{**c})^{-b} \right]^{u-1} \prod_{i=u+1}^n b[1 + x_{i(u+1)s}^{**c}]^{-bu} \right. \\ \left. \times \left[1 - (1 + x_{i(u+1)s}^{**c})^{-b} \right]^u \right],$$

by using binomial expansion,

$$L(\underline{x}^{**}|b) \propto b^p e^{-b[(u+1)\sum_{s=1}^r \sum_{i=1}^u \ln(1+x_{i(u)s}^{**c}) + u\sum_{s=1}^r \sum_{i=u+1}^n \ln(1+x_{i(u+1)s}^{**c})]} \prod_{s=1}^r \prod_{i=1}^u [\sum_{l^{**}=0}^{u-1} (-1)^{l^{**}} \binom{u}{l^{**}} (1+x_{i(u+1)s}^{**c})^{-bl^{**}}] \\ \times \binom{u-1}{l^{**}} (1+x_{i(u)s}^{**c})^{-bl^{**}} \prod_{s=1}^r \prod_{i=u+1}^n [\sum_{l^{**}=0}^u (-1)^{l^{**}} \binom{u}{l^{**}} (1+x_{i(u+1)s}^{**c})^{-bl^{**}}], \\ L(\underline{x}^{**}|b) \propto b^p \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \\ \sum_{L_{u+1}^{1**}=0}^u \sum_{L_{u+2}^{1**}=0}^u \cdots \sum_{L_n^{1**}=0}^u \sum_{L_{u+1}^{2**}=0}^u \sum_{L_{u+2}^{2**}=0}^u \cdots \sum_{L_n^{2**}=0}^u \cdots \sum_{L_{u+1}^{r**}=0}^u \sum_{L_{u+2}^{r**}=0}^u \cdots \sum_{L_n^{r**}=0}^u \cdots \\ \left[\prod_{s=1}^r \prod_{i=1}^u \left(\Phi_{L_i^{s**}}(i) \right) \right] \left[\prod_{s=1}^r \prod_{i=u+1}^n \left(\Phi_{l_i^{s**}}(i) \right) \right] e^{-b\sum_{s=1}^r \left[\sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{i=u+1}^n \left(T_{l_i^{s**}}(i) \right) \right]}, \quad (36)$$

where $\Phi_{L_i^{s**}}(i) = (-1)^{L_i^{s**}} \binom{u-1}{L_i^{s**}}$, $\Phi_{l_i^{s**}}(i) = (-1)^{l_i^{s**}} \binom{u}{l_i^{s**}}$,

$T_{L_i^{s**}}(i) = (u + L_i^{s**} + 1) \ln(1 + x_{i(u)s}^{**c})$ and $T_{l_i^{s**}}(i) = (u + l_i^{s**}) \ln(1 + x_{i(u+1)s}^{**c})$,

Combine the prior density in (1) and the likelihood function in (36), the posterior density of b based on MRSS in case of even set size will be obtained as follows

$$\pi_{MRSS}(b|\underline{x}^{**}) \propto b^{p-1} \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \\ \sum_{L_{u+1}^{1**}=0}^u \sum_{L_{u+2}^{1**}=0}^u \cdots \sum_{L_n^{1**}=0}^u \sum_{L_{u+1}^{2**}=0}^u \sum_{L_{u+2}^{2**}=0}^u \cdots \sum_{L_n^{2**}=0}^u \cdots \sum_{L_{u+1}^{r**}=0}^u \sum_{L_{u+2}^{r**}=0}^u \cdots \sum_{L_n^{r**}=0}^u \cdots \\ \left[\prod_{s=1}^r \prod_{i=1}^u \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{i=u+1}^n \Phi_{l_i^{s**}}(i) \right] e^{-b\sum_{s=1}^r \left[\sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{i=u+1}^n \left(T_{l_i^{s**}}(i) \right) \right]}, \quad (37)$$

Let $\{Y_{1(v)s}^{**}, Y_{2(v)s}^{**}, \dots, Y_{v(v)s}^{**}\} \cup \{Y_{v+1(v+1)s}^{**}, \dots, Y_{m(v+1)s}^{**}\}$ be a MRSS from Burr (c, a) , where $v = m/2$. Then the pdfs of $Y_{j(v)s}^{**}$ and $Y_{j(v+1)s}^{**}$ are given as follows:

$$f_v(y_{j(v)s}^{**}) = \frac{n!}{(u-1)!(u)!} ac y_{j(v)s}^{**c-1} [1 + y_{j(v)s}^{**c}]^{-(a(v+1)+1)} [1 - (1 + y_{j(v)s}^{**c})^{-a}]^{v-1}, y_{j(v)s}^{**} > 0$$

and

$$f_{v+1}(y_{j(v+1)s}^{**}) = \frac{m!}{(v-1)!(v)!} ac y_{j(v+1)s}^{**c-1} [1 + y_{j(v+1)s}^{**c}]^{-(av+1)} [1 - (1 + y_{j(v+1)s}^{**c})^{-a}]^v, y_{j(v+1)s}^{**} > 0,$$

Let $\underline{Y}^{**} = \{Y_{1(v)s}^{**}, Y_{2(v)s}^{**}, \dots, Y_{v(v)s}^{**}, Y_{v+1(v+1)s}^{**}, \dots, Y_{m(v+1)s}^{**}; s = 1, \dots, r\}$ be a median ranked set sample from Burr (c, a) , where $v = m/2$.

$$L_{MRSS}(\underline{y}^{**}|a) \propto a^q \sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \\ \sum_{\tau_{v+1}^{1**}=0}^v \sum_{\tau_{v+2}^{1**}=0}^v \cdots \sum_{\tau_m^{1**}=0}^v \sum_{\tau_{v+1}^{2**}=0}^v \sum_{\tau_{v+2}^{2**}=0}^v \cdots \sum_{\tau_m^{2**}=0}^v \cdots \sum_{\tau_{v+1}^{r**}=0}^v \sum_{\tau_{v+2}^{r**}=0}^v \cdots \sum_{\tau_m^{r**}=0}^v \cdots \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{\tau_j^{s**}}(j) \right] \\ \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{\tau_j^{s**}}(j) \right] e^{-a\sum_{s=1}^r \left[\sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{j=v+1}^m \left(T_{\tau_j^{s**}}(j) \right) \right]},$$

where $\Phi_{\tau_j^{s**}}(j) = (-1)^{\tau_j^{s**}} \binom{v-1}{\tau_j^{s**}}$, $\Phi_{\tau_j^{s**}}(j) = (-1)^{\tau_j^{s**}} \binom{v}{\tau_j^{s**}}$,

$$T_{\tau_j^{s**}}(j) = (v + \tau_j^{s**} + 1) \ln(1 + y_{j(v)s}^{**c}) \text{ and } T_{t_j^{s**}}(j) = (v + t_j^{s**}) \ln(1 + y_{j(v+1)s}^{**c}), \quad (38)$$

By combining the prior density (2) and the likelihood function (38), the posterior density of a denoted by $\pi_{MRSS}(a | \underline{y}^{**})$ based on MRSS in case of even set size will be obtained as follows

$$\begin{aligned} \pi_{MRSS}(a | \underline{y}^{**}) &\propto a^{q-1} \sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \\ &\sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{t_{v+1}^{1**}=0}^v \sum_{t_{v+2}^{1**}=0}^v \cdots \sum_{t_m^{1**}=0}^v \sum_{t_{v+1}^{2**}=0}^v \sum_{t_{v+2}^{2**}=0}^v \cdots \sum_{t_m^{2**}=0}^v \cdots \sum_{t_{v+1}^{r**}=0}^v \sum_{t_{v+2}^{r**}=0}^v \cdots \\ &\sum_{t_m^{r**}=0}^v \cdots \left[\prod_{s=1}^r \prod_{j=1}^v \Phi_{\tau_j^{s**}}(j) \right] \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{t_j^{s**}}(j) \right] e^{-a \sum_{s=1}^r \left[\sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{j=v+1}^m \left(T_{t_j^{s**}}(j) \right) \right]}. \end{aligned} \quad (39)$$

From posterior densities (37) and (39), the joint posterior density function of b and a will be as follows

$$\begin{aligned} \pi_{MRSS}(b, a | \underline{x}^{**}, \underline{y}^{**}) &\propto \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \\ &\sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{L_{u+1}^{1**}=0}^u \sum_{L_{u+2}^{1**}=0}^u \cdots \sum_{L_n^{1**}=0}^u \sum_{L_{u+1}^{2**}=0}^u \sum_{L_{u+2}^{2**}=0}^u \cdots \sum_{L_n^{2**}=0}^u \cdots \sum_{L_{u+1}^{r**}=0}^u \sum_{L_{u+2}^{r**}=0}^u \cdots \sum_{L_n^{r**}=0}^u \cdots \\ &\sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{t_{v+1}^{1**}=0}^v \sum_{t_{v+2}^{1**}=0}^v \\ &\cdots \sum_{t_m^{1**}=0}^v \sum_{t_{v+1}^{2**}=0}^v \cdots \sum_{t_{v+2}^{2**}=0}^v \cdots \sum_{t_m^{2**}=0}^v \cdots \sum_{t_{v+1}^{r**}=0}^v \sum_{t_{v+2}^{r**}=0}^v \cdots \sum_{t_m^{r**}=0}^v \cdots \left[\prod_{s=1}^r \prod_{i=1}^u \Phi_{L_i^{s**}}(i) \right] \\ &\times \left[\prod_{s=1}^r \prod_{i=u+1}^n \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{j=1}^v \Phi_{\tau_j^{s**}}(j) \right] \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{t_j^{s**}}(j) \right] b^{p-1} a^{q-1} \\ &\times e^{-b \sum_{s=1}^r \left[\sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{i=u+1}^n \left(T_{L_i^{s**}}(i) \right) \right]} e^{-a \sum_{s=1}^r \left[\sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{j=v+1}^m \left(T_{t_j^{s**}}(j) \right) \right]}. \end{aligned} \quad (40)$$

By substituting the transformation technique (5) in the posterior density function (40), the Bayes estimator of R based on MRSS with even set size under non-informative prior can be obtained as follows

$$\begin{aligned} \pi_{MRSS}(A_1, r_1 | \underline{x}^{**}, \underline{y}^{**}) &\propto \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \\ &\sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{L_{u+1}^{1**}=0}^u \sum_{L_{u+2}^{1**}=0}^u \cdots \sum_{L_n^{1**}=0}^u \sum_{L_{u+1}^{2**}=0}^u \sum_{L_{u+2}^{2**}=0}^u \cdots \sum_{L_n^{2**}=0}^u \cdots \sum_{L_{u+1}^{r**}=0}^u \sum_{L_{u+2}^{r**}=0}^u \cdots \sum_{L_n^{r**}=0}^u \cdots \\ &\sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{t_{v+1}^{1**}=0}^v \\ &\sum_{t_{v+2}^{1**}=0}^v \cdots \sum_{t_m^{1**}=0}^v \sum_{t_{v+1}^{2**}=0}^v \sum_{t_{v+2}^{2**}=0}^v \cdots \sum_{t_m^{2**}=0}^v \cdots \sum_{t_{v+1}^{r**}=0}^v \sum_{t_{v+2}^{r**}=0}^v \cdots \sum_{t_m^{r**}=0}^v \cdots (1 - r_1)^{p-1} r_1^{q-1} \\ &\times A_1^{p+q-1} \left[\prod_{s=1}^r \prod_{i=1}^u \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{i=u+1}^n \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{j=1}^v \Phi_{\tau_j^{s**}}(j) \right] \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{t_j^{s**}}(j) \right] \\ &e^{-A_1 \left[(1-r_1) \sum_{s=1}^r \left[\sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{i=u+1}^n \left(T_{L_i^{s**}}(i) \right) \right] + r_1 \sum_{s=1}^r \left[\sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{j=v+1}^m \left(T_{t_j^{s**}}(j) \right) \right] \right]}. \end{aligned}$$

Integrate out of A_1 , the marginal posterior density function of r_1 is as follows

$$\begin{aligned} \pi_{MRSS}(r_1 | \underline{x}^{**}, \underline{y}^{**}) &= \Psi^{**} \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \\ &\sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{L_{u+1}^{1**}=0}^u \sum_{L_{u+2}^{1**}=0}^u \cdots \sum_{L_n^{1**}=0}^u \sum_{L_{u+1}^{2**}=0}^u \sum_{L_{u+2}^{2**}=0}^u \cdots \sum_{L_n^{2**}=0}^u \cdots \sum_{L_{u+1}^{r**}=0}^u \sum_{L_{u+2}^{r**}=0}^u \cdots \sum_{L_n^{r**}=0}^u \cdots \sum_{\tau_1^{1**}=0}^{v-1} \\ &\sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{t_{v+1}^{1**}=0}^v \sum_{t_{v+2}^{1**}=0}^v \cdots \sum_{t_m^{1**}=0}^v \\ &\sum_{t_{v+1}^{2**}=0}^v \sum_{t_{v+2}^{2**}=0}^v \cdots \sum_{t_m^{2**}=0}^v \cdots \sum_{t_{v+1}^{r**}=0}^v \sum_{t_{v+2}^{r**}=0}^v \cdots \sum_{t_m^{r**}=0}^v \cdots (1 - r_1)^{p-1} r_1^{q-1} \end{aligned}$$

$$\begin{aligned} & \sum_{t_{v+1}^{2**}=0}^v \sum_{t_{v+2}^{2**}=0}^v \cdots \sum_{t_m^{2**}=0}^v \cdots \sum_{t_{v+1}^{r**}=0}^v \sum_{t_{v+2}^{r**}=0}^v \cdots \sum_{t_m^{r**}=0}^v \cdots \left[\prod_{s=1}^r \prod_{i=1}^u \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{i=u+1}^n \Phi_{L_i^{s**}}(i) \right] \\ & \left[\prod_{s=1}^r \prod_{j=1}^v \Phi_{\tau_j^{s**}}(j) \right] \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{\tau_j^{s**}}(j) \right] (1-r_1)^{p-1} r_1^{q-1} [(1-r_1) \left(\sum_{s=1}^r \sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) \right. \\ & \left. + \sum_{s=1}^r \sum_{i=u+1}^n \left(T_{L_i^{s**}}(i) \right) + r_1 \left(\sum_{s=1}^r \sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{s=1}^r \sum_{j=v+1}^m \left(T_{\tau_j^{s**}}(j) \right) \right)]^{-(p+q)}, \quad (41) \end{aligned}$$

where Ψ^{**} is a constant and it can be obtained as follows

$$\begin{aligned} \Psi^{**^{-1}} &= \int_0^1 \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{L_{u+1}^{1**}=0}^u \\ & \sum_{L_{u+2}^{1**}=0}^u \cdots \sum_{L_n^{1**}=0}^u \sum_{L_{u+1}^{2**}=0}^u \sum_{L_{u+2}^{2**}=0}^u \cdots \sum_{L_n^{2**}=0}^u \cdots \sum_{L_{u+1}^{r**}=0}^u \sum_{L_{u+2}^{r**}=0}^u \cdots \sum_{L_n^{r**}=0}^u \cdots \sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \\ & \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{\tau_{v+1}^{1**}=0}^v \sum_{\tau_{v+2}^{1**}=0}^v \cdots \sum_{\tau_m^{1**}=0}^v \sum_{\tau_{v+1}^{2**}=0}^v \sum_{\tau_{v+2}^{2**}=0}^v \\ & \cdots \sum_{\tau_m^{2**}=0}^v \cdots \sum_{\tau_{v+1}^{r**}=0}^v \sum_{\tau_{v+2}^{r**}=0}^v \cdots \sum_{\tau_m^{r**}=0}^v \cdots \left[\prod_{s=1}^r \prod_{i=1}^u \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{i=u+1}^n \Phi_{L_i^{s**}}(i) \right] \\ & \left[\prod_{s=1}^r \prod_{j=1}^v \Phi_{\tau_j^{s**}}(j) \right] \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{\tau_j^{s**}}(j) \right] (1-r_1)^{p-1} r_1^{-p-1} \left(\sum_{s=1}^r \sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) \right. \\ & \left. + \sum_{s=1}^r \sum_{j=v+1}^m \left(T_{\tau_j^{s**}}(j) \right) \right)^{-(p+q)} \left[\frac{(1-r_1) \left(\sum_{s=1}^r \sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{s=1}^r \sum_{i=u+1}^n \left(T_{L_i^{s**}}(i) \right) \right)}{r_1 \left(\sum_{s=1}^r \sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{s=1}^r \sum_{j=v+1}^m \left(T_{\tau_j^{s**}}(j) \right) \right)} + 1 \right]^{-(p+q)} dr_1, \quad (42) \end{aligned}$$

using (7), then (42) will be written as follows

$$\begin{aligned} \Psi^{**^{-1}} &= \int_0^\infty \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{L_{u+1}^{1**}=0}^u \\ & \sum_{L_{u+2}^{1**}=0}^u \cdots \sum_{L_n^{1**}=0}^u \sum_{L_{u+1}^{2**}=0}^u \sum_{L_{u+2}^{2**}=0}^u \cdots \sum_{L_n^{2**}=0}^u \cdots \sum_{L_{u+1}^{r**}=0}^u \sum_{L_{u+2}^{r**}=0}^u \cdots \sum_{L_n^{r**}=0}^u \cdots \sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \\ & \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{\tau_{v+1}^{1**}=0}^v \sum_{\tau_{v+2}^{1**}=0}^v \cdots \sum_{\tau_m^{1**}=0}^v \sum_{\tau_{v+1}^{2**}=0}^v \sum_{\tau_{v+2}^{2**}=0}^v \\ & \cdots \sum_{\tau_m^{2**}=0}^v \cdots \sum_{\tau_{v+1}^{r**}=0}^v \sum_{\tau_{v+2}^{r**}=0}^v \cdots \sum_{\tau_m^{r**}=0}^v \cdots \left[\prod_{s=1}^r \prod_{i=1}^u \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{i=u+1}^n \Phi_{L_i^{s**}}(i) \right] \\ & \left[\prod_{s=1}^r \prod_{j=1}^v \Phi_{\tau_j^{s**}}(j) \right] \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{\tau_j^{s**}}(j) \right] \left(\frac{\eta_1}{1+\eta_1} \right)^{p-1} \left(\frac{1}{1+\eta_1} \right)^{-p-1} \left(\sum_{s=1}^r \sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) \right. \\ & \left. + \sum_{s=1}^r \sum_{j=v+1}^m \left(T_{\tau_j^{s**}}(j) \right) \right)^{-(p+q)} \left[\eta_1 \left(\frac{\sum_{s=1}^r \left(\sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{i=u+1}^n \left(T_{L_i^{s**}}(i) \right) \right)}{\sum_{s=1}^r \left(\sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{j=v+1}^m \left(T_{\tau_j^{s**}}(j) \right) \right)} \right) + 1 \right]^{-(p+q)} \frac{d\eta_1}{(1+\eta_1)^2}, \end{aligned}$$

then,

$$\begin{aligned} \Psi^{**^{-1}} &= \int_0^\infty \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \\ & \sum_{L_{u+1}^{1**}=0}^u \sum_{L_{u+2}^{1**}=0}^u \cdots \sum_{L_n^{1**}=0}^u \sum_{L_{u+1}^{2**}=0}^u \sum_{L_{u+2}^{2**}=0}^u \cdots \sum_{L_n^{2**}=0}^u \cdots \sum_{L_{u+1}^{r**}=0}^u \sum_{L_{u+2}^{r**}=0}^u \cdots \sum_{L_n^{r**}=0}^u \cdots \sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \\ & \cdots \sum_{\tau_1^{2**}=0}^{v-1} \cdots \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \cdots \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{\tau_{v+1}^{1**}=0}^v \sum_{\tau_{v+2}^{1**}=0}^v \cdots \sum_{\tau_m^{1**}=0}^v \sum_{\tau_{v+1}^{2**}=0}^v \\ & \sum_{\tau_{v+2}^{2**}=0}^v \cdots \sum_{\tau_m^{2**}=0}^v \cdots \sum_{\tau_{v+1}^{r**}=0}^v \sum_{\tau_{v+2}^{r**}=0}^v \cdots \sum_{\tau_m^{r**}=0}^v \cdots \left[\prod_{s=1}^r \prod_{i=1}^u \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{i=u+1}^n \Phi_{L_i^{s**}}(i) \right] \end{aligned}$$

$$\times \left[\prod_{s=1}^r \prod_{j=1}^v \Phi_{\tau_j^{s**}}(j) \right] \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{t_j^{s**}}(j) \right] (\eta_1)^{p-1} \left(\sum_{s=1}^r \sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{s=1}^r \sum_{j=v+1}^m \left(T_{t_j^{s**}}(j) \right) \right)^{-(p+q)} \left[\eta_1 \left(\frac{\sum_{s=1}^r \sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{s=1}^r \sum_{i=u+1}^n \left(T_{l_i^{s**}}(i) \right)}{\sum_{s=1}^r \sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{s=1}^r \sum_{j=v+1}^m \left(T_{t_j^{s**}}(j) \right)} \right) + 1 \right]^{-p+q} d\eta_1, \quad (43)$$

$$\text{Let, } \eta_5 = \eta_1 \left(\frac{\sum_{s=1}^r \sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{s=1}^r \sum_{i=u+1}^n \left(T_{l_i^{s**}}(i) \right)}{\sum_{s=1}^r \sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{s=1}^r \sum_{j=v+1}^m \left(T_{t_j^{s**}}(j) \right)} \right), \text{ then (43) takes the following form}$$

$$\begin{aligned} \Psi^{**,-1} &= \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{l_{u+1}^{1**}=0}^u \cdots \sum_{l_n^{1**}=0}^u \sum_{l_{u+2}^{1**}=0}^u \sum_{l_{u+3}^{1**}=0}^u \cdots \sum_{l_n^{1**}=0}^u \cdots \sum_{l_{u+2}^{r**}=0}^u \cdots \sum_{l_n^{r**}=0}^u \cdots \sum_{l_{u+2}^{v-1**}=0}^{v-1} \sum_{l_{u+3}^{v-1**}=0}^{v-1} \cdots \sum_{l_n^{v-1**}=0}^{v-1} \cdots \sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \cdots \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{\tau_{v+1}^{1**}=0}^v \sum_{\tau_{v+2}^{1**}=0}^v \cdots \sum_{\tau_m^{1**}=0}^v \sum_{\tau_{v+2}^{2**}=0}^v \cdots \sum_{\tau_m^{2**}=0}^v \cdots \sum_{\tau_{v+1}^{r**}=0}^v \sum_{\tau_{v+2}^{r**}=0}^v \cdots \sum_{\tau_m^{r**}=0}^v \cdots \left[\prod_{s=1}^r \prod_{i=1}^u \Phi_{L_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{i=u+1}^n \Phi_{l_i^{s**}}(i) \right] \left[\prod_{s=1}^r \prod_{j=1}^v \Phi_{\tau_j^{s**}}(j) \right] \\ &\times \left[\prod_{s=1}^r \prod_{j=v+1}^m \Phi_{t_j^{s**}}(j) \right] \left(\sum_{s=1}^r \sum_{i=1}^u \left(T_{L_i^{s**}}(i) \right) + \sum_{s=1}^r \sum_{i=u+1}^n \left(T_{l_i^{s**}}(i) \right) \right)^{-p} B(p, q) \\ &\times \left(\sum_{s=1}^r \sum_{j=1}^v \left(T_{\tau_j^{s**}}(j) \right) + \sum_{s=1}^r \sum_{j=v+1}^m \left(T_{t_j^{s**}}(j) \right) \right)^{-q}. \end{aligned} \quad (44)$$

So, the posterior density function of R denoted by $\pi_{MRSS}(r_1 | \underline{x}^{**}, \underline{y}^{**})$, can be obtained by using (44) and (41), for $0 < r < 1$. Then Bayes estimator of R denoted by R_{SE}^{**} under SE loss function is as follows

$$R_{SE}^{**} = \int_0^1 r_1 \pi_{MRSS}(r_1 | \underline{x}^{**}, \underline{y}^{**}) dr_1, \quad (45)$$

Additionally, the Bayes estimator of R under LINEX loss function, denoted by R_{LIN}^{**} , Based on MRSS in case of even set size is given as follows

$$\begin{aligned} R_{LIN}^{**} &= \frac{-1}{\vartheta} \ln E(e^{-\vartheta r_1}), \\ E(e^{-\vartheta r_1}) &= \int_0^1 e^{-\vartheta r_1} \pi_{MRSS}(r_1 | \underline{x}^{**}, \underline{y}^{**}) dr_1. \end{aligned} \quad (46)$$

By using (45), Bayes estimator of R under SE in the simple case of a single cycle when ($r = 1$) is as follows

$$\begin{aligned} R_{SE}^{**} &= \int_0^1 \left[\sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \left[\prod_{i=1}^u (-1)^{L_i^{1**}} \binom{u-1}{L_i^{1**}} \right] \sum_{l_{u+1}^{1**}=0}^u \sum_{l_{u+2}^{1**}=0}^u \cdots \sum_{l_n^{1**}=0}^u \left[\prod_{i=u+1}^n (-1)^{l_i^{1**}} \binom{u}{l_i^{1**}} \right] \right. \\ &\sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \left[\prod_{j=1}^v \left((-1)^{\tau_j^{1**}} \binom{v-1}{\tau_j^{1**}} \right) \right] \sum_{t_{v+1}^{1**}=0}^v \sum_{t_{v+2}^{1**}=0}^v \cdots \sum_{t_m^{1**}=0}^v \left[\prod_{j=v+1}^m \left((-1)^{t_j^{1**}} \binom{v}{t_j^{1**}} \right) \right] \\ &\times (1 - r_1)^{n-1} r_1^m \left[(1 - r_1) \left[\sum_{i=1}^u (u + L_i^{1**} + 1) \ln(1 + x_{i(u)}^{**c}) + \sum_{i=u+1}^n (u + l_i^{1**}) \ln(1 + x_{i(u+1)}^{**c}) \right] \right. \\ &\left. + r_1 \left[\sum_{j=1}^v (v + \tau_j^{1**} + 1) \ln(1 + y_{j(v)}^{**c}) + \sum_{j=v+1}^m (v + t_j^{1**}) \ln(1 + y_{j(v+1)}^{**c}) \right] \right]^{-(n+m)} \left[\sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \right. \\ &\sum_{L_u^{1**}=0}^{u-1} \sum_{l_{u+1}^{1**}=0}^u \sum_{l_{u+2}^{1**}=0}^u \cdots \sum_{l_n^{1**}=0}^u \sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{t_{v+1}^{1**}=0}^v \sum_{t_{v+2}^{1**}=0}^v \cdots \sum_{t_m^{1**}=0}^v \end{aligned}$$

$$\begin{aligned}
 & \left[\prod_{i=1}^u (-1)^{L_i^{**}} \binom{u-1}{L_i^{**}} \right] \left[\prod_{i=u+1}^n (-1)^{l_i^{**}} \binom{u}{l_i^{**}} \right] \left[\prod_{j=1}^v \left((-1)^{\tau_j^{**}} \binom{v-1}{\tau_j^{**}} \right) \right] \left[\prod_{j=v+1}^m \left((-1)^{t_j^{**}} \binom{v}{t_j^{**}} \right) \right] \\
 & \times B(n, m) \left[\sum_{i=1}^u (u + L_i^{**} + 1) \ln(1 + x_{i(u)}^{**c}) + \sum_{i=u+1}^n (u + l_i^{**}) \ln(1 + x_{i(u+1)}^{**c}) \right]^{-n} \\
 & \times \left[\sum_{j=1}^v (v + \tau_j^{**} + 1) \ln(1 + y_{j(v)}^{**c}) + \sum_{j=v+1}^m (v + t_j^{**}) \ln(1 + y_{j(v+1)}^{**c}) \right]^{-m} dr_1. \quad (47)
 \end{aligned}$$

Bayes estimator of R under LINEX loss function, based on MRSS in case of even set size when $(r = 1)$ is given by

$$\begin{aligned}
 R_{LIN}^{**} &= \frac{-1}{\theta} \ln \left\{ \int_0^1 \left[\sum_{L_1^{**}=0}^{u-1} \sum_{L_2^{**}=0}^{u-1} \dots \sum_{L_u^{**}=0}^{u-1} \left[\prod_{i=1}^u (-1)^{L_i^{**}} \binom{u-1}{L_i^{**}} \right] \sum_{l_{u+1}^{**}=0}^u \sum_{l_{u+2}^{**}=0}^u \dots \right. \right. \\
 & \sum_{l_n^{**}=0}^u \left[\prod_{i=u+1}^n (-1)^{l_i^{**}} \binom{u}{l_i^{**}} \right] \sum_{\tau_1^{**}=0}^{v-1} \sum_{\tau_2^{**}=0}^{v-1} \dots \sum_{\tau_v^{**}=0}^{v-1} \left[\prod_{j=1}^v \left((-1)^{\tau_j^{**}} \binom{v-1}{\tau_j^{**}} \right) \right] \\
 & \times \sum_{\tau_{v+1}^{**}=0}^v \sum_{\tau_{v+2}^{**}=0}^v \dots \sum_{\tau_m^{**}=0}^v \left[\prod_{j=v+1}^m \left((-1)^{t_j^{**}} \binom{v}{t_j^{**}} \right) \right] e^{-\theta r_1 \ln((1-r_1)^{n-1} r_1^{m-1})} \\
 & \times \left[(1-r_1) \left[\sum_{i=1}^u (u + L_i^{**} + 1) \ln(1 + x_{i(u)}^{**c}) + \sum_{i=u+1}^n (u + l_i^{**}) \ln(1 + x_{i(u+1)}^{**c}) \right] \right. \\
 & \left. + r_1 \left[\sum_{j=1}^v (v + \tau_j^{**} + 1) \ln(1 + y_{j(v)}^{**c}) + \sum_{j=v+1}^m (v + t_j^{**}) \ln(1 + y_{j(v+1)}^{**c}) \right] \right]^{-(n+m)} \\
 & \times \left[\left[\sum_{L_1^{**}=0}^{u-1} \sum_{L_2^{**}=0}^{u-1} \dots \sum_{L_u^{**}=0}^{u-1} \left[\prod_{i=1}^u (-1)^{L_i^{**}} \binom{u-1}{L_i^{**}} \right] \sum_{l_{u+1}^{**}=0}^u \sum_{l_{u+2}^{**}=0}^u \dots \sum_{l_n^{**}=0}^u \left[\prod_{i=u+1}^n (-1)^{l_i^{**}} \binom{u}{l_i^{**}} \right] \right. \right. \\
 & \sum_{\tau_1^{**}=0}^{v-1} \sum_{\tau_2^{**}=0}^{v-1} \dots \sum_{\tau_v^{**}=0}^{v-1} \left[\prod_{j=1}^v \left((-1)^{\tau_j^{**}} \binom{v-1}{\tau_j^{**}} \right) \right] \sum_{\tau_{v+1}^{**}=0}^v \sum_{\tau_{v+2}^{**}=0}^v \dots \sum_{\tau_m^{**}=0}^v \left[\prod_{j=v+1}^m \left((-1)^{t_j^{**}} \binom{v}{t_j^{**}} \right) \right] \\
 & \times B(n, m) \left[\sum_{i=1}^u (u + L_i^{**} + 1) \ln(1 + x_{i(u)}^{**c}) + \sum_{i=u+1}^n (u + l_i^{**}) \ln(1 + x_{i(u+1)}^{**c}) \right]^{-n} \\
 & \times \left. \left[\sum_{j=1}^v (v + \tau_j^{**} + 1) \ln(1 + y_{j(v)}^{**c}) + \sum_{j=v+1}^m (v + t_j^{**}) \ln(1 + y_{j(v+1)}^{**c}) \right]^{-m} \right\} dr_1. \quad (48)
 \end{aligned}$$

Clearly, it is not easy to obtain a closed form solution for Bayesian estimators under non-informative Jeffery prior in (47) and (48) using SE and LINEX loss functions respectively, so an iterative procedure can be used to evaluate these equations.

5. Numerical Study

In this section, an extensive numerical investigation will be carried out using MathCAD (14) to compare the performance of the Bayes estimates $\hat{R}_{SE}$, $\hat{R}_{LIN}$, R_{SE}^* , R_{LIN}^* , R_{SE}^{**} and R_{LIN}^{**} in case of odd and even set sizes respectively, based on MRSS data under SE and LINEX loss functions. The investigated properties are biases and mean square errors (MSEs). Without loss of generality, the number of cycles r will be one in all cases.

Several studies have shown that if one chooses the parameters b and c within the ranges (1-10) and (1-20) respectively, along with a sample size (2-10), the cumulative distribution function of Burr (c, b) covers the three main Pearson system types I, IV and VI, as well as many traditional types such that gamma distribution (see Burr (1968) and Burr and Cislak (1968)), keeping the above points in mind, 1000 random samples of Burr (c, b) and

Burr (c, a) are generated with sample sizes $n = m = 2, 3, 4$. Without loss of generality, the shape parameter c will be assumed to be one in all experiments, the ratio ρ is selected as 0.25, 0.33, 0.5, 1, 2, 3, 4, 5, 6 depend on the different values of the shape parameters $b = 3, 4, 5, 6, 9, 12$ and $a = 2, 3, 6, 9, 10, 12$ where $\rho = \frac{b}{a}$.

Numerical results are summarized in Tables (1) – (9) and represented through Figures(1) – (4) from these tables and figures the following conclusions can be observed as follows

1. The biases of all estimators $\hat{R}_{SE}, \hat{R}_{LIN}, \dot{R}_{SE}, \dot{R}_{LIN}, R_{SE}^*, R_{LIN}^*, R_{SE}^{**}$ and R_{LIN}^{**} based on SRS, RSS, MRSS in case of odd and even set sizes respectively are small. (See Tables (1) – (9)).
2. MSEs of all estimators $\hat{R}_{SE}, \hat{R}_{LIN}, \dot{R}_{SE}, \dot{R}_{LIN}, R_{SE}^*, R_{LIN}^*, R_{SE}^{**}$ and R_{LIN}^{**} are increasing as ρ increases up to $\rho = 1$, then MSEs are decreasing as ρ increases, in almost all cases.
3. In almost all cases, MSEs of all estimators $\hat{R}_{SE}, \hat{R}_{LIN}, \dot{R}_{SE}, \dot{R}_{LIN}, R_{SE}^*, R_{LIN}^*, R_{SE}^{**}$ and R_{LIN}^{**} are decreasing as the set size increases at the same value of ρ . (See for example Figure (1)).

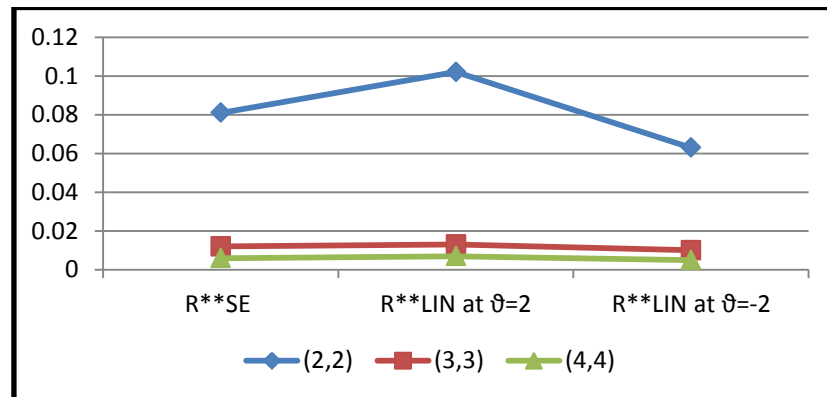


Figure (1): MSEs of R_{SE}^{**} and R_{LIN}^{**} when $\rho = 0.25$ based on MRSS data with even set size.

4. In almost all cases, in case of $\rho < 1$, MSEs of $\hat{R}_{LIN}, \dot{R}_{LIN}, R_{LIN}^*$ and R_{LIN}^{**} under LINEX loss function when $\vartheta = -2$, have the smallest MSEs comparing with the other estimators $\hat{R}_{SE}, \dot{R}_{SE}, R_{SE}^*$ and R_{SE}^{**} based on SE loss function and $\hat{R}_{LIN}, \dot{R}_{LIN}, R_{LIN}^*$ and R_{LIN}^{**} based on LINEX loss function when $\vartheta = 2$. (See for example Figures (2) and (3)).
5. In almost all cases, in case of $\rho > 1$, MSEs of $\hat{R}_{LIN}, \dot{R}_{LIN}, R_{LIN}^*$ and R_{LIN}^{**} under LINEX loss function when $\vartheta = 2$, have the smallest MSEs comparing with the other estimators $\hat{R}_{SE}, \dot{R}_{SE}, R_{SE}^*$ and R_{SE}^{**} based on SE and $\hat{R}_{LIN}, \dot{R}_{LIN}, R_{LIN}^*$ and R_{LIN}^{**} based on LINEX loss function when $\vartheta = -2$. (See for example Figures (2) and (3)).

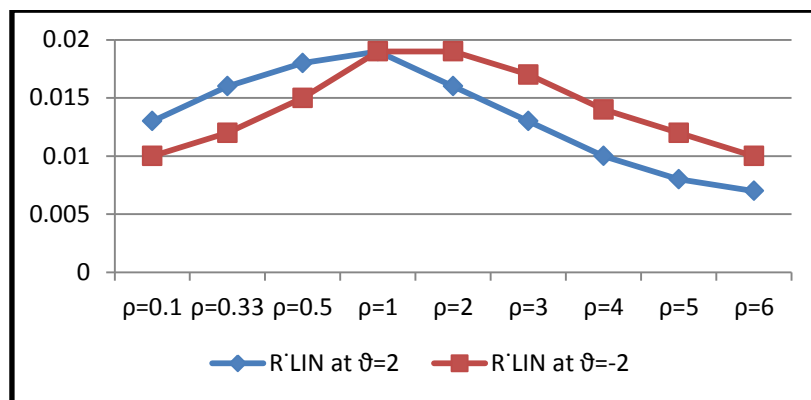


Figure (2): MSEs of the estimates $\dot{R}_{LIN}$ based on RSS data when $\vartheta = 2$ and $\vartheta = -2$.

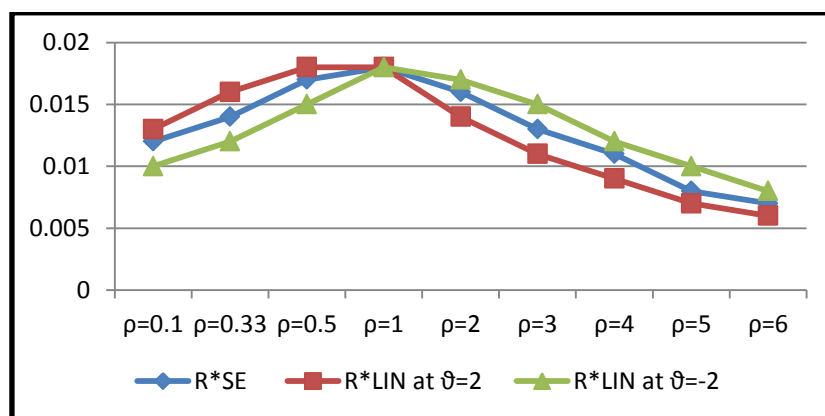


Figure (3):MSEs of the estimates $R_{SE}^*, R_{SE}^{**}, R_{LIN}^*$ and R_{LIN}^{**} based on MRSS data when $(n, m) = (3,3)$.

6. MSEs of the estimators $R_{SE}^*, R_{LIN}^*, R_{SE}^{**}$ and R_{LIN}^{**} , $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on MRSS and RSS data are smaller than MSEs of the estimators $\hat{R}_{SE}, \hat{R}_{LIN}$ based on SRS data, in almost all cases. (See for example Figure (4)).
7. MSEs of the estimators $R_{SE}^*, R_{LIN}^*, R_{SE}^{**}$ and R_{LIN}^{**} based on MRSS in case of odd and even set sizes respectively, are smaller than the estimators $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on RSS, in almost all cases. (See for example Figure (4)).

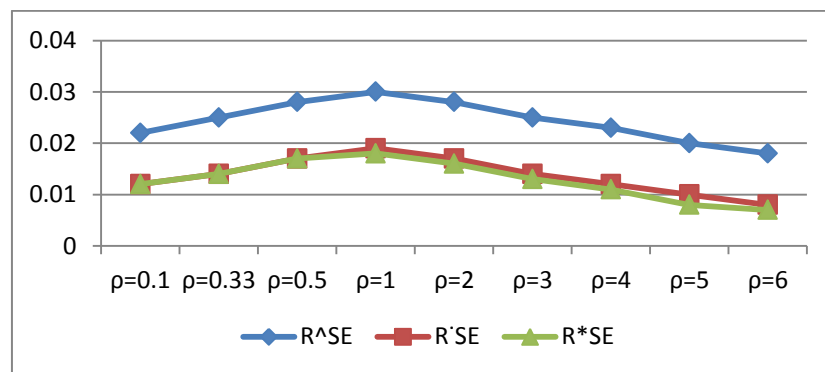


Figure (4): MSEs of the estimates $\hat{R}_{SE}, \dot{R}_{SE}$ and R_{SE}^* based on SRS, RSS and MRSS data at $(n, m) = (3,3)$.

6. Conclusions

In this article, the estimation problem of $R = P(Y < X)$ when X and Y follow Burr XII distribution using Bayesian approach based on SRS, RSS and MRSS techniques using non-informative Jeffery prior under symmetric and asymmetric loss functions is discussed.

From the numerical results, it is observed that, the MSEs of all estimators based on SRS, RSS and MRSS are decreasing as the set size increases. In almost all cases, MSEs of RSS and MRSS based on RSS and MRSS techniques are smaller than MSEs based on SRS. In almost all cases, the estimators based on MRSS have the smallest MSEs under SE and LINEX loss functions. The estimators based on MRSS approach dominate the corresponding based on RSS and SRS. Finally, this study revealed that the estimators based on MRSS are more efficient than the other two sampling techniques.

Table (1): Biases and MSEs for the estimates $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ based on SRS
at $(n, m) = (2, 2)$.

ρ	Biases			MSEs		
	$\hat{R}_{SE}$	$\hat{R}_{LIN}$		$\hat{R}_{SE}$	$\hat{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.087	-0.121	-0.059	0.036	0.047	0.028
0.33	-0.081	-0.117	-0.048	0.038	0.048	0.031
0.50	-0.062	-0.102	-0.026	0.039	0.047	0.033
1	-0.014	-0.054	0.026	0.038	0.031	0.038
2	0.036	0.001	0.075	0.035	0.030	0.041
3	0.056	0.025	0.091	0.032	0.026	0.041
4	0.064	0.037	0.097	0.030	0.023	0.039
5	0.068	0.043	0.097	0.027	0.021	0.036
6	0.068	0.046	0.069	0.025	0.019	0.033

Table (2): Biases and MSEs for the estimates $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ based on SRS
at $(n, m) = (3, 3)$.

ρ	Biases			MSEs		
	$\hat{R}_{SE}$	$\hat{R}_{LIN}$		$\hat{R}_{SE}$	$\hat{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.062	-0.085	-0.042	0.023	0.029	0.018
0.33	-0.0258	-0.084	-0.036	0.025	0.031	0.021
0.50	-0.004	-0.075	-0.019	0.028	0.033	0.024
1	-0.010	-0.040	0.020	0.028	0.029	0.028
2	0.027	0.001	0.055	0.026	0.023	0.030
3	0.041	0.019	0.066	0.023	0.019	0.028
4	0.046	0.028	0.068	0.020	0.017	0.025
5	0.048	0.032	0.067	0.018	0.014	0.022
6	0.048	0.034	0.065	0.016	0.013	0.020

Table (3): Biases and MSEs for the estimates $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ based on SRS data at $(n, m) = (4, 4)$.

ρ	Biases			MSEs		
	$\hat{R}_{SE}$	$\hat{R}_{LIN}$		$\hat{R}_{SE}$	$\hat{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.042	-0.058	-0.028	0.017	0.020	0.014
0.33	-0.038	-0.057	-0.021	0.019	0.023	0.017
0.50	-0.026	-0.048	-0.005	0.021	0.024	0.019
1	0.004	-0.020	0.028	0.024	0.024	0.024
2	0.033	0.013	0.056	0.023	0.020	0.026
3	0.043	0.026	0.063	0.020	0.017	0.023
4	0.046	0.032	0.063	0.017	0.014	0.021
5	0.047	0.034	0.061	0.015	0.012	0.018
6	0.046	0.035	0.058	0.013	0.011	0.016

Table (4): Biases and MSEs for the estimates $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on RSS data at $(n, m) = (2, 2)$.

ρ	Biases			MSEs		
	$\dot{R}_{SE}$	$\dot{R}_{LIN}$		$\dot{R}_{SE}$	$\dot{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.059	-0.080	-0.039	0.025	0.031	0.021
0.33	-0.054	-0.079	-0.031	0.028	0.034	0.023
0.50	-0.039	-0.068	-0.012	0.031	0.035	0.027
1	0.001	-0.030	0.033	0.030	0.031	0.031
2	0.039	0.012	0.068	0.030	0.026	0.035
3	0.053	0.030	0.079	0.027	0.022	0.033
4	0.058	0.038	0.081	0.024	0.019	0.030
5	0.060	0.042	0.080	0.021	0.017	0.027
6	0.059	0.043	0.078	0.019	0.015	0.024

Table (5): Biases and MSEs for the estimates $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on RSS data at $(n, m) = (3, 3)$.

ρ	Biases			MSEs		
	$\dot{R}_{SE}$	$\dot{R}_{LIN}$		$\dot{R}_{SE}$	$\dot{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.032	-0.043	-0.022	0.011	0.013	0.010
0.33	-0.030	-0.044	-0.017	0.013	0.016	0.012
0.50	-0.022	-0.039	-0.006	0.016	0.018	0.015
1	-0.0001	-0.018	0.018	0.018	0.018	0.018
2	0.023	0.006	0.039	0.016	0.015	0.018
3	0.030	0.017	0.044	0.014	0.012	0.016
4	0.032	0.022	0.044	0.011	0.009	0.013
5	0.032	0.023	0.042	0.009	0.008	0.011
6	0.031	0.024	0.039	0.008	0.006	0.009

Table (6): Biases and MSEs for the estimates $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on RSS data at $(n, m) = (4, 4)$.

ρ	Biases			MSEs		
	$\dot{R}_{SE}$	$\dot{R}_{LIN}$		$\dot{R}_{SE}$	$\dot{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.105	-0.114	-0.096	0.015	0.017	0.013
0.33	-0.093	-0.103	-0.083	0.011	0.015	0.007
0.50	-0.065	-0.076	-0.053	0.010	0.011	0.009
1	-0.001	-0.014	0.010	0.012	0.005	0.012
2	0.061	0.050	0.072	0.008	0.007	0.010
3	0.090	0.080	0.100	0.007	0.010	0.013
4	0.102	0.930	0.122	0.006	0.012	0.008
5	0.106	0.062	0.155	0.009	0.008	0.011
6	0.109	0.103	0.015	0.015	0.013	0.017

Table (7): Biases and MSEs for the estimates R_{SE}^{**} and R_{LIN}^{**} based on MRSS data at $(n, m) = (2, 2)$.

ρ	Biases			MSEs		
	R_{SE}^{**}	R_{LIN}^{**}		R_{SE}^{**}	R_{LIN}^{**}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.055	-0.078	-0.035	0.023	0.029	0.019
0.33	-0.050	-0.075	-0.027	0.026	0.032	0.021
0.50	-0.035	-0.064	-0.008	0.029	0.033	0.025
1	0.003	-0.027	0.033	0.030	0.031	0.031
2	0.041	0.014	0.070	0.029	0.025	0.034
3	0.055	0.031	0.081	0.026	0.021	0.032
4	0.059	0.039	0.083	0.023	0.018	0.029
5	0.060	0.043	0.081	0.020	0.016	0.026
6	0.060	0.044	0.079	0.018	0.014	0.023

Table (8): Biases and MSEs for the estimates R_{SE}^{**} and R_{LIN}^{**} based on MRSS data at $(n, m) = (3, 3)$.

ρ	Biases			MSEs		
	R_{SE}^{**}	R_{LIN}^{**}		R_{SE}^{**}	R_{LIN}^{**}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.031	-0.042	-0.021	0.010	0.012	0.009
0.33	-0.029	-0.042	-0.017	0.012	0.015	0.011
0.50	-0.022	-0.038	-0.007	0.015	0.017	0.014
1	-0.001	-0.018	0.017	0.017	0.018	0.017
2	0.021	0.006	0.037	0.015	0.014	0.017
3	0.028	0.016	0.041	0.013	0.011	0.015
4	0.030	0.020	0.041	0.010	0.008	0.012
5	0.030	0.022	0.039	0.008	0.007	0.010
6	0.029	0.023	0.007	0.006	0.005	0.008

Table (9): Biases and MSEs for the estimates R_{SE}^{**} and R_{LIN}^{**} based on MRSS data at $(n, m) = (4, 4)$.

ρ	Biases			MSEs		
	R_{SE}^{**}	R_{LIN}^{**}		R_{SE}^{**}	R_{LIN}^{**}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.020	-0.026	-0.014	0.006	0.007	0.005
0.33	-0.019	-0.007	-0.012	0.007	0.004	0.006
0.50	-0.015	-0.025	-0.005	0.010	0.010	0.008
1	-0.001	-0.012	-0.011	0.012	0.012	0.011
2	0.013	0.004	0.023	0.010	0.009	0.011
3	0.018	0.010	0.026	0.008	0.007	0.008
4	0.019	0.013	0.025	0.005	0.005	0.007
5	0.019	0.014	0.024	0.005	0.004	0.005
6	0.018	0.014	0.022	0.004	0.003	0.004

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Parametric inference on constant-stress life test model using Type-I progressively hybrid censored data

Ali A. Ismail

Abstract: In this article we consider the statistical inference problem on constant-stress partially accelerated life test model based on Type-I progressively hybrid censored data from Weibull distribution. Three different methods; maximum likelihood (ML), percentile bootstrap (PB) and bootstrap-t (BT) are used to make statistical inference on the Weibull distribution parameters and the acceleration factor. To evaluate the performance of these methods toward the obtained estimates, the associated mean squared errors (MSEs) are calculated. In addition, the confidence intervals of the model parameters as well as their coverage probabilities are computed. To illustrate the theoretical results based on this advanced censoring scheme, Monte Carlo simulation studies are presented.

Keywords: Reliability; statistical inference; constant-stress partially accelerated life test model; maximum likelihood; bootstrap; Type-I progressively hybrid censoring; Weibull distribution.

Subject Classifications: 62N01; 62N05; 62F10.

1. INTRODUCTION

In reliability analysis, it is important to analyze the time period over which components or systems function. Such analysis is called lifetime or failure time analysis. However, for highly reliable components, their lifetimes are usually longer under usual functioning circumstances. Consequently, it becomes time-consuming for acquiring enough failure time data or the cost of the experiment is increased. Therefore, the suitable method to assess highly reliable components is to test them under severe functioning circumstances, denoted as accelerated life test (ALT). In such experiment, testing units are exposed to higher than normal levels of stress, such as voltage, temperature, humidity, force, and so on, so that more failures can be noticed in a limited time.

In ALT, all test units are run under severe conditions. The information obtained from ALT is used to predict the actual product performance in usual circumstances via a life-stress model. However, in sometimes such model can't be known then ALT can't be applied. Accordingly, partially accelerated life test (PALT) comes to be a good alternative to make the required reliability analysis of highly reliable units, components or systems. The goal of a PALT is to gather more failure data in a restricted time without essentially spending high stresses to all test units. In PALT, the experimenter puts some of test units under normal (use) conditions and the remaining units under severe conditions. As Nelson [1] indicates "the stress can be applied in various ways, commonly used methods are step-stress and constant-stress". Under step-stress PALT, a test item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition (stress) until failure occurs or the observation is censored. But the constant-stress PALT runs each item at either use condition or accelerated condition only, i.e. each unit is run at a constant-stress level until the test is terminated.

PALT has been considered under traditional Type-I and Type-II censoring schemes by many authors, for example, see (Goel [2]; DeGroot and Goel [3]; Bhattacharyya and Soejoeti [4]; Bai and Chung [5]; Bai et al. [6]; Abdel-Ghaly et al. [7,8]; Ismail et al. [9]; Abdel-Ghani [10,11], Ismail [12]; Aly and Ismail [13]; Ismail and Sarhan [14]; and Ismail [15].

Cairo University, Faculty of Economics & Political Science, Department of Statistics, Giza 12613, Egypt

Under constant-stress PALT (CSPALT) there are some studies have been made using the usual Type-I and Type-II censoring schemes, for example, see Ismail [16,17]. Abushal and Soliman [18] have considered the CSPALT under progressive censoring scheme. Based on progressively Type-I hybrid censoring scheme there are some studies under ALT, for example, see Lin et al. [19], Ling et al. [20] and Zhao et al. [21]. But, specifically, under CSPALT there are no studies have been prepared before in this respect. Hence, this article will focus on CSPALT under progressively Type-I hybrid censoring scheme (Type-I PHCS). Such advanced censoring scheme under CSPALT will be described in the next Section.

The remainder of this article is prepared as follows. Section 2 presents the description of the model. In Section 3, the maximum likelihood (ML) estimators of the CSPALT model parameters are presented. In addition, confidence intervals for the model parameters based on three different methods of estimation are considered. Section 4 covers the simulation results. Concluding notes and further analyses are presented in Section 5.

2. MODEL DESCRIPTION

2.1. Weibull Distribution: A Lifetime Distribution

The lifetime of a test unit as a random variable say X_1 is assumed to has a two- parameter Weibull distribution (WD) with the shape and scale parameters β and η respectively. So, the probability density function (PDF) of X_1 is given by

$$f_1(x_1) \equiv f_{x_1}(x_1; \beta, \eta) = \frac{\beta}{\eta} \left(\frac{x_1}{\eta} \right)^{\beta-1} \exp\{-(x_1/\eta)^\beta\} \quad ; x_1 > 0, \beta > 0, \eta > 0, \quad (2.1)$$

WD is commonly used distribution in reliability analysis. It is used to investigate several lifetime data. Its hazard function can be increasing, decreasing and constant according to the shape parameter value. Therefore, this distribution is more popular than other distributions.

The reliability function of WD in (2.1) takes the form

$$S_1(x_1) = \exp\{-(x_1/\eta)^\beta\}, \quad (2.2)$$

The analogous failure rate function is given by

$$h_1(x_1) = \frac{\beta}{\eta} \left(\frac{x_1}{\eta} \right)^{\beta-1}. \quad (2.3)$$

2.2. Constant-Stress PALT

The test method of the constant-stress PALT and its assumptions are presented as follows.

Test Method

In a constant-stress PALT, the total sample size n of test units is apportioned into two portions such that:

1. n_1 units randomly selected among n test units sampled are assigned to run under use condition and $n_2 = n - n_1$ remaining units are subjected to run under accelerated condition.

2. Each test unit is run until a point time T is reached or the m th failure is occurred whichever is first attained or until the unit is removed, where T and m are pre-fixed values.

Assumptions

1. The lifetimes X_{1i} , $i = 1, \dots, n_1$ of items assigned to run under use condition are independent and identically distributed random variables (i.i.d. r.v.'s).
2. The lifetimes X_{2i} , $i = 1, \dots, n_2$ of items assigned to run under accelerated condition are i.i.d r.v.'s.
3. The hazard rate of an item tested at accelerated condition is given by $h_2(x_2) = \lambda h_1(x_1)$, where λ is an acceleration factor satisfying $\lambda > 1$.

In simple constant-stress PALT, the test unit is run either at use condition or accelerated condition only. Since the lifetime of the test unit has Weibull distribution, the PDF of a unit run at use condition is given as in (2.1). While for a unit run at accelerated condition, the PDF is presented by

$$f_2(x_2) \equiv f_{x_2}(x_2; \beta, \eta, \lambda)$$

$$= \frac{\lambda \beta}{\eta} \left(\frac{\lambda x_2}{\eta} \right)^{\beta-1} \exp\{-(\lambda x_2/\eta)^\beta\} \quad ; x_2 > 0, \lambda > 1, \beta > 0, \eta > 0, \quad (2.4)$$

where $X_2 = \lambda^{-1} X_1$.

The Type-I PHCS under CSPALT can be applied as follows. Assume that n_j ($j = 1, 2$) identical items are set on a life test with progressive censoring scheme $(R_{j1}, R_{j2}, \dots, R_{jm})$, $1 \leq m \leq n$, such that n_1 of items are assigned to run under use condition and the remaining items n_2 are assigned to run under accelerated condition. It is assumed that R_{ji} 's ($j = 1, 2$; $i = 1, 2, \dots, m$) are pre-fixed integer numbers. At the time of the first failure X_{j1} , R_{j1} units are randomly withdrawn from the remaining $n_j - 1$ surviving units. Similarly, at the time of the second failure X_{j2} , R_{j2} units are randomly withdrawn from the remaining $n_j - 2 - R_{j1}$ surviving units and so on. If the m th failure occurs at time X_{jm} before the time point T , the experiment finishes at the time point X_{jm} . But if the m th failure does not occur before the time point T and only k_j ($j = 1, 2$) failures occur before the time point T , where $0 \leq k_j < m$; then all the remaining $R_{k_j}^* = n_j - K_j - (R_{j1} + R_{j2} + \dots + R_{jk_j})$ surviving units are withdrawn and the experiment terminates at the time T . It is observed that compared with the standard Type-I censoring scheme, the completion time of the Type-I PHCS is at most T .

Thus, under Type-I PHCS, we can deal with one of the following main types of observations.

Set 1: $(x_{11}, x_{12}, \dots, x_{1m}; x_{21}, x_{22}, \dots, x_{2m})$, if $x_{jm} < T$, $j = 1, 2$;

Set 2: $(x_{11}, x_{12}, \dots, x_{1k_1}; x_{21}, x_{22}, \dots, x_{2m})$, if $x_{1k_1} < T < x_{1k_1+1}$, $x_{2m} < T$;

Set 3: $(x_{11}, x_{12}, \dots, x_{1k_1}; x_{21}, x_{22}, \dots, x_{2k_2})$, if $x_{1k_1} < T < x_{1k_1+1}$, $x_{2k_2} < T < x_{2k_2+1}$.

3. MAXIMUM LIKELIHOOD ESTIMATION

This section considers the process of finding the ML estimates of the parameters β , η and λ based on Type-I PHCS data under CSPALT model. In addition, interval estimations of the parameters are explored by using three different methods of estimation.

3.1. Point Estimation

Here, the ML point estimates of the unknown parameters are given. Based on the observed Type-I PHCS data from WD, the likelihood functions under CSPALT for the three sets of data indicated above are expressed as follows.

The likelihood function for the data set 1 is given by

$$L(\beta, \eta, \lambda) \propto \prod_{i=1}^m f_1(x_{1i}) [S_1(x_{1i})]^{R_{1i}} f_2(x_{2i}) [S_2(x_{2i})]^{R_{2i}} \quad (3.1)$$

where

$$S_2(x_2) = \exp\{-(\lambda x_2 / \eta)^\beta\},$$

For the data set 2, the likelihood function is set by

$$L(\beta, \eta, \lambda) \propto \prod_{i=1}^{k_1} f_1(x_{1i}) [S_1(x_{1i})]^{R_{1i}} [S_1(T)]^{R_{k_1}^*} \cdot \prod_{l=1}^m f_2(x_{2l}) [S_2(x_{2l})]^{R_{2l}} \quad (3.2)$$

The likelihood function for the data set 3 is expressed by

$$L(\beta, \eta, \lambda) \propto \prod_{i=1}^{k_1} f_1(x_{1i}) [S_1(x_{1i})]^{R_{1i}} [S_1(T)]^{R_{k_1}^*} \\ \times \prod_{l=1}^{k_2} f_2(x_{2l}) [S_2(x_{2l})]^{R_{2l}} [S_2(T)]^{R_{k_2}^*} \quad (3.3)$$

This article will consider only the case of data set 1. To get the ML estimates of the model parameters, the natural logarithm of the likelihood function for the data set 1 is obtained by

$$\ln L(\beta, \eta, \lambda) = m \ln \beta - m \beta \ln \eta + m \beta \ln \lambda \\ + (\beta - 1) \left[\sum_{i=1}^m \ln x_{1i} + \sum_{i=1}^m \ln x_{2i} \right] \\ - \frac{1}{\eta^\beta} \left[\sum_{i=1}^m x_{1i}^\beta + \sum_{i=1}^m R_{1i} x_{1i}^\beta \right] \\ - \left(\frac{\lambda}{\eta} \right)^\beta \left[\sum_{i=1}^m x_{2i}^\beta + \sum_{i=1}^m R_{2i} x_{2i}^\beta \right], \quad (3.4)$$

Deriving the first partial derivatives of Equation (3.4) with respect to β, η and λ and equating each to zero, we get the likelihood equations as

$$\frac{\partial \ln L}{\partial \beta} = \frac{m}{\beta} + m \ln \lambda - m \ln \eta + \left[\sum_{i=1}^m \ln x_{1i} + \sum_{i=1}^m \ln x_{2i} \right] \\ - \frac{1}{\eta^\beta} \left\{ \left[\sum_{i=1}^m x_{1i}^\beta \ln x_{1i} + \sum_{i=1}^m R_{1i} x_{1i}^\beta \ln x_{1i} \right] - \left[\sum_{i=1}^m x_{1i}^\beta + \sum_{i=1}^m R_{1i} x_{1i}^\beta \right] \ln \eta \right\} \\ - \left(\frac{\lambda}{\eta} \right)^\beta \left\{ \ln \left(\frac{\lambda}{\eta} \right) \left[\sum_{i=1}^m x_{2i}^\beta + \sum_{i=1}^m R_{2i} x_{2i}^\beta \right] + \left[\sum_{i=1}^m x_{2i}^\beta \ln x_{2i} + \sum_{i=1}^m R_{2i} x_{2i}^\beta \ln x_{2i} \right] \right\} = 0, \quad (3.5)$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \eta} = & -\frac{m\beta}{\eta} + \frac{\beta}{\eta^{\beta+1}} \left[\sum_{i=1}^m x_{1i}^{\beta} + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} \right] \\ & + \frac{\beta\lambda^{\beta}}{\eta^{\beta+1}} \left[\sum_{i=1}^m x_{2i}^{\beta} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \right] = 0, \end{aligned} \quad (3.6)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{m\beta}{\lambda} - \frac{\beta\lambda^{\beta-1}}{\eta^{\beta}} \left[\sum_{i=1}^m x_{2i}^{\beta} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \right] = 0. \quad (3.7)$$

From (3.7) we can obtain $\hat{\lambda}$ as a function of η and β as

$$\hat{\lambda} = \frac{m^{1/\hat{\beta}} \eta}{\left[\sum_{i=1}^m x_{2i}^{\hat{\beta}} + \sum_{i=1}^m R_{2i} x_{2i}^{\hat{\beta}} \right]^{1/\hat{\beta}}}. \quad (3.8)$$

Now, the system is reduced to two non-linear likelihood equations of β and η . It can be solved iteratively using an iterative method such as Newton-Raphson to obtain the ML estimate of β and η . Therefore, the ML estimate of λ can be easily obtained from (3.8).

3.2. Approximate Confidence Intervals

In this subsection, the approximate confidence intervals (CIs) of the parameters are derived based on the asymptotic distribution of the ML estimators of the elements of the vector of unknown parameters $\Omega = (\beta, \eta, \lambda)$. It is known that the asymptotic distribution of the ML estimators of Ω is given by, see Miller [28],

$$((\hat{\beta} - \beta), (\hat{\eta} - \eta), (\hat{\lambda} - \lambda)) \rightarrow N(0, \mathbf{I}^{-1}(\beta, \eta, \lambda))$$

where $\mathbf{I}^{-1}(\beta, \eta, \lambda)$ is the variance-covariance matrix of the unknown parameters $\Omega = (\beta, \eta, \lambda)$. The elements of the 3×3 matrix $\mathbf{I}^{-1}$, $I_{ij}(\beta, \eta, \lambda)$, $i, j = 1, 2, 3$; can be approximated by $I_{ij}(\hat{\beta}, \hat{\eta}, \hat{\lambda})$, where

$$I_{ij}(\hat{\Omega}) = -\frac{\partial^2 \ln L(\Omega)}{\partial \Omega_i \partial \Omega_j} \Big|_{\Omega=\hat{\Omega}}$$

Now, we get the following

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta^2} = & -\frac{m}{\beta^2} \\ & -\frac{1}{\eta^{\beta}} \left\{ \sum_{i=1}^m x_{1i}^{\beta} (\ln x_{1i})^2 + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} (\ln x_{1i})^2 - \left(\sum_{i=1}^m x_{1i}^{\beta} \ln x_{1i} + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} \ln x_{1i} \right) \ln \eta \right\} \\ & + \frac{1}{\eta^{\beta}} \left\{ \left[\sum_{i=1}^m x_{1i}^{\beta} \ln x_{1i} + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} \ln x_{1i} - \left(\sum_{i=1}^m x_{1i}^{\beta} + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} \right) \ln \eta \right] \ln \eta \right\} \\ & - \left(\frac{\lambda}{\eta} \right)^{\beta} \left\{ \left[\left(\ln \left(\frac{\lambda}{\eta} \right) \right)^2 \left(\sum_{i=1}^m x_{2i}^{\beta} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \right) \right] + 2 \ln \left(\frac{\lambda}{\eta} \right) \left[\sum_{i=1}^m x_{2i}^{\beta} \ln x_{2i} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \ln x_{2i} \right] \right\} \\ & - \left(\frac{\lambda}{\eta} \right)^{\beta} \left\{ \left[\sum_{i=1}^m x_{2i}^{\beta} (\ln x_{2i})^2 + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} (\ln x_{2i})^2 \right] \right\}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta \partial \eta} = & -\frac{m}{\eta} - \eta^{\beta-1} \left(\sum_{i=1}^m x_{1i}^{\beta} + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} \right) \\ & - \eta^{\beta-1} \left\{ \beta \left[\sum_{i=1}^m x_{1i}^{\beta} \ln x_{1i} + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} \ln x_{1i} - \left(\sum_{i=1}^m x_{1i}^{\beta} + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} \right) \ln \eta \right] \right\} \\ & - \frac{\lambda^{\beta}}{\eta^{\beta+1}} \left\{ \left[\beta \ln \left(\frac{\lambda}{\eta} \right) + 1 \right] \left[\sum_{i=1}^m x_{2i}^{\beta} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \right] \right\} \\ & + \frac{\beta \lambda^{\beta}}{\eta^{\beta+1}} \left[\sum_{i=1}^m x_{2i}^{\beta} \ln x_{2i} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \ln x_{2i} \right], \end{aligned} \quad (3.10)$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = \frac{m}{\lambda} - \frac{\lambda^{\beta-1}}{\eta^{\beta}} \left\{ (1+\beta \ln \lambda) - \beta \ln \eta \left[\sum_{i=1}^m x_{2i}^{\beta} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \right] + \beta \left[\sum_{i=1}^m x_{2i}^{\beta} \ln x_{2i} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \ln x_{2i} \right] \right\}, \quad (3.11)$$

$$\frac{\partial^2 \ln L}{\partial \eta^2} = \frac{m\beta}{\eta^2} - \frac{\beta(\beta+1)}{\eta^{\beta+2}} \left\{ \left[\sum_{i=1}^m x_{1i}^{\beta} + \sum_{i=1}^m R_{1i} x_{1i}^{\beta} \right] + \lambda^{\beta} \left[\sum_{i=1}^m x_{2i}^{\beta} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \right] \right\}, \quad (3.12)$$

$$\frac{\partial^2 \ln L}{\partial \eta \partial \lambda} = \frac{\beta^2 \lambda^{\beta-1}}{\eta^{\beta+1}} \left[\sum_{i=1}^m x_{2i}^{\beta} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \right], \quad (3.13)$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\frac{m\beta}{\lambda^2} - \frac{\beta(\beta-1)\lambda^{\beta-2}}{\eta^{\beta}} \left[\sum_{i=1}^m x_{2i}^{\beta} + \sum_{i=1}^m R_{2i} x_{2i}^{\beta} \right]. \quad (3.14)$$

Hence, the approximate 100(1 - γ)% two sided confidence intervals for β , η and λ are, respectively, given by

$$\hat{\beta} \pm Z_{\gamma/2} \sqrt{I_{11}^{-1}(\hat{\beta}, \hat{\eta}, \hat{\lambda})}, \quad \hat{\eta} \pm Z_{\gamma/2} \sqrt{I_{22}^{-1}(\hat{\beta}, \hat{\eta}, \hat{\lambda})} \text{ and } \hat{\lambda} \pm Z_{\gamma/2} \sqrt{I_{33}^{-1}(\hat{\beta}, \hat{\eta}, \hat{\lambda})}. \quad (3.15)$$

where $Z_{\gamma/2}$ is the upper ($\gamma/2$)th percentile of a standard normal distribution.

When the sample size is small, the normal approximation may be poor. However, another transformation of the MLE can be applied to correct the insufficient performance of the normal approximation. Meeker and Escobar [23] suggested the use of the normal approximation for the log-transformed MLE.

Let $\hat{\omega}_i, i = 1, 2, 3$, with $(\hat{\omega}_1 = \hat{\beta}, \hat{\omega}_2 = \hat{\eta}, \hat{\omega}_3 = \hat{\lambda})$. A two-sided 100(1- γ) % normal approximation CIs for $\hat{\omega}_i, i = 1, 2, 3$, are given by

$$(\hat{\omega}_i \cdot \exp(-\frac{Z_{\gamma/2} \sqrt{var(\hat{\omega}_i)}}{\hat{\omega}_i}), \hat{\omega}_i \cdot \exp(\frac{Z_{\gamma/2} \sqrt{var(\hat{\omega}_i)}}{\hat{\omega}_i}))$$

3.3. Bootstrap Confidence Intervals

In this section, we use the parametric bootstrap method to construct CIs for the unknown parameters β , η and λ . The bootstrap is a resampling method for statistical inference. It is commonly used to estimate CIs. It can also be used to estimate bias and variance of an estimator. It has the advantage of computational ease especially for large sample sizes. We present two parametric bootstrap methods, percentile bootstrap CI (PBCI) proposed by Efron [24] and bootstrap-t CI (BTCL) proposed by Hall [25]. The following steps can be proceeded to obtain bootstrap samples for both methods.

(1) Using the original Type-I progressively hybrid censored sample, $(x_{j1} < x_{j2} < \dots < x_{jm}), x_{jm} < T, j = 1, 2$, obtain $\hat{\beta}, \hat{\eta}$ and $\hat{\lambda}$.

(2) Using the values of n_j and m ($1 \leq m \leq n_j$) with the same censoring scheme (CS) in step 1 ($R_{j1}, R_{j2}, \dots, R_{jm}$), $j = 1, 2$, generate two independent samples of sizes n_1 and n_2 from Weibull distribution, $\underline{x}^* = (x_{j1}^* < x_{j2}^* < \dots < x_{jm}^*)$, $j = 1, 2$.

(3) As in step 1 based on $\underline{x}^*$ compute the bootstrap sample estimates of $\hat{\beta}, \hat{\eta}$ and $\hat{\lambda}$ say, $\hat{\beta}^*, \hat{\eta}^*$ and $\hat{\lambda}^*$.

(4) Repeat the above steps 2 and 3 N ($=10,000$) times representing N different bootstrap samples.

(5) Arrange all $\hat{\beta}^*, \hat{\eta}^*$ and $\hat{\lambda}^*$ in an ascending order to obtain the bootstrap sample $\hat{\phi}_{\ell}^{*[1]}, \hat{\phi}_{\ell}^{*[2]}, \dots, \hat{\phi}_{\ell}^{*[N]}$, $\ell=1,2,3$, where $\phi_1^* = \beta^*, \phi_2^* = \eta^*$ and $\phi_3^* = \lambda^*$.

3.3.1. Percentile Bootstrap Confidence Intervals

Let $G(z) = P(\hat{\phi}_{\ell}^* \leq z)$ be CDF of $\hat{\phi}_{\ell}^*$. Define $\hat{\phi}_{\ell \text{boot}}^* = G^{-1}(z)$ for given z . The approximate bootstrap $100(1-\gamma)\%$ CI of $\hat{\phi}_{\ell}^*$ is given by

$$(\hat{\phi}_{\ell \text{boot}}^* (\frac{\gamma}{2}), \hat{\phi}_{\ell \text{boot}}^* (1 - \frac{\gamma}{2}))$$

3.3.2. Bootstrap-t Confidence Intervals

First, find the order statistics $\psi_{\ell}^{*[1]} < \psi_{\ell}^{*[2]} < \dots < \psi_{\ell}^{*[N]}$, where

$$\psi_{\ell}^{*[j]} = \frac{\hat{\phi}_{\ell}^{*[j]} - \hat{\phi}_{\ell}}{\sqrt{\text{Var} \hat{\phi}_{\ell}^{*[j]}}, j = 1, 2, \dots, N, \ell=1, 2, 3,$$

where $\hat{\phi}_1 = \hat{\beta}$, $\hat{\phi}_2 = \hat{\eta}$ and $\hat{\phi}_3 = \hat{\lambda}$.

Let $H(z) = P(\psi_{\ell}^* < z)$ be the CDF of ψ_{ℓ}^* . For a given z , define

$$\hat{\phi}_{\ell \text{boot-t}} = \hat{\phi}_{\ell} + \sqrt{\text{Var}(\hat{\phi}_{\ell})} H^{-1}(z).$$

The approximate $100(1-\gamma)\%$ CI of $\hat{\phi}_{\ell}$ is given by

$$(\hat{\phi}_{\ell \text{boot-t}} (\frac{\gamma}{2}), \hat{\phi}_{\ell \text{boot-t}} (1 - \frac{\gamma}{2})).$$

4. SIMULATION STUDIES

Simulation studies are conducted to examine and compare the performance of the proposed methods; ML, percentile bootstrap and bootstrap-t for different sample sizes, different censoring schemes (CSs) and different parameter values.

Simulation studies are presented in this section to investigate the performance of the estimators of the three proposed methods in terms of their mean squared errors (MSEs) for different choices of n_j ($j = 1, 2$), m and T values. Also, 95 % CIs of the model parameters based on the three proposed methods are constructed and their average lengths as well as their coverage probabilities (CP) are computed.

Here, three different progressive CSs are considered.

CS 1: $R_1 = \cdot \cdot \cdot = R_{m-1} = 0$ and $R_m = n - m$;

CS 2: $R_1 = n - m$ and $R_2 = \cdot \cdot \cdot = R_m = 0$; and

CS 3: $R_1 = \cdot \cdot \cdot = R_{m-1} = 1$ and $R_m = n - 2m + 1$.

For each setting, based on 10,000 replications, the average values of MSEs for the three proposed methods are computed and reported in Tables 1-4. Moreover, average confidence intervals lengths and their CP are also computed and presented in Tables 5-8.

The simulation studies are applied according to the following algorithm.

- 1) Specify the values of n_j ($j = 1, 2$), m and T .
- 2) Specify the true values of the distribution parameters β , η and λ .
- 3) Generate two random samples of sizes n_1 and n_2 from Weibull distribution and sort it. The Weibull random variable can be easily generated. For example, if U represents a uniform random variable from $[0, 1]$, then $x_1 = -\eta [\ln(1-U)]^{1/\beta}$ has Weibull distribution with pdf given by (2.1), and $x_2 = -\frac{\eta}{\lambda} [\ln(1-U)]^{1/\beta}$ has Weibull distribution with pdf given by (2.4).
- 4) Generate progressively Type-I hybrid censored data for given n_j ($j = 1, 2$), m and T with Weibull distribution parameters β , η and λ .
For the case when $x_{jm} < T$, $j = 1, 2$, the sample is

$$\begin{cases} x_{11}, x_{12}, \dots, x_{1m} < T \\ x_{21}, x_{22}, \dots, x_{2m} < T \end{cases};$$
- 5) Replicate the steps 3-4 10,000 times.
- 6) Compute the average values of MSEs associated with each estimation method.
- 7) Compute the average values of intervals lengths (ILs) as well as their CP for each parameter associated with each method at confidence level $1 - \gamma = 0.95$.
- 8) Steps 1-7 are done with different values of n_j ($j = 1, 2$), m , T , β , η and λ .

Applying the above algorithm, the average values of MSEs, ILs and CP are obtained using 10,000 replications to avoid randomness.

From the results, we can observe the following.

The bootstrap-t method provides more accurate results than both methods of the percentile bootstrap and ML. It gives smaller values of MSEs compared to those based on the other two methods. The bootstrap-t CIs are better than those of both the percentile bootstrap CIs and the approximate CIs. The bootstrap-t method yields smaller widths and closer coverage probabilities to the nominal level than those of the other two methods. In general, the performance of the approximate CIs is the lowest compared with those of the two methods of bootstrap. The performance is improved as the sample size increases.

Concerning the comparison of the performance among the three different censoring Schemes, it is observed that the performance of CS 1 is the best while it is the lowest in the case of CS 2 because of the heavy amount of censoring at the early stages of the experiment.

In addition, the following comments can be made.

- 1) For fixed n and T , the MSEs decrease as m increases.
- 2) For fixed T , the MSEs considerably decrease as n and m increase at the same time.
- 3) For fixed n and m , the MSEs decrease as T increases.

Table 1. Average values of the MSEs of the MLE, percentile bootstrap (Boot-P) and bootstrap-t (Boot-t) estimators when β , η , λ and T set at 0.6, 0.8, 1.4 and 6, respectively

(n, m)	CS	MLE			Boot-P			Boot-t		
		$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$
(20, 10)	1	0.356	0.384	0.506	0.281	0.299	0.394	0.234	0.249	0.332
	2	0.466	0.493	0.636	0.367	0.388	0.501	0.306	0.323	0.417
	3	0.393	0.418	0.609	0.309	0.329	0.479	0.258	0.274	0.399
(40, 10)	1	0.252	0.271	0.434	0.199	0.213	0.342	0.166	0.177	0.285
	2	0.338	0.365	0.471	0.266	0.283	0.371	0.222	0.236	0.309
	3	0.287	0.304	0.437	0.226	0.239	0.344	0.188	0.232	0.287
(60, 10)	1	0.164	0.176	0.339	0.126	0.134	0.267	0.105	0.111	0.222
	2	0.216	0.229	0.399	0.173	0.181	0.314	0.142	0.151	0.262
	3	0.202	0.214	0.353	0.159	0.169	0.278	0.133	0.141	0.232
(20, 15)	1	0.138	0.143	0.197	0.108	0.113	0.155	0.091	0.094	0.129
	2	0.163	0.175	0.534	0.129	0.138	0.42	0.107	0.115	0.353
	3	0.162	0.173	0.273	0.128	0.136	0.215	0.106	0.113	0.179
(40, 15)	1	0.094	0.094	0.128	0.074	0.074	0.101	0.062	0.062	0.084
	2	0.133	0.143	0.491	0.103	0.113	0.387	0.086	0.092	0.322
	3	0.099	0.097	0.138	0.078	0.076	0.109	0.065	0.064	0.092
(60, 15)	1	0.039	0.042	0.062	0.031	0.033	0.049	0.026	0.027	0.041
	2	0.083	0.089	0.106	0.065	0.075	0.084	0.054	0.058	0.074
	3	0.072	0.077	0.098	0.054	0.062	0.077	0.047	0.053	0.064

Table 2. Average values of the MSEs of the MLE, percentile bootstrap (Boot-P) and bootstrap-t (Boot-t) estimators when β , η , λ and T set at 0.6, 0.8, 1.4 and 11, respectively

(n, m)	CS	MLE			Boot-P			Boot-t		
		$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$
(20, 10)	1	0.296	0.333	0.448	0.267	0.301	0.402	0.215	0.239	0.314
	2	0.371	0.407	0.558	0.335	0.367	0.514	0.272	0.292	0.391
	3	0.358	0.392	0.536	0.322	0.354	0.481	0.261	0.281	0.376
(40, 10)	1	0.255	0.278	0.383	0.234	0.251	0.344	0.185	0.202	0.269
	2	0.275	0.302	0.414	0.248	0.273	0.371	0.201	0.217	0.292
	3	0.256	0.285	0.385	0.231	0.253	0.345	0.186	0.201	0.273
(60, 10)	1	0.207	0.226	0.311	0.187	0.204	0.279	0.151	0.162	0.218
	2	0.233	0.257	0.351	0.213	0.232	0.315	0.173	0.184	0.246
	3	0.212	0.231	0.324	0.191	0.209	0.287	0.154	0.166	0.224
(20, 15)	1	0.115	0.126	0.173	0.104	0.114	0.155	0.084	0.094	0.121
	2	0.163	0.179	0.246	0.147	0.162	0.224	0.118	0.129	0.172
	3	0.164	0.175	0.239	0.144	0.158	0.215	0.116	0.126	0.168
(40, 15)	1	0.075	0.083	0.113	0.068	0.075	0.102	0.055	0.059	0.079
	2	0.094	0.104	0.144	0.084	0.091	0.129	0.068	0.072	0.101
	3	0.081	0.088	0.121	0.073	0.082	0.109	0.059	0.063	0.085
(60, 15)	1	0.037	0.041	0.059	0.033	0.037	0.053	0.027	0.032	0.041
	2	0.063	0.068	0.093	0.057	0.062	0.083	0.046	0.049	0.065
	3	0.058	0.063	0.089	0.056	0.059	0.078	0.042	0.045	0.061

Table 3. Average values of the MSEs of the MLE, percentile bootstrap (Boot-P) and bootstrap-t (Boot-t) estimators when β , η , λ and T set at 1.6, 0.8, 1.4 and 11, respectively

(n, m)	CS	MLE			Boot-P			Boot-t		
		$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$
(20, 10)	1	0.376	0.422	0.547	0.309	0.346	0.458	0.215	0.233	0.304
	2	0.472	0.509	0.682	0.388	0.421	0.572	0.272	0.284	0.378
	3	0.454	0.493	0.654	0.374	0.409	0.547	0.261	0.275	0.364
(40, 10)	1	0.323	0.347	0.469	0.267	0.288	0.396	0.185	0.194	0.260
	2	0.349	0.378	0.506	0.287	0.314	0.423	0.199	0.211	0.281
	3	0.325	0.349	0.472	0.267	0.290	0.399	0.186	0.196	0.262
(60, 10)	1	0.262	0.283	0.379	0.216	0.234	0.318	0.151	0.158	0.211
	2	0.296	0.322	0.429	0.244	0.272	0.358	0.169	0.185	0.239
	3	0.269	0.294	0.391	0.221	0.239	0.326	0.154	0.162	0.217
(20, 15)	1	0.146	0.157	0.211	0.125	0.131	0.176	0.088	0.088	0.118
	2	0.206	0.226	0.301	0.169	0.186	0.251	0.119	0.126	0.167
	3	0.203	0.219	0.293	0.167	0.189	0.245	0.116	0.122	0.162
(40, 15)	1	0.099	0.104	0.139	0.078	0.085	0.116	0.055	0.058	0.077
	2	0.118	0.125	0.179	0.101	0.104	0.147	0.068	0.071	0.097
	3	0.104	0.110	0.147	0.085	0.091	0.123	0.059	0.061	0.082
(60, 15)	1	0.046	0.051	0.072	0.038	0.043	0.061	0.026	0.029	0.041
	2	0.085	0.085	0.114	0.066	0.071	0.095	0.046	0.052	0.064
	3	0.074	0.078	0.106	0.063	0.067	0.088	0.042	0.044	0.059

Table 4. Average values of the MSEs of the MLE, percentile bootstrap (Boot-P) and bootstrap-t (Boot-t) estimators when β , η , λ and T set at 1.6, 0.8, 1.4 and 6, respectively

(n, m)	CS	MLE			Boot-P			Boot-t		
		$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$
(20, 10)	1	0.711	0.798	0.956	0.503	0.565	0.744	0.297	0.323	0.421
	2	0.892	0.963	1.206	0.631	0.684	0.931	0.375	0.392	0.522
	3	0.859	0.934	1.152	0.608	0.665	0.889	0.362	0.385	0.503
(40, 10)	1	0.612	0.656	0.821	0.434	0.468	0.644	0.255	0.268	0.359
	2	0.663	0.714	0.895	0.466	0.514	0.693	0.275	0.291	0.388
	3	0.614	0.664	0.826	0.434	0.471	0.649	0.256	0.272	0.361
(60, 10)	1	0.495	0.535	0.641	0.351	0.382	0.516	0.208	0.221	0.291
	2	0.559	0.609	0.755	0.396	0.443	0.581	0.233	0.255	0.332
	3	0.509	0.556	0.667	0.359	0.389	0.532	0.213	0.223	0.301
(20, 15)	1	0.275	0.301	0.373	0.204	0.216	0.286	0.122	0.122	0.163
	2	0.389	0.427	1.011	0.278	0.303	0.408	0.164	0.174	0.231
	3	0.383	0.414	0.516	0.271	0.308	0.399	0.163	0.168	0.223
(40, 15)	1	0.188	0.197	0.242	0.126	0.139	0.189	0.076	0.083	0.106
	2	0.223	0.236	0.927	0.164	0.169	0.239	0.094	0.098	0.133
	3	0.197	0.208	0.261	0.139	0.148	0.203	0.082	0.084	0.113
(60, 15)	1	0.086	0.096	0.117	0.061	0.074	0.099	0.036	0.042	0.056
	2	0.161	0.161	0.210	0.108	0.115	0.155	0.063	0.071	0.089
	3	0.142	0.147	0.185	0.103	0.109	0.143	0.059	0.061	0.082

Table 5. Average values of the confidence intervals lengths and their coverage probabilities between brackets based on the MLE, percentile bootstrap (Boot-P) and bootstrap-t (Boot-t) methods when β , η , λ and T set at 0.6, 0.8, 1.4 and 6, respectively

(n, m)	CS	MLE			Boot-P			Boot-t		
		β	η	λ	β	η	λ	β	η	λ
(20, 10)	1	2.972 (92.7)	3.189 (92.2)	3.688 (92.1)	2.675 (94.1)	3.106 (93.8)	3.353 (93.9)	1.989 (94.7)	3.024 (94.6)	3.134 (94.8)
	2	5.538 (89.6)	5.942 (89.2)	6.863 (89)	4.984 (93.2)	5.788 (93)	6.239 (92.9)	3.706 (95.9)	5.634 (96)	5.838 (95.9)
	3	4.636 (91.8)	4.974 (91.3)	5.745 (91.2)	4.172 (93.9)	4.845 (94)	5.223 (93.8)	3.102 (95.6)	4.716 (95.8)	4.886 (95.5)
(40, 10)	1	2.497 (93.2)	2.679 (92.7)	3.095 (92.6)	2.247 (94.8)	2.609 (94.4)	2.813 (94.2)	1.671 (95.5)	2.541 (95.2)	2.632 (95)
	2	4.861 (91.4)	5.216 (90.9)	6.024 (90.8)	4.375 (94)	5.081 (93.6)	5.477 (93.6)	3.253 (95.8)	4.946 (95.5)	5.125 (95.5)
	3	3.488 (92.3)	3.743 (91.8)	4.323 (91.7)	3.141 (94.4)	3.646 (94)	3.932 (93.9)	2.334 (95.6)	3.549 (95.3)	3.677 (95.3)
(60, 10)	1	1.855 (92.4)	1.991 (91.9)	2.301 (91.8)	1.670 (94.3)	1.939 (93.9)	2.091 (93.8)	1.242 (95.3)	1.888 (95)	1.956 (94.9)
	2	4.771 (91)	5.122 (90.5)	5.913 (90.4)	4.294 (93.8)	4.987 (93.4)	5.376 (93.5)	3.192 (95.7)	4.854 (95.4)	5.029 (95.7)
	3	2.854 (92.2)	3.062 (91.7)	3.536 (91.6)	2.568 (94.2)	2.983 (93.9)	3.215 (93.6)	1.913 (95.4)	2.903 (95.3)	3.009 (94.7)
(20, 15)	1	1.469 (93)	1.578 (92.5)	1.827 (92.4)	1.322 (94.6)	1.536 (94.2)	1.655 (94)	0.983 (95.3)	1.494 (95)	1.549 (94.8)
	2	3.919 (90)	4.205 (89.6)	4.857 (89.4)	3.527 (93.2)	4.096 (92.8)	4.416 (92.4)	2.622 (95.5)	3.988 (95.2)	4.132 (94.6)
	3	1.537 (92.1)	1.649 (91.6)	1.905 (91.5)	1.383 (94.2)	1.606 (93.8)	1.732 (93.6)	1.029 (95.4)	1.563 (95.1)	1.624 (94.9)
(40, 15)	1	1.292 (93.3)	1.386 (92.8)	1.626 (92.7)	1.163 (94.7)	1.351 (94.3)	1.478 (94.2)	0.865 (95.3)	1.315 (95)	1.363 (94.8)
	2	3.358 (92)	3.603 (91.5)	4.161 (91.4)	3.022 (94.2)	3.509 (93.8)	3.782 (93.3)	2.247 (95.5)	3.416 (95.2)	3.543 (94.4)
	3	1.376 (92.4)	1.477 (91.9)	1.706 (91.8)	1.239 (94)	1.439 (93.6)	1.551 (93.7)	0.921 (94.8)	1.401 (94.5)	1.451 (94.7)
(60, 15)	1	1.111 (93.4)	1.193 (92.9)	1.377 (92.8)	1.013 (94.7)	1.161 (94.3)	1.252 (94.2)	0.744 (95.2)	1.132 (94.9)	1.172 (94.7)
	2	2.982 (91.8)	3.199 (91.3)	3.696 (91.2)	2.684 (94)	3.116 (93.9)	3.364 (93.2)	1.995 (95.4)	3.034 (95.7)	3.144 (94.3)
	3	1.182 (92.5)	1.268 (92)	1.465 (91.9)	1.064 (94.2)	1.235 (93.8)	1.332 (93.6)	0.791 (95)	1.202 (94.7)	1.246 (94.5)

Table 6. Average values of the confidence intervals lengths and their coverage probabilities between brackets based on the MLE, percentile bootstrap (Boot-P) and bootstrap-t (Boot-t) methods when β , η , λ and T set at 0.6, 0.8, 1.4 and 11, respectively

(n, m)	CS	MLE			Boot-P			Boot-t		
		β	η	λ	β	η	λ	β	η	λ
(20, 10)	1	2.565 (91.2)	4.151 (90.4)	4.364 (90)	2.368 (93.3)	3.832 (92.8)	4.029 (92.7)	1.973 (95.3)	3.193 (95.1)	3.357 (95.4)
	2	4.781 (88.1)	7.726 (87.5)	8.131 (87)	4.413 (92.3)	7.132 (92)	7.506 (91.8)	3.677 (96.6)	5.943 (96.6)	6.255 (96.6)
	3	4.007 (90.3)	6.468 (89.5)	6.805 (89.1)	3.699 (93.3)	5.974 (92.9)	6.282 (92.6)	3.082 (96.3)	4.975 (96.4)	5.235 (96.2)
(40, 10)	1	2.155 (91.6)	3.485 (90.9)	3.667 (90.5)	1.989 (93.9)	3.217 (93.3)	3.384 (93.1)	1.658 (96.2)	2.681 (95.7)	2.822 (95.6)
	2	4.197 (89.9)	6.782 (89.1)	7.139 (88.8)	3.874 (93.2)	6.261 (92.6)	6.589 (92.5)	3.229 (96.5)	5.217 (96.1)	5.491 (96.2)
	3	3.012 (90.8)	4.868 (90)	5.122 (89.6)	2.784 (93.5)	4.493 (92.9)	4.728 (92.8)	2.317 (96.3)	3.744 (95.8)	3.944 (96)
(60, 10)	1	1.603 (90.9)	2.591 (90.1)	2.724 (89.7)	1.482 (93.4)	2.391 (92.8)	2.515 (92.6)	1.233 (96)	1.992 (95.5)	2.096 (95.5)
	2	4.126 (89.5)	6.657 (88.7)	7.006 (88.4)	3.808 (92.9)	6.145 (92.3)	6.467 (92.4)	3.174 (96.4)	5.121 (96)	5.389 (96.4)
	3	2.464 (90.7)	3.981 (89.9)	4.191 (89.5)	2.274 (93.4)	3.675 (92.9)	3.869 (92.4)	1.895 (96.1)	3.062 (95.8)	3.224 (95.3)
(20, 15)	1	1.273 (91.5)	2.049 (90.7)	2.156 (90.3)	1.172 (93.7)	1.892 (93.1)	1.991 (92.9)	0.977 (96)	1.576 (95.5)	1.659 (95.4)
	2	3.383 (88.5)	5.468 (87.9)	5.756 (87.4)	3.123 (92.3)	5.048 (91.8)	5.313 (91.3)	2.603 (96.2)	4.206 (95.7)	4.428 (95.2)
	3	1.327 (90.6)	2.192 (89.8)	2.258 (89.4)	1.225 (93.3)	2.024 (92.7)	2.084 (92.5)	1.021 (96.1)	1.686 (95.6)	1.737 (95.5)
(40, 15)	1	1.165 (91.7)	1.803 (91)	1.899 (90.6)	1.076 (93.8)	1.665 (93.3)	1.753 (93)	0.897 (96)	1.387 (95.6)	1.461 (95.4)
	2	2.899 (90.5)	4.685 (89.7)	4.931 (89.3)	2.676 (93.3)	4.324 (92.7)	4.551 (92.2)	2.231 (96.2)	3.604 (95.7)	3.793 (95)
	3	1.188 (90.9)	1.923 (90.1)	2.021 (89.7)	1.097 (93.2)	1.773 (92.6)	1.865 (92.5)	0.914 (95.4)	1.477 (95)	1.554 (95.3)
(60, 15)	1	0.965 (91.8)	1.552 (91.1)	1.632 (90.7)	0.886 (93.8)	1.431 (93.3)	1.506 (93)	0.738 (95.8)	1.192 (95.4)	1.255 (95.3)
	2	2.577 (90.3)	4.167 (89.5)	4.379 (89.1)	2.379 (93.2)	3.846 (92.9)	4.042 (92)	1.982 (96.1)	3.205 (96.3)	3.368 (94.9)
	3	1.025 (91)	1.649 (90.2)	1.736 (89.8)	0.946 (93.5)	1.522 (92.7)	1.602 (92.5)	0.789 (96)	1.268 (95.2)	1.335 (95.1)

Table 7. Average values of the confidence intervals lengths and their coverage probabilities between brackets based on the MLE, percentile bootstrap (Boot-P) and bootstrap-t (Boot-t) methods when β , η , λ and T set at 1.6, 0.8, 1.4 and 11, respectively

(n, m)	CS	MLE			Boot-P			Boot-t		
		β	η	λ	β	η	λ	β	η	λ
(20, 10)	1	2.162 (93.3)	3.527 (92.7)	3.769 (91.5)	1.996 (94.6)	3.253 (94.2)	3.479 (93.6)	1.663 (95.8)	2.708 (95.6)	2.899 (95.6)
	2	4.031 (90.2)	6.547 (89.7)	7.027 (88.4)	3.721 (93.6)	6.043 (93.4)	6.486 (92.6)	3.101 (97)	5.036 (97.1)	5.405 (96.8)
	3	3.377 (92.4)	5.483 (91.8)	5.877 (90.6)	3.118 (94.3)	5.058 (94.1)	5.425 (93.5)	2.598 (96.2)	4.215 (96.4)	4.521 (96.4)
(40, 10)	1	1.817 (93.8)	2.954 (93.2)	3.166 (92)	1.678 (95.1)	2.726 (94.7)	2.922 (93.9)	1.398 (96.4)	2.272 (96.3)	2.435 (95.8)
	2	3.537 (92)	5.747 (91.4)	6.171 (90.2)	3.265 (94.4)	5.305 (94)	5.696 (93.3)	2.721 (96.9)	4.421 (96.6)	4.747 (96.4)
	3	2.541 (92.9)	4.124 (92.3)	4.423 (91.1)	2.345 (94.7)	3.806 (94.3)	4.082 (93.6)	1.954 (96.5)	3.172 (96.4)	3.402 (96.2)
(60, 10)	1	1.355 (93)	2.193 (92.4)	2.353 (91.2)	1.251 (94.6)	2.024 (94.2)	2.172 (93.5)	1.042 (96.3)	1.687 (96.1)	1.808 (95.7)
	2	3.472 (91.6)	5.643 (91)	6.049 (89.8)	3.205 (94.2)	5.209 (93.7)	5.584 (93.2)	2.671 (96.8)	4.341 (96.5)	4.653 (96.6)
	3	2.076 (92.8)	3.374 (92.2)	3.631 (91)	1.916 (94.6)	3.114 (94.3)	3.351 (93.3)	1.597 (96.5)	2.595 (96.4)	2.792 (95.5)
(20, 15)	1	1.074 (93.6)	1.736 (93)	1.862 (91.8)	0.988 (95)	1.602 (94.5)	1.718 (93.7)	0.823 (96.4)	1.335 (96.1)	1.432 (95.6)
	2	2.854 (90.6)	4.635 (90.1)	4.972 (88.8)	2.634 (93.5)	4.278 (93.2)	4.588 (92.1)	2.195 (96.4)	3.565 (96.3)	3.823 (95.4)
	3	1.121 (92.7)	1.817 (92.1)	1.949 (90.9)	1.034 (94.6)	1.678 (94.1)	1.799 (93.3)	0.862 (96.5)	1.398 (96.2)	1.499 (95.7)
(40, 15)	1	0.983 (93.9)	1.528 (93.3)	1.641 (92.1)	0.907 (95.1)	1.412 (94.7)	1.514 (93.9)	0.756 (96.3)	1.175 (96.2)	1.262 (95.6)
	2	2.448 (92.6)	3.969 (92)	4.258 (90.8)	2.264 (94.4)	3.664 (94.1)	3.934 (93)	1.883 (96.3)	3.053 (96.3)	3.275 (95.2)
	3	1.019 (93)	1.628 (92.4)	1.754 (91.2)	0.941 (94.4)	1.502 (94.1)	1.619 (93.4)	0.784 (95.8)	1.252 (95.7)	1.349 (95.5)
(60, 15)	1	0.811 (94)	1.314 (93.4)	1.409 (92.2)	0.748 (95.1)	1.213 (94.7)	1.301 (93.9)	0.623 (96.2)	1.011 (96)	1.084 (95.5)
	2	2.172 (92.4)	3.526 (91.8)	3.78 (90.6)	2.003 (94.4)	3.254 (94.3)	3.492 (92.9)	1.669 (96.5)	2.712 (96.8)	2.908 (95.1)
	3	0.863 (93.1)	1.398 (92.5)	1.502 (91.3)	0.797 (94.7)	1.291 (93.9)	1.385 (93.3)	0.664 (96.3)	1.075 (95.3)	1.154 (95.3)

Table 8. Average values of the confidence intervals lengths and their coverage probabilities between brackets based on the MLE, percentile bootstrap (Boot-P) and bootstrap-t (Boot-t) methods when β , η , λ and T set at 1.6, 0.8, 1.4 and 6, respectively

(n, m)	CS	MLE			Boot-P			Boot-t		
		β	η	λ	β	η	λ	β	η	λ
(20, 10)	1	2.205 (93.6)	3.588 (91.9)	3.844 (90.6)	2.035 (94.6)	3.312 (93.7)	3.548 (93.2)	1.696 (95.6)	2.763 (95.5)	2.957 (95.7)
	2	4.112 (90.5)	6.682 (88.9)	7.161 (87.5)	3.794 (93.6)	6.168 (92.9)	6.611 (92)	3.161 (96.8)	5.142 (96.9)	5.509 (96.5)
	3	3.445 (92.7)	5.592 (91)	5.993 (89.7)	3.180 (94.3)	5.161 (93.7)	5.532 (92.9)	2.652 (96)	4.304 (96.5)	4.611 (96.1)
(40, 10)	1	1.853 (94.1)	3.012 (92.4)	3.229 (91.1)	1.711 (95.1)	2.783 (94.2)	2.981 (93.4)	1.426 (96.2)	2.317 (96.1)	2.484 (95.6)
	2	3.608 (92.3)	5.862 (90.6)	6.286 (89.3)	3.332 (94.5)	5.411 (93.5)	5.803 (92.7)	2.775 (96.7)	4.509 (96.4)	4.836 (96.1)
	3	2.591 (93.2)	4.207 (91.5)	4.512 (90.2)	2.392 (94.7)	3.883 (93.8)	4.165 (93)	1.993 (96.3)	3.236 (96.2)	3.471 (95.8)
(60, 10)	1	1.379 (93.3)	2.238 (91.6)	2.398 (90.3)	1.272 (94.7)	2.066 (93.7)	2.214 (92.9)	1.061 (96.1)	1.722 (95.8)	1.845 (95.4)
	2	3.542 (91.9)	5.754 (90.2)	6.172 (88.9)	3.274 (94.2)	5.312 (93.2)	5.696 (92.6)	2.725 (96.6)	4.426 (96.3)	4.747 (96.3)
	3	2.118 (93.1)	3.441 (91.4)	3.691 (90.1)	1.955 (94.7)	3.176 (93.8)	3.407 (92.7)	1.629 (96.3)	2.647 (96.2)	2.839 (95.2)
(20, 15)	1	1.091 (93.9)	1.773 (92.2)	1.899 (90.9)	1.007 (95)	1.634 (94)	1.753 (93.1)	0.839 (96.2)	1.362 (95.8)	1.461 (95.3)
	2	2.913 (90.9)	4.726 (89.3)	5.069 (87.9)	2.686 (93.5)	4.363 (92.7)	4.679 (91.5)	2.239 (96.2)	3.636 (96.1)	3.902 (95.1)
	3	1.141 (93)	1.853 (91.3)	1.991 (90)	1.053 (94.6)	1.711 (93.6)	1.837 (92.7)	0.878 (96.3)	1.426 (96)	1.531 (95.4)
(40, 15)	1	1.002 (94.2)	1.557 (92.5)	1.687 (91.2)	0.925 (95.1)	1.437 (94.2)	1.551 (93.3)	0.771 (96.1)	1.198 (96)	1.293 (95.3)
	2	2.492 (92.9)	4.048 (91.2)	4.343 (89.9)	2.301 (94.5)	3.737 (93.6)	4.009 (92.4)	1.917 (96.1)	3.114 (96.1)	3.341 (94.9)
	3	1.021 (93.3)	1.664 (91.6)	1.779 (90.3)	0.942 (94.5)	1.533 (93.6)	1.642 (92.8)	0.785 (95.6)	1.277 (95.5)	1.368 (95.2)
(60, 15)	1	0.825 (94.3)	1.341 (92.6)	1.437 (91.3)	0.762 (95.1)	1.237 (94.2)	1.327 (93.3)	0.635 (96)	1.031 (95.7)	1.106 (95.2)
	2	2.212 (92.7)	3.595 (91)	3.857 (89.7)	2.042 (94.5)	3.318 (93.8)	3.562 (92.3)	1.702 (96.3)	2.765 (96.6)	2.967 (94.8)
	3	0.881 (93.4)	1.426 (91.7)	1.534 (90.4)	0.813 (94.7)	1.316 (93.6)	1.416 (92.7)	0.678 (96.1)	1.097 (95.5)	1.182 (95)

5. CONCLUDING REMARKS AND FURTHER STUDIES

In this article, we have considered the problem of estimation of Weibull distribution parameters and the acceleration factor using three different methods when the data are progressively Type-I hybrid censored under CSPALT using data set 1. The CIs of the model parameters are constructed based on the three different methods (ML, percentile bootstrap and bootstrap-t methods). The performance of the estimators for each method is investigated using the MSEs by Monte Carlo simulations.

It is observed that in small and moderate sample sizes cases, bootstrap-t method works better than both the approximate method and percentile bootstrap method. It provides smaller MSEs, narrower CIs with closest CP to the nominal level. The methods of bootstrap are recommended, especially bootstrap-t method to construct CIs, even for large samples, for computational ease and high precision. Finally, as a future work, the Bayesian inference in the case of CSPALT under the same advanced censoring scheme proposed in this article will be considered. Also, reliability analysis of both data set 2 and data set 3 is now under study.

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Inverted Kumaraswamy Distribution: Properties and Estimation

A. M. Abd AL-Fattah, A. A. EL-Helbawy and G. R. AL-Dayian

Abstract

In this paper inverted Kumaraswamy distribution is introduced as a new distribution. Some of its properties are presented through, some models of stress strength, measures of central tendency and dispersion and order statistics. Some sub-models, limiting distributions and the relation between inverted Kumaraswamy and other distributions are derived. Several inverted Kumaraswamy families of distributions are suggested using the composition of the cumulative distribution functions. The maximum likelihood, Bayes estimators and confidence intervals for the parameters, the reliability and the hazard rate functions of the inverted Kumaraswamy distribution based on Type II censored samples are obtained. A numerical study is carried out to illustrate the theoretical results for both approaches, the maximum likelihood estimation and the Bayesian estimation.

Keywords: *Kumaraswamy distribution; Stress-strength models; Maximum likelihood estimation; Type II censored; Bayesian estimation; Monte Carlo simulation; Markov chain Monte Carlo (MCMC).*

1. Introduction

The inverted distributions are important in a wide range of applications; for example, in problems related to econometrics, biological sciences, survey sampling, engineering sciences, medical research and life testing problems. In addition, it is employed in financial literature, environmental studies, survival and reliability theory [See Abd EL-Kader (2013)].

Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt

Many researchers studied the inverted distributions and its applications; for example, Calabria and Pulcini (1990) presented the inverse Weibull, AL-Dayian (1999) introduced the inverted Burr Type XII distribution, Abd El-Kader *et al.* (2003) described the inverted Pareto Type I distribution, AL-Dayian (2004) discussed inverted Pareto Type II distribution, Prakash (2012) studied the inverted exponential model and Aljuaid (2013) presented exponentiated inverted Weibull distribution.

Kumaraswamy (1980) presented a distribution, which has many similarities to the beta distribution but it has a number of advantages of an invertible closed-form cumulative distribution function, particularly straightforward distribution and quantile functions which do not depend on special functions. This distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the height of individuals, scores obtained on a test, atmospheric temperatures and hydrological data such as daily rain fall, daily stream flow. For more details [See Kumaraswamy (1980), Jones (2009), Golizadeh *et al.* (2011), Sindhu *et al.* (2013) and Sharaf El-Deen *et al.* (2014)].

The inverted Kumaraswamy distribution can be derived from *Kumaraswamy* (Kum) distribution using the transformation $T = \frac{1}{x} - 1$, when X has a *Kum* distribution with probability density function (pdf)

$$f(x; \alpha, \beta) = \alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}, \quad 0 < x < 1; \quad \alpha, \beta > 0 \quad (1)$$

Then, the distribution of $T = \frac{1}{x} - 1$ is referred to as the *inverse or inverted Kumaraswamy* (IKum) distribution and the domain of T is $(0, \infty)$. Using the previous transformation is better than using $T = \frac{1}{x}$, which let the pdf more flexible, also using this transformation is analogous to finding inverted beta distribution (beta Type II) from beta distribution Type I.

Assuming T is a random variable distribution as IKum with shape parameters, $\alpha > 0$ and $\beta > 0$ denoted by $T \sim \text{IKum}(\alpha, \beta)$, the pdf and cumulative distribution function (cdf) are given, respectively, by

$$f(t; \alpha, \beta) = \alpha\beta(1+t)^{-(\alpha+1)}(1-(1+t)^{-\alpha})^{\beta-1}, \quad t > 0; \quad \alpha, \beta > 0, \quad (2)$$

and

$$F(t; \alpha, \beta) = (1 - (1+t)^{-\alpha})^\beta, \quad t > 0; \quad \alpha, \beta > 0. \quad (3)$$

This paper is organized as follows. In Section 2 the main properties of the IKum distribution are presented. Some sub-models, limiting distributions and the relation between inverted Kumaraswamy and other distributions are derived in Section 3. Maximum likelihood estimation for the parameters, reliability (rf) and hazard rate functions (hrf) of the IKum distribution based on Type II censored samples are obtained, also Bayesian estimation of the parameters, rf and hrf of the IKum distribution based on Type II censored samples are developed in Section 4. A numerical study is presented in Section 5 to illustrate the application procedures of the various results developed in this paper.

2. The Main Properties of the Inverted Kumaraswamy Distribution

This section is devoted to illustrate the statistical properties of IKum distribution, through rf, some models of the stress-strength, hrf and reversed hazard (rhrf), measures of central tendency and dispersion, graphical and order statistics.

a. Reliability function

$$R_1(t) = P(T > t) = 1 - F(t) = 1 - (1 - (1 + t)^{-\alpha})^\beta, \quad t > 0. \quad (4)$$

b. Some stress-strength models

- 1) Let T be the stress component subject strength Y , the random variables T and Y are independent distributions from IKum (α, β_1) and (α, β_2) respectively, then the reliability function is given by;

$$\begin{aligned} R_2(t) &= P(T < Y) = \int_0^\infty \int_0^y f(y)f(t) dt dy \\ &= \int_0^\infty f(y)F_T(y)dy \\ &= \int_0^\infty \alpha\beta_2(1+y)^{-(\alpha+1)}(1-(1+y)^{-\alpha})^{\beta_1+\beta_2-1}dy \\ &= \frac{\beta_2}{\beta_1+\beta_2} \end{aligned} \quad (5)$$

- 2) Let T and Z be two independent random stress variables with known cdfs $H_T(t), G_Z(z)$, and follow IKum (α, β_1) and IKum (α, β_3) , respectively, and let Y be independent of T and Z be a random strength variable with known cdf, $F_Y(y)$, and follows IKum (α, β_2) . Then it is given by;

$$R_3(t) = P(T < Y < Z)$$

$$\begin{aligned}
 &= \int_0^{\infty} H_T(y) dF_Y(y) - \int_0^{\infty} H_T(y) G_Z(y) dF_Y(y) \\
 &= \int_0^{\infty} [(1 - (1 + y)^{-\alpha})^{\beta_1} - (1 - (1 + y)^{-\alpha})^{\beta_1 + \beta_3}] f(y) dy \\
 &= \int_0^{\infty} \alpha \beta_2 (1 + y)^{-(\alpha+1)} (1 - (1 + y)^{-\alpha})^{\beta_1 + \beta_2 - 1} dy \\
 &\quad - \int_0^{\infty} \alpha \beta_2 (1 + y)^{-(\alpha+1)} (1 - (1 + y)^{-\alpha})^{\beta_1 + \beta_2 + \beta_3 - 1} dy \\
 &= \frac{\beta_2}{\beta_1 + \beta_2} - \frac{\beta_2}{\beta_1 + \beta_2 + \beta_3} \\
 &= \frac{\beta_2 \beta_3}{(\beta_1 + \beta_2)(\beta_1 + \beta_2 + \beta_3)} .
 \end{aligned} \tag{6}$$

c. Hazard and reversed hazard rate functions

The hrf denoted by $h_1(t)$ and rhrf denoted by $h_2(t)$ are given, respectively, by

$$h_1(t) = \frac{f(t)}{R(t)} = \frac{\alpha \beta (1+t)^{-(\alpha+1)} (1 - (1+t)^{-\alpha})^{\beta-1}}{1 - (1 - (1+t)^{-\alpha})^{\beta}} , \quad t > 0; \quad \alpha, \beta > 0, \tag{7}$$

and

$$h_2(t) = \frac{f(t)}{F(t)} = \alpha \beta (1 + t)^{-(\alpha+1)} (1 - (1 + t)^{-\alpha})^{-1} , \quad t > 0; \quad \alpha, \beta > 0. \tag{8}$$

Plots of pdf, hrf and rhrf of IKum are given, respectively in Figures 1-3. The plots, in Figures 1 and 2, indicate that the curves of the pdf and hrf are monotone decreasing at the values $\alpha = 2.36$ and $\beta = 0.5$, the curves of the pdf and hrf are increasing and then decreasing at the values $\alpha = 5.18$ and $\beta = 5.18$ and when $\alpha = 0.77$ and $\beta = 0.39$ the curves of the pdf and hrf are right skewed. From Figure 3 one can observe that the curves of the rhrf at the all values are decreasing and then constant.

d. The mode of the inverted Kumaraswamy distribution

The mode of the IKum distribution is given by

$$\text{Mode} = t_{mode} = \left[\left(\frac{\alpha+1}{\alpha\beta+1} \right)^{-\frac{1}{\alpha}} - 1 \right] . \tag{9}$$

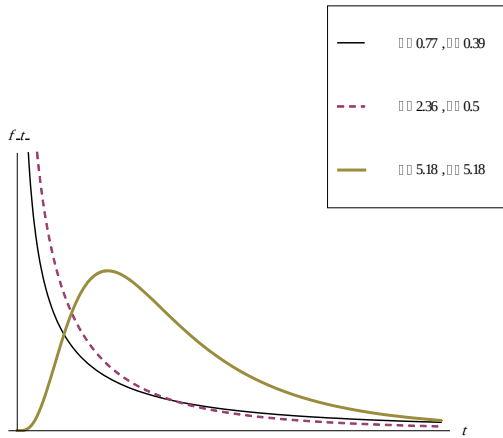


Figure 1
The plots of the probability density function

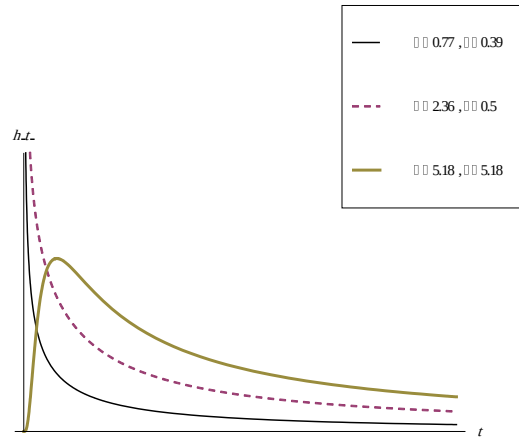


Figure 2
The plots of the hazard rate function

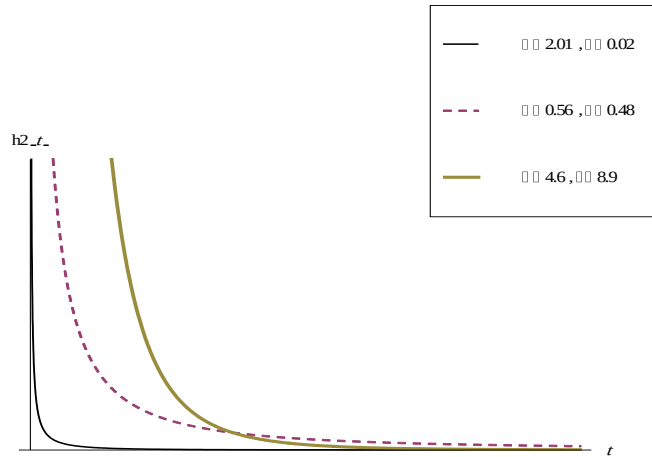


Figure 3

The plots of the reversed hazard rate function

e. Quantiles of the inverted Kumaraswamy distribution

The quantile function of the IKum is given by;

$$t_q = \left[\left(1 - (q)^{\frac{1}{\beta}} \right)^{-\frac{1}{\alpha}} - 1 \right], \quad 0 < q < 1. \quad (10)$$

Special cases can be obtained using (10) such as the second quartile (median), when $q = 0.5$

$$\text{Median} = t_{\text{median}} = \left[\left(1 - (0.5)^{\frac{1}{\beta}} \right)^{-\frac{1}{\alpha}} - 1 \right], \quad (11)$$

and the inter-quantile range can be expressed as

$$IQR(T) = \left(1 - \left(\frac{3}{4}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} - \left(1 - \left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}. \quad (12)$$

f. The central and non-central moments

The r^{th} non central moment of the IKum (α, β) distribution is given by

$$\mu_r = \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} \beta B\left(1 - \frac{j}{\alpha}, \beta\right), \quad r = 1, 2, \dots, \alpha > j, j = 0, \dots, r, \quad (13)$$

where $B(., .)$ is the beta function.

Thus, the mean and the variance of IKum are given by;

$$\mu = \beta B\left(1 - \frac{1}{\alpha}, \beta\right) - 1, \quad \alpha \geq 1, \quad (14)$$

and

$$\mu_2 = \beta B\left(1 - \frac{2}{\alpha}, \beta\right) - \left[\beta B\left(1 - \frac{1}{\alpha}, \beta\right)\right]^2, \quad \alpha > 1. \quad (15)$$

It is noticed that the r th moment depends on beta function, therefore, it is suitable to find the approximate mean and the approximate variance.

g. The approximate mean and variance

If $W \sim \text{Exponential}(\beta)$, with $\mu = E(W) = \frac{1}{\beta}$ and $\sigma^2 = V(W) = \frac{1}{\beta^2}$ then the variable $T = g(w) = \left[(1 - e^{-w})^{-\frac{1}{\alpha}} - 1\right] \sim \text{IKum}(\alpha, \beta)$, [See Appendix B]. This relation can be used to derive the approximate mean and variance of IKum (α, β) using the method of statistical differentials as given below:

$$E(T) \approx g(\mu) + \frac{1}{2} \sigma^2 g''(\mu), \quad (16)$$

and

$$V(T) \approx \sigma^2 (g'(\mu))^2, \quad (17)$$

where

$$g(\mu) = \left[(1 - e^{-\mu})^{-\frac{1}{\alpha}} - 1\right], \quad (18)$$

$$g'(\mu) = -\frac{1}{\alpha} (1 - e^{-\mu})^{-\left(\frac{1}{\alpha} + 1\right)} e^{-\mu}, \quad (19)$$

and

$$g''(\mu) = \frac{1}{\alpha^2} e^{-2\mu} (1 - e^{-\mu})^{-\left(\frac{1}{\alpha}+2\right)} [\alpha e^{\mu} + 1] , \quad (20)$$

where $\mu = \frac{1}{\beta}$ and $\sigma^2 = \frac{1}{\beta^2}$

Substituting (18), (19) and (20) in (16) and (17) it can be shown that the approximate mean and variance of T are

$$E(T) \approx \frac{1}{2\alpha^2\beta^2} \left[e^{-\frac{2}{\beta}} (1 - e^{-\frac{1}{\beta}})^{-\left(\frac{1}{\alpha}+2\right)} (\alpha e^{\frac{1}{\beta}} + 1) \right] + \left[(1 - e^{-\frac{1}{\beta}})^{-\frac{1}{\alpha}} - 1 \right] , \quad (21)$$

and

$$V(T) \approx \frac{1}{\alpha^2\beta^2} e^{-\frac{2}{\beta}} (1 - e^{-\frac{1}{\beta}})^{-2\left(\frac{1}{\alpha}+1\right)} . \quad (22)$$

h. Order statistics

The i^{th} order statistic, $t_{(i)}$ of a random sample of size n from the IKum (α, β) distribution has the following density

$$g(t_{(i)}) = \frac{n!}{(i-1)!(n-i)!} \alpha \beta (1 + t_{(i)})^{-(\alpha+1)} (1 - (1 + t_{(i)})^{-\alpha})^{\beta(i-1)} \\ \left(1 - (1 - (1 + t_{(i)})^{-\alpha})^{\beta} \right)^{(n-i)} , \quad t_{(i)} > 0 \quad (23)$$

Special cases

I. When $i = 1$, one obtains the pdf of the first order statistic

$$g(t_{(1)}) = n\alpha\beta(1 + t_{(1)})^{-(\alpha+1)} [1 - (1 - (1 + t_{(1)})^{-\alpha})^{\beta}]^{(n-1)} , \quad t_{(1)} > 0. \quad (24)$$

II. When $i = n$, the pdf of the largest order statistic can be obtained

$$g(t_{(n)}) = n\alpha\beta(1 + t_{(n)})^{-(\alpha+1)} (1 - (1 + t_{(n)})^{-\alpha})^{\beta(n-1)} , \quad t_{(n)} > 0 . \quad (25)$$

III. The median distribution of a random sample of size n from the IKum (α, β) distribution (when n is odd) has the density

$$g\left(t_{\left(\frac{n+1}{2}\right)}\right) = \frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2} \alpha \beta (1 + t_{\left(\frac{n+1}{2}\right)})^{-(\alpha+1)} \left(1 - (1 + t_{\left(\frac{n+1}{2}\right)})^{-\alpha} \right)^{\beta\left(\frac{n-1}{2}\right)} \\ \left(1 - \left(1 - (1 + t_{\left(\frac{n+1}{2}\right)})^{-\alpha} \right)^{\beta} \right)^{\left(\frac{n-1}{2}\right)} , \quad t_{\left(\frac{n+1}{2}\right)} > 0. \quad (26)$$

3. Related Distributions

This section discussed some sub-models of IKum distribution, some relations of the IKum distribution to other distributions, limiting and several IKum G families of distributions.

3.1 Some sub-models

I. Lomax (Pareto Type II) distribution

The Lomax distribution is a special case from IKum distribution, when $\beta = 1$ in (2) with the following pdf

$$f(t; \alpha) = \frac{\alpha}{(1+t)^{\alpha+1}}, \quad t > 0, \alpha > 0. \quad (27)$$

II. Beta Type II (inverted beta) distribution

The inverted beta Type II ($\beta, 1$) is a special case from IKum distribution, when $\alpha = 1$ in (2) as follows

$$f(t; \beta) = \frac{1}{B(\beta, 1)} t^{\beta-1} (1+t)^{-(\beta+1)}, \quad t > 0, \beta > 0. \quad (28)$$

Also, when $Y = k(1 - (1 + T)^{-1})^{-1}$ then $Y \sim$ Pareto Type I(β, k),

and, when $Y = \ln((1 - T)^{-1})^{-1}$ then $Y \sim$ exponential (β).

III. The log-logistic (Fisk) distribution

The log-logistic (Fisk) distribution is a special case from IKum distribution, when $\alpha = \beta = 1$ in (2), with the following form

$$f(t) = \frac{1}{(1+t)^2}, \quad 0 < t < \infty. \quad (29)$$

3.2 Some relations between the inverted Kumaraswamy distribution and other distributions

The inverted distribution can be transformed to several distributions using appropriate transformations such as exponentiated Weibull (exponentiated exponential, Weibull, Burr Type X, exponential and Rayleigh), generalized uniform (beta Type I, inverted generalized Pareto Type I and uniform (0, 1)), left truncated exponentiated exponential (left truncated exponential and exponential), exponentiated Burr Type XII (Burr Type XII, generalized Lomax, beta Type II and F- distribution), Kumaraswamy-Dagum (Dagum, Kumaraswamy-Burr Type III, Burr Type III and log logistic) and Kumaraswamy-inverse Weibull (Kumaraswamy-inverse exponential and inverse exponential). Table 1 summarizes the transformations from IKum to other distributions [See Appendix A].

3.3 Limiting distributions

I. If $T \sim \text{IKum}(\alpha, \beta)$ and $Y_1 = \beta^{-\frac{1}{\alpha}}(1 + T)$ on $(\beta^{-\frac{1}{\alpha}}, \infty)$, then the pdf of y_1 is

$f(y_1; \alpha, \beta) = \alpha y_1^{-(\alpha+1)} \left(1 - \frac{y_1^{-\alpha}}{\beta}\right)^{\beta-1}$. As $\beta \rightarrow \infty$, the pdf of y_1 tends to

$f(y_1; \alpha) = \alpha y_1^{-(\alpha+1)} e^{-y_1^{-\alpha}}$, $y_1 > 0$, which is the pdf of the inverted Weibull distribution.

II. If $T \sim \text{IKum}(\alpha, \beta)$ and $Y_2 = \alpha(1 - (1 + T)^{-1})$ on $(0, \alpha)$, then the pdf of y_2 is

$f(y_2; \alpha, \beta) = \beta \left(1 - \frac{y_2}{\alpha}\right)^{\alpha-1} \left(1 - \left(1 - \frac{y_2}{\alpha}\right)^{\alpha}\right)^{\beta-1}$. As $\alpha \rightarrow \infty$ the pdf of y_2 tends to $f(y_2; \beta) = \beta e^{-y_2} (1 - e^{-y_2})^{\beta-1}$, $y_2 > 0$, which is the pdf of the generalized exponential distribution.

III. If $T \sim \text{IKum}(\alpha, \beta)$ and $Y_3 = \alpha \left(1 - \beta^{\frac{1}{\alpha}} (1 + T)^{-1}\right)$ on $\left(\alpha \left(1 - \beta^{\frac{1}{\alpha}}\right), \alpha\right)$,

then the pdf of y_3 is $f(y_3; \alpha, \beta) = \left(1 - \frac{y_3}{\alpha}\right)^{\alpha-1} \left(1 - \frac{\left(1 - \frac{y_3}{\alpha}\right)^{\alpha}}{\beta}\right)^{\beta-1}$.

As both $\beta \rightarrow \infty$ and $\alpha \rightarrow \infty$, the pdf of y_3 tends to $f(y_3) = e^{-y_3} \exp[-e^{-y_3}]$, $y_3 > 0$, which is the pdf of the standard extreme value distribution of the first Type. [For proofs, See Appendix B].

3.4 Composition of cumulative distribution functions

Composition of cumulative distribution functions (cdfs) with other cdfs, or functions of cdfs, produces new families of distributions[See AL-Hussaini (2012)]. Applying this to IKum distribution, several IKum G families of distributions can be generated such as IKum exponential, IKum Weibull, IKum Pareto Type I, IKum inverted Weibull, IKum Rayleigh, IKum Burr Type XII and IKum Dagum distributions. Table 2 summarizes some IKum G distributions, [See Appendix A], where $F_G(t)$ is defined as;

$$F_G(t) = (1 - (1 + G(t))^{-\alpha})^{\beta}. \quad (30)$$

4. Estimation Based on Type II Censored

This section develops the estimation of the shape parameters, rf and hrf based on Type II censored samples using maximum likelihood and Bayesian approaches. Also the confidence intervals of the shape parameters, rf and hrf are obtained.

4.1 Maximum likelihood estimation

In this subsection, the maximum likelihood (ML) estimation is used to estimate the parameters, rf and hrf of IKum based on Type II censored samples.

Suppose that $(T_{(1)} \leq T_{(2)} \leq \dots \leq T_{(r)})$ is a censored sample of size r obtained from a life test on n items, the likelihood function based on Type II censored sample is given by

$$L(\alpha, \beta; \underline{t}) = [\prod_{i=1}^r f(t_i, \alpha, \beta)] [R(t_r, \alpha, \beta)]^{(n-r)} \\ \propto \alpha^r \beta^r \prod_{i=1}^r (1 + t_{(i)})^{-(\alpha+1)} \prod_{i=1}^r (1 - (1 + t_{(i)})^{-\alpha})^{\beta-1} \left[1 - (1 - (1 + t_{(r)})^{-\alpha})^{\beta} \right]^{n-r}. \quad (31)$$

The natural logarithm of the likelihood function is given by

$$\ell = \ln L(\alpha, \beta; \underline{t}) \\ = r \ln \alpha + r \ln \beta - (\alpha + 1) \sum_{i=1}^r \ln(1 + t_{(i)}) + (\beta - 1) \sum_{i=1}^r \ln(1 - (1 + t_{(i)})^{-\alpha}) \\ + (n - r) \ln \left[1 - (1 - (1 + t_{(r)})^{-\alpha})^{\beta} \right]. \quad (32)$$

Considering the two parameters α and β are unknown and differentiating the log likelihood function in (32) with respect to α and β as follows

$$\frac{\partial \ell}{\partial \beta} = \frac{r}{\beta} + \sum_{i=1}^r \ln(1 - (1 + t_{(i)})^{-\alpha}) - \frac{(n-r)(1 - (1 + t_{(r)})^{-\alpha})^{\beta} \ln[(1 - (1 + t_{(r)})^{-\alpha})^{\beta}]}{1 - (1 - (1 + t_{(r)})^{-\alpha})^{\beta}}, \quad (33)$$

and

$$\frac{\partial \ell}{\partial \alpha} = \frac{r}{\alpha} - \sum_{i=1}^r \ln(1 + t_{(i)}) + (\beta - 1) \sum_{i=1}^r \frac{(1 + t_{(i)})^{-\alpha} \ln(1 + t_{(i)})}{(1 - (1 + t_{(i)})^{-\alpha})} - \frac{(n-r)\beta(1 - (1 + t_{(r)})^{-\alpha})^{\beta-1} (1 + t_{(r)})^{-\alpha} \ln(1 + t_{(r)})}{1 - (1 - (1 + t_{(r)})^{-\alpha})^{\beta}} \quad (34)$$

Equating the two nonlinear likelihood Equations (33) and (34) to zero, and solving numerically one obtains the maximum likelihood estimates (MLEs) of α and β .

The MLEs of the $R_1(t)$, $R_2(t)$, $R_3(t)$, $h_1(t)$ and $h_2(t)$ can be obtained, using the invariance property of MLE, by replacing the parameters α and β in (4) - (7) by their MLEs. The MLEs of $R_1(t)$, $R_2(t)$, $R_3(t)$, $h_1(t)$ and $h_2(t)$ are given, respectively by

$$\hat{R}_1(t) = 1 - (1 - (1 + t)^{-\hat{\alpha}})^{\hat{\beta}}, \quad t > 0, \quad \alpha, \beta > 0, \quad (35)$$

$$\hat{R}_2(t) = \frac{\hat{\beta}_2}{\hat{\beta}_1 + \hat{\beta}_2} \quad , \quad (36)$$

$$\hat{R}_3(t) = \frac{\hat{\beta}_2 \hat{\beta}_3}{(\hat{\beta}_1 + \hat{\beta}_2)(\hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3)} \quad , \quad (37)$$

$$\hat{h}_1(t) = \frac{\hat{\alpha} \hat{\beta} t^{-(\hat{\alpha}+1)} (1-(1+t)^{-\hat{\alpha}})^{\hat{\beta}-1}}{1-(1-(1+t)^{-\hat{\alpha}})^{\hat{\beta}}} \quad , \quad t > 0, \quad \alpha, \beta > 0 \quad , \quad (38)$$

and

$$\hat{h}_2(t) = \hat{\alpha} \hat{\beta} (1+t)^{-(\hat{\alpha}+1)} (1-(1+t)^{-\hat{\alpha}})^{-1} \quad , \quad t > 0, \quad \alpha, \beta > 0. \quad (39)$$

Asymptotic variance–covariance matrix of maximum likelihood estimators

The asymptotic variance-covariance matrix of the MLEs for the two parameters α and β is the inverse of the Fisher information matrix as follows

$$\tilde{I}^{-1} \approx \begin{bmatrix} \widehat{var}(\hat{\alpha}) & \widehat{cov}(\hat{\alpha}, \hat{\beta}) \\ \widehat{cov}(\hat{\alpha}, \hat{\beta}) & \widehat{var}(\hat{\beta}) \end{bmatrix} \approx \frac{1}{|I|} \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ell}{\partial \alpha^2} \end{bmatrix} \bigg|_{\hat{\alpha}, \hat{\beta}} \quad ,$$

with

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{-r}{\beta^2} - \frac{(n-r)[1-(1+t_{(r)})^{-\alpha}]^{2\beta} \ln[1-(1+t_{(r)})^{-\alpha}]^2}{\left[(1-(1+t_{(r)})^{-\alpha})^\beta - 1\right]^2} + \frac{(n-r)(1-(1+t_{(r)})^{-\alpha})^\beta \ln[1-(1+t_{(r)})^{-\alpha}]^2}{(1-(1+t_{(r)})^{-\alpha})^\beta - 1} \quad , \quad (40)$$

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \alpha^2} = & \frac{-r}{\alpha^2} - (\beta - 1) \sum_{i=1}^r \left[\frac{(1+t_{(i)})^{-2\alpha} \ln(1+t_{(i)})^2}{((1+t_{(i)})^{-\alpha} - 1)^2} - \frac{(1+t_{(i)})^{-\alpha} \ln(1+t_{(i)})^2}{(1+t_{(i)})^{-\alpha} - 1} \right] \\ & + \frac{\beta(n-r)(\beta-1)(1+t_{(r)})^{-2\alpha} (1-(1+t_{(r)})^{-\alpha})^{\beta-2} \ln(1+t_{(r)})^2}{(1-(1+t_{(r)})^{-\alpha})^\beta - 1} \\ & - \frac{\beta^2(n-r)(1+t_{(r)})^{-2\alpha} (1-(1+t_{(r)})^{-\alpha})^{2\beta-2} \ln(1+t_{(r)})^2}{\left[(1-(1+t_{(r)})^{-\alpha})^\beta - 1\right]^2} \\ & - \frac{\beta(n-r)(1+t_{(r)})^{-\alpha} (1-(1+t_{(r)})^{-\alpha})^{\beta-1} \ln(1+t_{(r)})^2}{(1-(1+t_{(r)})^{-\alpha})^\beta - 1} \quad , \quad (41) \end{aligned}$$

and

$$\begin{aligned}
 & \frac{\partial^2 \ell}{\partial \beta \partial \alpha} \\
 &= \frac{(n-r)\beta (1 - (1 + t_{(r)})^{-\alpha})^{\beta-1} \ln(1 + t_{(r)}) \ln(1 - (1 + t_{(r)})^{-\alpha})}{(1 + t_{(r)})^\alpha [(1 - (1 + t_{(r)})^{-\alpha})^\beta - 1]} \\
 & - \sum_{i=1}^r \frac{(1 + t_{(i)})^{-\alpha} \ln(1 + t_{(i)})}{[(1 + t_{(i)})^{-\alpha} - 1]} - \frac{(n-r)(1 + t_{(r)})^{-\alpha} (1 - (1 + t_{(r)})^{-\alpha})^\beta \ln(1 + t_{(r)})}{((1 + t_{(r)})^{-\alpha} - 1) [(1 - (1 + t_{(r)})^{-\alpha})^\beta - 1]} \\
 & - \frac{(n-r)\beta (1 - (1 + t_{(r)})^{-\alpha})^{\beta-1} (1 - (1 + t_{(r)})^{-\alpha})^\beta \ln(1 + t_{(r)}) \ln(1 - (1 + t_{(r)})^{-\alpha})}{(1 + t_{(r)})^\alpha [(1 - (1 + t_{(r)})^{-\alpha})^\beta - 1]^2}. \quad (42)
 \end{aligned}$$

The asymptotic normality of MLE can be used to compute the asymptotic 100 (1- ω)% confidence intervals for α and β as follows

$$\hat{\alpha} \pm Z_{(1-\frac{\omega}{2})} \sqrt{\widehat{var}(\hat{\alpha})} \quad \text{and} \quad \hat{\beta} \pm Z_{(1-\frac{\omega}{2})} \sqrt{\widehat{var}(\hat{\beta})}. \quad (43)$$

Also, the asymptotic 100(1- ω)% confidence intervals for rf and hrf are given by

$$\hat{R}_1(t) \pm Z_{(1-\frac{\omega}{2})} \sqrt{\widehat{var}(\hat{R}_1(t))} \quad \text{and} \quad \hat{h}_1(t) \pm Z_{(1-\frac{\omega}{2})} \sqrt{\widehat{var}(\hat{h}_1(t))}. \quad (44)$$

where $Z_{(1-\frac{\omega}{2})}$ is standard normal and (1 - ω) is the confidence coefficient.

4.2 Bayesian estimation

In this subsection, the Bayesian approach is considered, under squared error loss function to estimate the parameters, rf and hrf of the IKum distribution based on Type II censored samples, using the non-informative prior. Also credible intervals for the parameters, rf and hrf are obtained.

Assuming that both of the parameters α and β are unknown and independent distributions, the joint non-informative prior on α and β is

$$\pi(\alpha, \beta) \propto \frac{1}{\alpha\beta}, \quad \alpha, \beta > 0, \quad (45)$$

The joint posterior distribution of α and β can be obtained using (31) and (45) as follows

$$\pi(\alpha, \beta | \underline{t}) = L(\alpha, \beta | \underline{t}) \pi(\alpha, \beta). \quad (46)$$

$$\propto \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)}, \quad (47)$$

$$= \rho \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)}, \quad (48)$$

where

$$u = (1 - (1 + t_{(r)})^{-\alpha}) \quad \text{and} \quad u_i = (1 - (1 + t_{(i)})^{-\alpha}),$$

and

$$\rho^{-1} = \int_0^\infty \int_0^\infty \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\alpha d\beta,$$

which is a normalizing constant.

Considering the squared error (SE) loss function as a symmetric loss function, then the Bayes estimators of the parameters, based on Type II censoring, is the mean of the posterior density and using the joint non-informative prior, one can obtain the Bayes estimators for the parameters, which are given by their marginal posterior expectations using (48) as follows

$$\alpha^* = E(\alpha|\underline{t}) = \int_0^\infty \int_0^\infty \rho \alpha^r \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\beta d\alpha, \quad (49)$$

and

$$\beta^* = E(\beta|\underline{t}) = \int_0^\infty \int_0^\infty \rho \alpha^{r-1} \beta^r e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\alpha d\beta. \quad (50)$$

And, the Bayes estimators of the rf and hrf under SE loss function can be obtained using (4), (7) and (48) as follows

$$\begin{aligned} R^*_1(t) &= E(R_1(t)|\underline{t}) \\ &= 1 - \int_0^\infty \int_0^\infty (1 - (1+t)^{-\alpha})^\beta \rho \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\alpha d\beta, \end{aligned} \quad (51)$$

and

$$\begin{aligned} h^*_1(t) &= E(h_1(t)|\underline{t}) \\ &= \int_0^\infty \int_0^\infty \frac{\alpha \beta (1+t)^{-(\alpha+1)} (1-(1+t)^{-\alpha})^{\beta-1}}{1-(1-(1+t)^{-\alpha})^\beta} \rho \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\alpha d\beta. \end{aligned} \quad (52)$$

To obtain the Bayes estimates of the parameters, rf and hrf, the Equations (49)-(52) should be solved numerically.

The credible intervals for the Bayes estimators of α and β are given from IKum based on Type II censored data. In general, $L(\underline{t}), U(\underline{t})$ is a 100 (1- ω) % credible interval for θ if

$$P[L(\underline{t}) < \theta < U(\underline{t})] = \int_{L(\underline{t})}^{U(\underline{t})} \pi(\theta|\underline{t}) d\theta = 1 - \omega,$$

where $L(\underline{t})$ and $U(\underline{t})$ are the lower and upper limits.

Since, the posterior distribution is given by (47), then a 100 (1- ω) % credible interval for α is $(L_1(\underline{t}), U_1(\underline{t}))$

where

$$P[\alpha > L_1(\underline{t})|\underline{t}] = \int_{L_1(\underline{t})}^{\infty} \int_0^{\infty} \rho \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\beta d\alpha = 1 - \frac{\omega}{2}, \quad (53)$$

and

$$P[\alpha > U_1(\underline{t})|\underline{t}] = \int_{U_1(\underline{t})}^{\infty} \int_0^{\infty} \rho \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\beta d\alpha = \frac{\omega}{2}. \quad (54)$$

Also, a 100 (1- ω) % credible interval for parameter β is $(L_2(\underline{t}), U_2(\underline{t}))$ where

$$\begin{aligned} P[\beta > L_2(\underline{t})|\underline{t}] &= \int_{L_2(\underline{t})}^{\infty} \int_0^{\infty} \rho \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\alpha d\beta \\ &= 1 - \frac{\omega}{2}, \end{aligned} \quad (55)$$

and

$$P[\beta > U_2(\underline{t})|\underline{t}] = \int_{U_2(\underline{t})}^{\infty} \int_0^{\infty} \rho \alpha^{r-1} \beta^{r-1} e^{-\alpha \sum_{i=1}^r \ln(1+t_{(i)})} e^{(\beta-1) \sum_{i=1}^r \ln(u_i)} e^{(n-r) \ln(1-u^\beta)} d\alpha d\beta = \frac{\omega}{2} \quad (56)$$

5. Numerical Study

- a. In this section, a simulation study is presented to illustrate the application of the various theoretical result developed in the previous section on basis of generated data from IKum(α, β) distribution, for different sample sizes ($n=20, 30, 60$ and 100) and using number of replications $N=10000$. The computations are performed using Mathematica 9.
- b. The relative absolute biases (RABs), relative mean square errors (RMSEs), variances of ML and Bayes estimates of the shape parameters, rf and hrf are computed as follows:
 - 1) RABs (estimator) = $\frac{|\text{bias}(\text{estimator})|}{\text{true value}}$,
 - 2) Relative mean square errors (RMSEs) = $\frac{\text{MSE}(\text{estimator})}{\text{true value}}$,
 - 3) Variances (estimator) = $\text{MSE}(\text{estimator}) - \text{bias}^2(\text{estimator})$.
- c. Table 3 shows the (RABs), (RMSEs) and variances of MLEs and 95% confidence intervals (CIs) where the initial parameter values are $\alpha = 0.4$, $\beta = 0.8$ based on three levels of Type II censoring 70%, 90% and 100%. Table 4 displays the same computational results but for different initial parameter values $\alpha = 1$, $\beta = 0.8$ and for $\alpha = 0.4$, $\beta = 1$ in Table 5.

- d. Table 6 presents RABs, RMSEs of the MLEs and 95% CIs of the rf and hrf at different mission time t_0 where ($t_0 = 0.3, 0.6$ and 1) from IKum distribution for different sample sizes where ($n=20, 60$ and 100) and level of censoring $r = 90\%$ and repetitions $N = 10000$.
- e. Table 7 shows the Bayes estimates of the parameters, rf and hrf and their mean squared errors (MSEs), standard deviations (sd) and credible intervals based on Type II censoring using the joint non-informative prior. The computations are performed using MCMC method.

Concluding remarks

- (i) From Tables 3, 4 and 5, one can observe that the RABs, variances and RMSEs of the MLEs of the shape parameters α and β decrease when the sample size n increases. Also, it is observed that as the level of censoring decreases the RAB, variances and RMSEs of the estimates decrease. The lengths of the confidence intervals becomes narrower as the sample size increases.
- (ii) Table 6 indicated that the rf decreases and the hrf increases when the mission time t_0 increases, also the RABs and RMSEs of the MLEs of the rf and the hrf decrease when the sample size is increases.
- (iii) It is clear from Table 7 that the MSEs and sds of the Bayes estimates of the parameters, rf and hrf performs better and the credible intervals get shorter when the sample size increases.
- (iv) These results are expected since decreasing the level of censoring means that more information is provided by the sample and hence increase the accuracy of the estimates.
- (v) In general, when $r = n$, all the results obtained for Type II censored sample reduce to those of the complete sample.

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APPENDIX A

Table 1

Summary of some transformations applied to the inverted Kumaraswamy and the resulting distributions

Transformation	The resulting Distribution	Pdf
$[\ln(1+T)]^{\frac{1}{\theta}}$ Special cases:	Exponentiated Weibull (α, β, θ)	$f(y_1) = \alpha\beta\theta y_1^{\theta-1} e^{-\alpha y_1^\theta} \left(1 - e^{-\alpha y_1^\theta}\right)^{\beta-1},$ $0 < y_1 < \infty,$ $\alpha, \beta, \theta > 0$
(i) $\theta = 1$	Exponentiated exponential (α, β)	
(ii) $\beta = 1$	Weibull (α, θ)	
(iii) $\theta = 2$	Burr Type X (generalized Rayleigh) (α, β)	
(iv) $\beta = \theta = 1$	Exponential (α)	
(v) $\beta = 1$ and $\theta = 2$	Rayleigh (α)	
$w^{\frac{1}{\alpha}}(1+T)^{-1} + c$ Special cases:	Generalized uniform (α, β, w, c)	$f(y_2) = \frac{\alpha\beta w(y_2-c)^{\alpha-1}}{(1-w(y_2-c)^\alpha)^{-\beta+1}},$ $c < y_2 < c + w^{\frac{1}{\alpha}},$ $\alpha, \beta, w, c > 0$
(i) $\alpha = w = 1$ and $c = 0$	Beta Type I $(1, \beta)$	
(ii) $c = 0$	Inverted generalized Pareto Type I (α, β, w)	
(iii) $\alpha = \beta = w = 1$	Uniform $(0,1)$	
$\ln(1+T)^\theta + b$ Special cases:	Left truncated exponentiated exponential $(\alpha, \beta, \theta, b)$	$f(y_3) = \frac{\alpha\beta}{\theta} e^{-\alpha(\frac{y_3-b}{\theta})} \left(1 - e^{-\alpha(\frac{y_3-b}{\theta})}\right)^{\beta-1},$ $b < y_3 < \infty$
(i) $\beta = 1$	Left truncated exponential (α, θ, b)	
(ii) $\beta = 1$ and $b = 0$	Exponential (α, θ)	

Table 1 (continued)

Transformation	The resulting Distribution	Pdf
$sT^{\frac{1}{c}}$ Special cases:	Exponentiated Burr Type XII (α, β, s, c)	$f(y_4) = \frac{\alpha\beta c}{s^c} y_4^{c-1} \left(1 + \left(\frac{y_4}{s}\right)^c\right)^{-(\alpha+1)} \left(1 - \left(1 + \left(\frac{y_4}{s}\right)^c\right)^{-\alpha}\right)^{\beta-1},$ $y_4 > 0,$ $\alpha, \beta, s, c > 0$
(i) $\beta = 1$	Burr Type XII (α, s, c)	
(ii) $\beta = c = 1$	Generalized Lomax (Pareto Type II) (α, s)	
(iii) $\beta = c = 1$ and $s = 1$	Beta Type II $(1, \alpha)$	
(iv) $\beta = c = 1$ and $s = \alpha$	F-distribution $(2, 2\alpha)$	
$\left(\frac{T}{\lambda}\right)^{-\frac{1}{s}}$ Special cases:	Kumaraswamy-Dagum $(\alpha, \beta, \lambda, s)$	$f(y_5) = \frac{\alpha\beta\lambda s}{y_5^{s+1}} (1 + \lambda y_5^{-s})^{-(\alpha+1)} (1 - (1 + \lambda y_5^{-s})^{-\alpha})^{\beta-1},$ $y_5 > 0,$ $\alpha, \beta, \lambda, s > 0$
(i) $\beta = 1$	Dagum (α, λ, s)	
(ii) $\lambda = 1$	Kumaraswamy-Burr Type III (α, β, s)	
(iii) $\beta = \lambda = 1$	Burr Type III (α, s)	
(iv) $\alpha = 1$	Kumaraswamy-Fisk (log logistic) (λ, β, s)	
$\left(\frac{\ln(1+T)}{\theta}\right)^{-\frac{1}{b}}$ Special cases:	Kumaraswamy-inverse Weibull $(\alpha, \beta, \theta, b)$	$f(t) = \frac{\alpha\beta\theta b}{y_6^{b+1}} e^{-\frac{\alpha\theta}{y_6^b}} \left(1 - e^{-\frac{\alpha\theta}{y_6^b}}\right)^{\beta-1},$ $y_6 > 0$
(i) $b = 1$	Kumaraswamy-inverse exponential (α, β, θ)	
(ii) $b = \beta = 1$	Inverse exponential (α, θ)	

Table 2
Some IKum G distributions

G(t)	cdf	Pdf
Exponential (θ) $1 - e^{-\frac{t}{\theta}}$	$F_G(t) = \left(1 - \left(1 + (1 - e^{-\frac{t}{\theta}})\right)^{-\alpha}\right)^{\beta}$	$f(t) = \frac{\alpha\beta}{\theta} e^{-\frac{t}{\theta}} (2 - e^{-\frac{t}{\theta}})^{-(\alpha+1)} \left(1 - (2 - e^{-\frac{t}{\theta}})^{-\alpha}\right)^{\beta-1}$, $t > 0$.
Weibull (c, k) $1 - e^{-\left(\frac{t}{c}\right)^k}$	$F_G(t) = \left(1 - \left(1 + (1 - e^{-\left(\frac{t}{c}\right)^k})\right)^{-\alpha}\right)^{\beta}$	$f(t) = \frac{\alpha\beta k}{c^k} t^{k-1} e^{-\left(\frac{t}{c}\right)^k} \left(2 - e^{-\left(\frac{t}{c}\right)^k}\right)^{-(\alpha+1)} \left(1 - (2 - e^{-\left(\frac{t}{c}\right)^k})^{-\alpha}\right)^{\beta-1}$, $t > 0$.
Pareto Type I (b, a) $1 - \left(\frac{b}{t}\right)^a$	$F_G(t) = \left(1 - \left(1 + (1 - \left(\frac{b}{t}\right)^a)\right)^{-\alpha}\right)^{\beta}$	$f(t) = ab^{a-1}\alpha\beta t^{1-a} \left(2 - \left(\frac{b}{t}\right)^a\right)^{-(\alpha+1)} \left(1 - (2 - \left(\frac{b}{t}\right)^a)^{-\alpha}\right)^{\beta-1}$, $t > b$.
Inverted Weibull (a, c) $e^{-at^{-c}}$	$F_G(t) = \left(1 - (1 + e^{-at^{-c}})^{-\alpha}\right)^{\beta}$	$f(t) = ac\alpha\beta t^{-(c+1)} e^{-at^{-c}} (1 + e^{-at^{-c}})^{-(\alpha+1)} (1 - (1 + e^{-at^{-c}})^{-\alpha})^{\beta-1}$, $t > 0$.
Rayleigh (σ) $1 - e^{-\frac{t^2}{2\sigma^2}}$	$F_G(t) = \left(1 - \left(1 + (1 - e^{-\frac{t^2}{2\sigma^2}})\right)^{-\alpha}\right)^{\beta}$	$f(t) = \frac{\alpha\beta}{\sigma^2} t e^{-\frac{t^2}{2\sigma^2}} \left(2 - e^{-\frac{t^2}{2\sigma^2}}\right)^{-(\alpha+1)} \left(1 - (2 - e^{-\frac{t^2}{2\sigma^2}})^{-\alpha}\right)^{\beta-1}$, $t > 0$.
Burr Type XII (c,k) $1 - (1 + t^c)^{-k}$	$F_G(t) = (1 - (1 + (1 - (1 + t^c)^{-k}))^{-\alpha})^{\beta}$	$f(t) = ck\alpha\beta t^{c-1} (1 + t^c)^{-(k+1)} (2 - (1 + t^c)^{-k})^{-(\alpha+1)} (1 - (2 - (1 + t^c)^{-k})^{-\alpha})^{\beta-1}$, $t > 0$.
Dagum (λ, c, k) $(1 + \lambda t^{-c})^{-k}$	$F_G(t) = (1 - (1 + (1 + \lambda t^{-c})^{-k})^{-\alpha})^{\beta}$	$f(t) = ck\alpha\beta\lambda t^{-(c+1)} (1 + \lambda t^{-c})^{-(k+1)} (1 + (1 + \lambda t^{-c})^{-k})^{-(\alpha+1)} (1 - (1 + (1 + \lambda t^{-c})^{-k})^{-\alpha})^{\beta-1}$, $t > 0$.

Table 3

Relative absolute biases, relative mean square errors and variances of ML estimates and 95% confidence intervals of the shape parameters α and β from IKum for different sample size n, censoring size r and replications N = 10000 ($\alpha = 0.4$, $\beta = 0.8$)

n	r	Estimate s	RAB	RMSE	Variance	Lower	Upper	Length
30	21	$\hat{\alpha}$	0.0019	0.0323	0.0327	0.1204	0.8296	0.7092
		$\hat{\beta}$	0.0044	0.0799	0.0818	0.3545	1.4755	1.1209
	27	$\hat{\alpha}$	0.0007	0.0172	0.0182	0.1818	0.7119	0.5301
		$\hat{\beta}$	0.0025	0.0585	0.0619	0.3986	1.3745	0.9759
	30	$\hat{\alpha}$	0.0004	0.0127	0.0138	0.2044	0.6656	0.4613
		$\hat{\beta}$	0.0018	0.0489	0.0527	0.4227	1.3229	0.9002
60	42	$\hat{\alpha}$	0.0004	0.0120	0.0132	0.2098	0.6595	0.4496
		$\hat{\beta}$	0.0009	0.0269	0.0292	0.5174	1.1875	0.6701
	54	$\hat{\alpha}$	0.0002	0.0069	0.0077	0.2506	0.5953	0.3447
		$\hat{\beta}$	0.0007	0.0208	0.0228	0.5466	1.1389	0.5923
	60	$\hat{\alpha}$	0.0001	0.0056	0.0063	0.2634	0.5735	0.3099
		$\hat{\beta}$	0.0004	0.0188	0.0212	0.5487	1.1196	0.5709
100	70	$\hat{\alpha}$	0.0001	0.0062	0.0071	0.2546	0.5839	0.3293
		$\hat{\beta}$	0.0003	0.0134	0.0151	0.5879	1.0692	0.4813
	90	$\hat{\alpha}$	0.0001	0.0037	0.0043	0.2849	0.5409	0.2561
		$\hat{\beta}$	0.0002	0.0114	0.0130	0.5991	1.0463	0.4472
	100	$\hat{\alpha}$	0.0001	0.0029	0.0034	0.2964	0.5238	0.2274
		$\hat{\beta}$	0.0001	0.0099	0.0114	0.6103	1.0284	0.4181

Table 4

Relative absolute biases, relative mean square errors and variances of ML estimates and 95% confidence intervals of the shape parameters α and β from IKum for different sample size n, censoring size r and repetitions N = 10000 ($\alpha = 1$, $\beta = 0.8$)

n	r	Estimates	RAB	RMSE	Variance	Lower	Upper	Length
30	21	$\hat{\alpha}$	0.0169	0.1403	0.1787	0.3282	1.9853	1.6571
		$\hat{\beta}$	0.0072	0.0622	0.0797	0.3484	1.4552	1.1068
	27	$\hat{\alpha}$	0.0077	0.0841	0.1105	0.4541	1.7574	1.3032
		$\hat{\beta}$	0.0042	0.0439	0.0574	0.4083	1.3480	0.9397
	30	$\hat{\alpha}$	0.0057	0.0660	0.0872	0.5127	1.6704	1.1576
		$\hat{\beta}$	0.0037	0.0397	0.0522	0.4257	1.3211	0.8953
60	42	$\hat{\alpha}$	0.0058	0.0618	0.0810	0.5342	1.6501	1.1158
		$\hat{\beta}$	0.0023	0.0246	0.0322	0.5070	1.2104	0.7034
	54	$\hat{\alpha}$	0.0021	0.0330	0.0448	0.6403	1.4700	0.8297
		$\hat{\beta}$	0.0011	0.0168	0.0228	0.5446	1.1367	0.5920
	60	$\hat{\alpha}$	0.0013	0.0271	0.0373	0.6658	1.4233	0.7574
		$\hat{\beta}$	0.0005	0.0140	0.0196	0.5534	1.1025	0.5491
100	70	$\hat{\alpha}$	0.0028	0.0332	0.0440	0.6524	1.4753	0.8229
		$\hat{\beta}$	0.0007	0.0114	0.0155	0.5893	1.0775	0.4881
	90	$\hat{\alpha}$	0.0010	0.0198	0.0271	0.7165	1.3630	0.6465
		$\hat{\beta}$	0.0004	0.0093	0.0130	0.6009	1.0479	0.4470
	100	$\hat{\alpha}$	0.0003	0.0149	0.0211	0.7382	1.3081	0.5698
		$\hat{\beta}$	0.0001	0.0077	0.0109	0.6106	1.0201	0.4095

Table 5

Relative absolute biases, relative mean square errors and variances of ML estimates and 95% confidence intervals of the shape parameters α and β from IKum for different sample size n, censoring size r and repetitions N = 10000 ($\alpha = 0.4$, $\beta = 1$)

n	r	estimates	RAB	RMSE	Variance	Lower	Upper	Length
30	21	$\hat{\alpha}$	0.0012	0.0256	0.0251	0.1456	0.7678	0.6222
		$\hat{\beta}$	0.0059	0.1275	0.1249	0.4360	1.8217	1.3856
	27	$\hat{\alpha}$	0.0005	0.0149	0.0152	0.1952	0.6791	0.4839
		$\hat{\beta}$	0.0042	0.0976	0.0969	0.4989	1.7179	1.2189
	30	$\hat{\alpha}$	0.0004	0.0122	0.0126	0.2110	0.6525	0.4405
		$\hat{\beta}$	0.0035	0.0861	0.0860	0.5231	1.6729	1.14987
60	42	$\hat{\alpha}$	0.00024	0.0104	0.0109	0.2216	0.6308	0.4092
		$\hat{\beta}$	0.0015	0.0479	0.0491	0.6299	1.4982	0.8682
	54	$\hat{\alpha}$	0.0001	0.0063	0.0066	0.2585	0.5779	0.3194
		$\hat{\beta}$	0.0011	0.0383	0.0396	0.6642	1.4439	0.7797
	60	$\hat{\alpha}$	0.0001	0.0049	0.0053	0.2715	0.5576	0.2861
		$\hat{\beta}$	0.0007	0.0325	0.0341	0.6815	1.4061	0.7246
100	70	$\hat{\alpha}$	0.0000	0.0057	0.0061	0.2628	0.5680	0.3058
		$\hat{\beta}$	0.0006	0.0253	0.0267	0.7177	1.3582	0.6406
	90	$\hat{\alpha}$	0.0000	0.0033	0.0035	0.2941	0.5288	0.2346
		$\hat{\beta}$	0.0004	0.0190	0.0201	0.7541	1.3097	0.5556
	100	$\hat{\alpha}$	0.0000	0.0029	0.0031	0.3006	0.5189	0.2183
		$\hat{\beta}$	0.0003	0.0180	0.0192	0.7566	1.2998	0.5431

Table 6

Relative absolute biases, relative mean square errors of ML estimates and 95% confidence intervals of the reliability and hazard rate at $t_0 = (0.3, 0.6, 1)$, from IKum for different sample size n, censoring size r =90% and repetitions N = 10000

n	r	t_0	Estimates	MLE	RAB	RMSE	Lower	Upper	Length
30	27	0.3	$\hat{R}(t_0)$	0.9929	0.0031	0.0081	0.9514	1.0321	0.0820
			$\hat{h}(t_0)$	0.0241	0.0289	0.0763	0.0000	0.1526	0.2570
		0.6	$\hat{R}(t_0)$	0.9783	0.0074	0.0208	0.8761	1.0803	0.2042
			$\hat{h}(t_0)$	0.1217	0.0054	0.1352	0.0000	0.8726	1.5017
		1	$\hat{R}(t_0)$	0.8888	0.0052	0.0267	0.6250	1.1526	0.5276
			$\hat{h}(t_0)$	0.4780	0.0234	0.3617	0.0000	1.7176	2.4794
60	54	0.3	$\hat{R}(t_0)$	0.9985	0.0033	0.0085	0.9672	1.0250	0.0587
			$\hat{h}(t_0)$	0.0115	0.0316	0.0811	0.0000	0.1061	0.1891
		0.6	$\hat{R}(t_0)$	0.9916	0.0088	0.0226	0.9386	1.0450	0.1059
			$\hat{h}(t_0)$	0.0496	0.0133	0.0463	0.0000	0.2963	0.4932
		1	$\hat{R}(t_0)$	0.9245	0.0084	0.0265	0.7576	1.0913	0.3336
			$\hat{h}(t_0)$	0.3548	0.0073	0.1333	0.0000	1.1187	1.5272
100	90	0.3	$\hat{R}(t_0)$	0.9989	0.0035	0.0088	0.9822	1.0157	0.0334
			$\hat{h}(t_0)$	0.0034	0.0333	0.0840	0.0000	0.0567	0.1066
		0.6	$\hat{R}(t_0)$	0.9960	0.0093	0.0234	0.9634	1.0285	0.0651
			$\hat{h}(t_0)$	0.0308	0.0159	0.0427	0.0000	0.1472	0.2327
		1	$\hat{R}(t_0)$	0.9427	0.0104	0.0291	0.8158	1.0690	0.2538
			$\hat{h}(t_0)$	0.2906	0.0024	0.0753	0.0000	0.8830	1.1841

Table 7

Bayes estimates of the parameters, reliability and hazard rate at $t_0 = 0.5$ and their mean squared errors, standard deviations and credible intervals based on Type II censoring

n	r	estimates	Sd	MSE	Mean	LL	UL	CI
20	14	α^*	0.0052	5.132E-5	0.9947	0.9808	0.9999	0.0191
		β^*	0.0051	4.946E-5	0.9949	0.9809	0.9999	0.0190
		$R^*_1(t)$	0.0023	2.374E-5	0.6662	0.6607	0.6707	0.0100
		$h^*_1(t)$	0.0041	4.159E-5	0.6654	0.6555	0.6727	0.0172
	18	α^*	0.0033	3.048E-5	0.9966	0.9876	0.9999	0.0123
		β^*	0.0029	2.964E-5	0.997	0.9891	0.9999	0.0108
		$R^*_1(t)$	0.0014	1.329E-5	0.6665	0.6633	0.6693	0.0060
		$h^*_1(t)$	0.0025	2.336E-5	0.6657	0.6594	0.6701	0.0107
30	21	α^*	0.0024	2.496E-5	0.9976	0.9913	0.9999	0.0086
		β^*	0.0021	2.108E-5	0.9978	0.992	0.9999	0.0079
		$R^*_1(t)$	0.0010	9.658E-6	0.6665	0.6642	0.6686	0.0044
		$h^*_1(t)$	0.0018	1.799E-5	0.666	0.6615	0.6691	0.0076
	27	α^*	0.0015	1.447E-5	0.9985	0.9944	1.0000	0.0056
		β^*	0.0013	1.295E-5	0.9987	0.9951	1.0000	0.0049
		$R^*_1(t)$	6.445E-4	6.252E-6	0.6666	0.6652	0.6679	0.0027
		$h^*_1(t)$	0.0011	1.13E-5	0.6662	0.6633	0.6682	0.0049
60	42	α^*	6.599E-4	6.842E-6	0.9993	0.9976	1.0000	0.0024
		β^*	4.893E-4	4.660E-6	0.9995	0.9982	1.0000	0.0018
		$R^*_1(t)$	2.525E-4	2.449E-6	0.6667	0.6661	0.6672	0.0011
		$h^*_1(t)$	4.873E-4	4.886E-6	0.6664	0.6652	0.6672	0.0020
	54	α^*	3.739E-4	3.694E-6	0.9996	0.9986	1.0000	0.0014
		β^*	3.314E-4	3.154E-6	0.9997	0.9988	1.0000	0.0012
		$R^*_1(t)$	1.575E-4	1.569E-6	0.6666	0.6663	0.6670	0.0007
		$h^*_1(t)$	2.867E-4	2.880E-6	0.6666	0.6659	0.6671	0.0012
100	70	α^*	2.407E-4	2.474E-6	0.9998	0.9991	1.0000	0.0009
		β^*	1.697E-4	1.759E-6	0.9999	0.9994	1.0000	0.0006
		$R^*_1(t)$	8.990E-5	9.803E-7	0.6667	0.6665	0.6669	0.0004
		$h^*_1(t)$	1.764E-4	1.893E-6	0.6666	0.6661	0.6669	0.0008
	80	α^*	1.313E-4	1.432E-6	0.999	0.999	1.0000	0.0010
		β^*	1.157E-4	1.230E-6	0.999	0.999	1.0000	0.0010
		$R^*_1(t)$	5.534E-5	6.043E-7	0.6667	0.6664	0.6666	0.0002
		$h^*_1(t)$	1.009E-4	1.109E-6	0.6666	0.6665	0.6666	0.0001

APPENDIX B

- 1- If W has exponential pdf with parameter (β) , then the random variable $T = \left[(1 - e^{-w})^{-\frac{1}{\alpha}} - 1 \right]$ has IKum distribution with parameters (α, β) and given in Equation (2).

Proof : Let $f(w)$ be the pdf of exponential (β) distribution, given by

$$f(w, \beta) = \beta e^{-\beta w}, \quad w > 0.$$

Assuming $T = \left[(1 - e^{-w})^{-\frac{1}{\alpha}} - 1 \right]$, then $w = -\ln(1 - (1 + t)^{-\alpha})$,

$$\text{and } \left| \frac{dw}{dt} \right| = \frac{\alpha(1+t)^{-(\alpha+1)}}{(1-(1+t)^{-\alpha})},$$

hence, the pdf of T is given by

$$f_T(t) = f_W(-\ln(1 - (1 + t)^{-\alpha})) \left| \frac{dw}{dt} \right|,$$

then

$$f(t, \alpha, \beta) = \alpha \beta (1 + t)^{-(\alpha+1)} (1 - (1 + t)^{-\alpha})^{\beta-1}, \quad t > 0; \alpha, \beta > 0,$$

which is the pdf for IKum distribution (α, β) .

2- Limiting distributions

- a) Let $f_T(t)$ be the pdf of IKum (α, β) distribution, given by (2)

Considering $Y_1 = \beta^{-\frac{1}{\alpha}}(1 + T)$,

Then the interval $(0, \infty)$ is transformed to $(\beta^{-\frac{1}{\alpha}}, \infty)$,

$$\text{and } t = \beta^{\frac{1}{\alpha}} y_1 - 1, \quad \left| \frac{dt}{dy_1} \right| = \beta^{\frac{1}{\alpha}}.$$

Therefore, the pdf of y_1 is given by

$$f_{Y_1}(y_1) = f_T(\beta^{\frac{1}{\alpha}} y_1 - 1) \left| \frac{dt}{dy_1} \right|,$$

hence

$$f(y_1) = \alpha y_1^{-(\alpha+1)} \left(1 - \frac{y_1^{-\alpha}}{\beta} \right)^{\beta-1}. \quad \text{As } \beta \rightarrow \infty, \text{ the pdf of } y_1 \text{ tends to}$$

$$f(y_1; \alpha) = \alpha y_1^{-(\alpha+1)} e^{-y_1^{-\alpha}}, \quad y_1 > 0, \text{ which is the pdf of the inverted Weibull distribution.}$$

- b) Let $f_T(t)$ be the pdf of IKum (α, β) distribution, given by (2)

Considering $Y_2 = \alpha(1 - (1 + T)^{-1})$,

then the interval $(0, \infty)$ is transformed to $(0, \alpha)$,

$$\text{and } t = \left(1 - \frac{y_2}{\alpha} \right)^{-1} - 1, \quad \left| \frac{dt}{dy_2} \right| = \frac{1}{\alpha} \left(1 - \frac{y_2}{\alpha} \right)^{-2}.$$

Therefore, the pdf of y_2 is given by

$$f_{Y_2}(y_2) = f_T\left(\left(1 - \frac{y_2}{\alpha}\right)^{-1} - 1\right) \left| \frac{dt}{dy_2} \right|,$$

$$\text{then } f(y_2; \alpha, \beta) = \beta \left(1 - \frac{y_2}{\alpha} \right)^{\alpha-1} \left(1 - \left(1 - \frac{y_2}{\alpha} \right)^{\alpha} \right)^{\beta-1}. \quad \text{As } \alpha \rightarrow \infty \text{ the pdf of } y_2 \text{ tends to}$$

$$f(y_2; \beta) = \beta e^{-y_2} (1 - e^{-y_2})^{\beta-1}, \quad y_2 > 0, \text{ which is the pdf of the generalized exponential distribution.}$$

- c) Let $f_T(t)$ be the pdf of IKum (α, β) distribution, given by (2)

Let $Y_3 = \alpha \left(1 - \beta^{\frac{1}{\alpha}} (1 + T)^{-1} \right)$, then the interval $(0, \infty)$ is transformed to

$$\left(\alpha \left(1 - \beta^{\frac{1}{\alpha}} \right), \alpha \right), \text{ and } t = \beta^{\frac{1}{\alpha}} \left(1 - \frac{y_3}{\alpha} \right)^{-1} - 1, \quad \left| \frac{dt}{dy_3} \right| = \frac{\beta^{\frac{1}{\alpha}}}{\alpha} \left(1 - \frac{y_3}{\alpha} \right)^{-2}.$$

Therefore, the pdf of y_3 is given by $f_{Y_3}(y_3) = f_T\left(\beta^{\frac{1}{\alpha}} \left(1 - \frac{y_3}{\alpha} \right)^{-1} - 1\right) \left| \frac{dt}{dy_3} \right|$,

$$\text{hence, the pdf of } y_3 \text{ is } f(y_3; \alpha, \beta) = \left(1 - \frac{y_3}{\alpha} \right)^{\alpha-1} \left(1 - \frac{\left(1 - \frac{y_3}{\alpha} \right)^{\alpha}}{\beta} \right)^{\beta-1}.$$

As both $\beta \rightarrow \infty$ and $\alpha \rightarrow \infty$, the pdf of y_3 tends to

$$f(y_3) = e^{-y_3} \exp[-e^{-y_3}], \quad y_3 > 0,$$

which is the pdf of the standard extreme value distribution of the first Type.

Goodness of Fit for the Inverse Flexible Weibull Distribution

Waleed M. Afify⁽¹⁾

Ahmed Ramzy⁽²⁾

Abstract Goodness of fit tests are considered for testing the two-parameter inverse flexible Weibull distribution based on complete and type-II censored sampling. The maximum likelihood method of estimation is used for estimating the parameters. Critical values are obtained for the modified Kolmogorov-Smirnov, Anderson-Darling and Cramer-von Mises test statistics through Monte Carlo simulation. The power study of these tests will conduct for several of alternative distributions. Meanwhile, in case of Type-II right censored samples the power comparisons among these modified tests, a goodness of fit tests based on data transformations which have been introduced by Goldmann et al. (2015) and the exact goodness of fit test which has been suggested by Grane' (2012) are investigated for a number of alternative distributions. Moreover, real data set will be used as an illustrative example for goodness of fit tests for the inverse flexible Weibull distribution.

Keywords Goodness of fit; Censored samples; Inverse flexible Weibull; Monte Carlo simulation.

1. Introduction

Latterly, Bebbington et al. (2007) introduced a distribution as a generalization of Weibull distribution with just two-parameter which is so simple and very flexible to deal with survival analysis and model reliability data. It is called the flexible Weibull distribution (FW). A flexible Weibull distribution has probability density function (pdf) given by

$$g(y; \alpha, \beta) = \left(\alpha + \frac{\beta}{y^2} \right) e^{\left(\alpha y - \frac{\beta}{y} \right)} \exp[-e^{\left(\alpha y - \frac{\beta}{y} \right)}], \quad y, \alpha, \beta > 0, \quad (1.1)$$

and its cumulative distribution function (cdf) takes the following form

$$G(y; \alpha, \beta) = 1 - \exp[-e^{\left(\alpha y - \frac{\beta}{y} \right)}], \quad y, \alpha, \beta > 0, \quad (1.2)$$

and the survival function is given by

(1) Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Kafer EL-sheikh University, Egypt.

(2) Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Kafer EL-sheikh University, Egypt.

$$\bar{G}(y; \alpha, \beta) = \exp[-e^{\left(\frac{\alpha y - \beta}{y}\right)}], y, \alpha, \beta > 0. \quad (1.3)$$

When $\beta = 0$, distribution (1.3) reduces to type I extreme value distribution. Consequently, one can transform this distribution to the standard Weibull and hence the name flexible Weibull was given. In case of $\beta = \alpha$, this distribution reduces to a special case of the Slymen-Lachenbruch modified Weibull distribution, see Slymen and Lachenbruch (1984).

Recently, EL-Gohary et al. (2015) proposed a new two-parameter distribution which is the distribution of the reciprocal of a random variable has the flexible Weibull distribution. It is called the inverse flexible Weibull distribution (IFW). Suppose a non negative random variable X has the flexible Weibull distribution with the probability density function and the cumulative distribution function as in (1.1) and (1.2) respectively. Define a new random variable $Y = 1/X$, then the random variable Y has the inverse flexible Weibull distribution. The probability density function and the cumulative distribution function of Y are given consecutively by

$$f(y; \alpha, \beta) = \left(\beta + \frac{\alpha}{y^2}\right) e^{\left(\frac{\alpha}{y} - \beta y\right)} \exp[-e^{\left(\frac{\alpha}{y} - \beta y\right)}], y, \alpha, \beta > 0, \quad (1.4)$$

and

$$F(y; \alpha, \beta) = \exp[-e^{\left(\frac{\alpha}{y} - \beta y\right)}], y, \alpha, \beta > 0, \quad (1.5)$$

and the survival function is given by

$$\bar{F}(y; \alpha, \beta) = 1 - \exp[-e^{\left(\frac{\alpha}{y} - \beta y\right)}], y, \alpha, \beta > 0.$$

EL-Gohary et al. (2015) pointed out that, the inverse flexible Weibull distribution has the better fit with the comparison of various well-known classes of life time distributions. Therefore, the inverse flexible Weibull distribution play an important role in variety fields of research and development such as, reliability, biology, medicine and engineering.

Often one wants to determine the level of conformity among the data and a hypothesized distribution, in hopes of detecting a model that strictly depicts the population from which the data arose. This can be executed by finding suitable goodness of fit test statistic. One simplest approach for testing goodness of fit is based on intelligible measures of disparity among the empirical distribution function (EDF) and the hypothesized distribution function ($F_0(y)$). The three most important and known tests in this band are Kolmogorov-Smirnov (KS), Cramer-von Mises (CvM) and Anderson-Darling (AD) test statistics. The exact distributions of theses tests are more difficult to infer (extremely conservative) when the data are censored or the hypothesized distribution function relies on unknown parameters which must be estimated from the sample data.

Do not forget that, censored data arises in a natural way when there is a cost or a time constraint or there are difficulties in measuring the failure time of a

test items particularly for survival data. Possible censoring samples may be from right, or left. Right censoring occurs when the event has not experienced the failure at the last censoring time. If the event of interest has already occurred for the individuals before that person is observed in the study, it is left censoring. On the other hand, this paper deals with one type of censoring, Type-II censoring scheme. In Type-II censoring scheme, (failure-censoring) the experiment is achieved as soon as a predetermined number (say, k) of items have been failed. Therefore, put n items on a test, only first k failures will be noted, where, $k < n$. The data gained from such a restricted life test will be noted as a Type-II censored sample.

In this respect, many studies have been discussed some of which are concerned with modification KS, CvM, and AD tests for goodness of fit when the parameters are estimated and others when the data are censored. Lilliefors (1967,1969) established tables of critical values using Monte Carlo technique for the KS test in case of the mean and the variance of a normal distribution are estimated and in case of the mean of an exponential distribution is estimated. The expanded studies for testing goodness of fit of a normal and exponential distribution when the parameters are estimated have been discussed by Stephens (1970,1974), Durbin (1975), and Green and Hegazy(1976). Smith and Bain (1976), Stephens (1977), Littell et al. (1979) have discussed goodness of fit tests for the two parameters Weibull distribution. A modified version of the KS statistic for testing normality when the parameters are estimated has been proposed by Michael(1983). Also, Woodruff et al. (1984) modified the KS for gamma distribution in case of unknown location and scale parameters, but with known shape parameter. Furthermore, Yen and Moore (1988) constructed tables of critical values for modified CvM and AD statistics for the Laplace distribution in case of unknown location and scale estimators. Also, Hassan (2005) introduced modified goodness of fit tests(KS, CvM, and AD) for the generalized exponential distribution when the scale and shape parameters are estimated in case of complete samples.

When the data are censored various nice papers have been proposed. Barr and Davidson (1973) proposed a goodness of fit test for censored samples which is the censored version of the KS test. Tables of asymptotic distribution of Barr and Davidson's statistic have been introduced by Koziol and Byar (1975). Also, Pettitt and Stephens (1976) introduced modified CvM and AD goodness of fit tests for censored samples. They also proposed tables of the asymptotic distributions for these modified tests. Dufour and Maag (1978) introduced tables of critical values of Barr and Davidson's test statistic. In addition, Ashour and Abd-Elfattah (1994) proposed tables of critical values using Monte Carlo simulation for modified KS, CvM, and AD statistics for Burr type XII distribution when the shape parameters are estimated in case of complete and type-II censored samples. Also, Castro-Kuriss (2011) presented a goodness of fit test with type-II censored samples from a general location-scale distribution when the parameters are estimated. This test consider a generalization in case of censored samples of that introduced by Michael (1983). Moreover, when the data are type-II right censored Goldmann et al. (2015) introduced number of goodness of fit procedures that based on data transformations.

Another approach to test the goodness of fit with censored samples was developed by Grane' (2012). She obtained an exact goodness of fit tests for Type-I right, Type II, and Type-II doubly censored data.

In this paper, we drive the maximum likelihood estimators of the unknown parameters in case of Type-II right censored samples and modify the KS, CVM, AD tests for the inverse flexible Weibull distribution with unknown parameters based on complete and Type-II right censored samples. Tables of critical values for these modified tests can be obtained using Monte Carlo simulation. The power comparison between these modified tests is discussed. Also to compare the efficiency between these modified tests, a goodness of fit tests based on data transformations which have been introduced by Goldmann et al. (2015) and the test of Grane' (2012), the power study for different sample sizes is obtained. In addition to, areal data set as an illustrative example for goodness of fit tests for the inverse flexible Weibull distribution has been analyzed.

This paper is a systematic as follows. In Section 2 we present the maximum likelihood estimators (MLE) of the unknown parameters in case of complete samples and we drive the maximum likelihood estimators of the unknown parameters in case of Type-II right censored samples. Section 3 presents the modified KS, CVM, and AD tests for the inverse flexible Weibull distribution when the parameters are estimated in both cases complete and Type-II right censored samples. Moreover, we introduce a goodness of fit tests based on data transformations which have been proposed by Goldmann et al. (2015) and the exact goodness of fit test with Type-II right censored samples which has been suggested by Grane' (2012). In section 4 critical values for these modified tests are produced. In section 5 the power comparison among these modified tests is investigated. On the other hand, this comparison is also conducted between these tests, a goodness of fit tests based on data transformations and the exact test of Grane'. Finally, in section 6 we use a real data set as an illustrative example for goodness of fit tests for the inverse flexible Weibull distribution.

2. Maximum Likelihood Estimation

Let $y_1, y_2, \dots, y_n$ represent independent and identically distributed (iid) random sample from (1.4) with order statistics $y_{(1)}, y_{(2)}, \dots, y_{(n)}$ and let $y_{(1)}, y_{(2)}, \dots, y_{(k)}$ where, $k \leq n$ represent the ordered sample obtained from Type-II right censoring sample. In this section, one can introduce the maximum likelihood estimators of the two unknown parameters α and β for the inverse flexible Weibull distribution in two cases, complete and Type-II right censored samples.

(2.1) Maximum Likelihood Estimators Based on Complete Samples

When the data are complete the maximum likelihood estimates of the two parameters of the inverse flexible Weibull distribution can be obtained through maximizing the following likelihood function

$$L = \prod_{i=1}^n f(y_i; \alpha, \beta)$$

Clearly, EL-Gohary et al. (2015) showed that the maximum likelihood estimators of α and β can be obtained by solving the following two equations

$$\sum_{i=1}^n \left[\frac{1}{y_i} - \frac{e^{\left[\frac{\alpha}{y_i} - \beta y_i \right]}}{y_i} + \frac{1}{\beta y_i^2 + \alpha} \right] = 0 \quad (2.1)$$

and

$$\sum_{i=1}^n \left[-y_i + y_i e^{\left[\frac{\alpha}{y_i} - \beta y_i \right]} + \frac{y_i^2}{\beta y_i^2 + \alpha} \right] = 0 \quad (2.2)$$

(2.2) Maximum Likelihood Estimators Based on Type-II Right Censored Samples

When the data come from Type-II right censoring scheme the likelihood function for this type of censored sample is given by

$$L = \frac{n!}{(n-k)!} \prod_{i=1}^k f(y_{(i)}) [1 - F(y_{(k)})]^{n-k} \quad (2.3)$$

we can drive the maximum likelihood estimators of the unknown parameters in this case as follow:

Substituting (1.4) and (1.5) in (2.3), we have

$$L = \frac{n!}{(n-k)!} \prod_{i=1}^k \left(\beta + \frac{\alpha}{y_{(i)}^2} \right) e^{\sum_{i=1}^k \left[\frac{\alpha}{y_{(i)}} - \beta y_{(i)} \right]} e^{\left[-\sum_{i=1}^k \left[\frac{\alpha}{y_{(i)}} - \beta y_{(i)} \right] \right]} \left[1 - e^{\left[\frac{\alpha}{y_{(k)}} - \beta y_{(k)} \right]} \right]^{n-k}$$

The log-likelihood function is given by

$$\begin{aligned} \log L = & \log \left(\frac{n!}{(n-k)!} \right) + \sum_{i=1}^k \log \left(\beta + \frac{\alpha}{y_{(i)}^2} \right) + \sum_{i=1}^k \left[\frac{\alpha}{y_{(i)}} - \beta y_{(i)} \right] - \sum_{i=1}^k e^{\left[\frac{\alpha}{y_{(i)}} - \beta y_{(i)} \right]} \\ & + (n-k) \log \left[1 - e^{\left[\frac{\alpha}{y_{(k)}} - \beta y_{(k)} \right]} \right] \end{aligned}$$

The MLEs $\hat{\alpha}$ and $\hat{\beta}$ of α and β can be obtained as the simultaneous solution of the following two nonlinear equations:

$$\sum_{i=1}^k \frac{1/y_{(i)}^2}{\left(\beta + \alpha/y_{(i)}^2 \right)} + \sum_{i=1}^k \frac{1}{y_{(i)}} - \sum_{i=1}^k \frac{e^{\left[\frac{\alpha}{y_{(i)}} - \beta y_{(i)} \right]}}{y_{(i)}} + (n-k) \frac{\left(e^{\left[\frac{\alpha}{y_{(k)}} - \beta y_{(k)} \right]} \right) \left(e^{\left[\frac{\alpha}{y_{(k)}} - \beta y_{(k)} \right]} \frac{1}{y_{(k)}} \right)}{1 - \left(e^{\left[\frac{\alpha}{y_{(k)}} - \beta y_{(k)} \right]} \right)} = 0,$$

(2.4)

$$\sum_{i=1}^k \frac{1}{(\beta + \alpha/y_{(i)}^2)} - \sum_{i=1}^k y_{(i)} + \sum_{i=1}^k y_{(i)} e^{\left[\frac{\alpha}{y_{(i)}} - \beta y_{(i)}\right]} - (n-k) \frac{\left(e^{-e^{\left[\frac{\alpha}{y_{(k)}} - \beta y_{(k)}\right]}} \right) \left(e^{\left[\frac{\alpha}{y_{(k)}} - \beta y_{(k)}\right]} y_{(k)} \right)}{1 - e^{-e^{\left[\frac{\alpha}{y_{(k)}} - \beta y_{(k)}\right]}}} = 0$$

(2.5)

From the previous two subsections, it is obvious that, the preceding equations are in implicit form, hence one can use numerical iteration to solve them.

The following table shows the values of absolute relative bias (ARB) and relative root mean square error (RRMSE) of the maximum likelihood estimators (MLEs) which are obtained by Monte Carlo simulation (based on 5000 Monte Carlo runs) for different sample sizes ($n=10, 20, 30, 40, 50$, and 100) based on Type-II censoring scheme from inverse flexible Weibull distribution which censoring fractions are $k/n = 0.8, 1$ using MathCAD (14) program. The true parameter values are chosen as ($\alpha = 0.375, \beta = 1.125$), where

$$ARB(\hat{\psi}) = \left| \frac{(\hat{\psi} - \psi)}{\psi} \right| \text{ and } RRMSE(\hat{\psi}) = \frac{\sqrt{MSE(\hat{\psi} - \psi)}}{\psi}.$$

Table 1: The ARB and RRMSE of the parameters $\hat{\psi} \equiv (\alpha, \beta)$

n	k	MLE of α		MLE of β	
		ARB	RRMSE	ARB	RRMSE
10	10	2.781363	3.009116	1.202548	1.385352
	8	2.862416	3.156097	1.260645	1.514649
20	20	2.417069	2.506594	1.010041	1.090791
	16	2.42008	2.525018	1.008395	1.111716
30	30	2.309029	2.363317	0.955649	1.006499
	24	2.316544	2.382749	0.943535	1.010382
40	40	2.272203	2.312	0.935129	0.973471
	32	2.242328	2.287646	0.906019	0.955157
50	50	2.244555	2.276351	0.921	0.951996
	40	2.201501	2.235611	0.884434	0.922983
100	100	2.183981	2.199159	0.891393	0.906659
	90	2.141939	2.158887	0.851317	0.870495

From the previous table, one can conclude that, as the sample size increases the values of absolute relative bias (ARB) and relative root mean square error (RRMSE) of the estimated parameters decrease in almost of the times. This means that the maximum likelihood estimates provide asymptotically normally distributed and consistent estimators for the parameters. In additional to these

values are small, which is main the MLEs are a good estimators for the parameters of inverse flexible Weibull distribution.

3. Goodness of Fit Tests

Let Y define a random variable with distribution function $F(y)$. It is worth mentioning that, in goodness of fit testing one can strive to verify the compatibility between set of data and the hypothesized distribution $F_0(y)$. This problem can be cast as $H_0: F(y) = F_0(y)$ against the alternative $H_1: F(y) \neq F_0(y)$. In this section, we introduce the modified KS, CVM, and AD tests for the inverse flexible Weibull distribution when the parameters are estimated in both cases complete and Type-II right censored samples. Furthermore, we introduce a goodness of fit tests based on data transformations which have been proposed by Goldmann et al. (2015) and the exact goodness of fit test with Type-II right censored samples which has been suggested by Grane' (2012).

(3.1) Modified Goodness of Fit Tests for Complete Samples with Unknown Parameters

Suppose a given random sample of size n from inverse flexible Weibull distribution with cumulative distribution function given in (1.5). Let $y_{(1)}, y_{(2)}, \dots, y_{(n)}$ be the order statistics, and the null hypothesis is

H_0 : the complete sample $y_{(1)} < y_{(2)} < \dots < y_{(n)}$ comes from the inverse flexible Weibull distribution with unknown parameters, then

1- The modified Kolmogorov-Smirnov test

$$\begin{aligned} D_n^+ &= \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - F_0(y_{(i)}; \hat{\alpha}; \hat{\beta}) \right\}; \\ D_n^- &= \max_{1 \leq i \leq n} \left\{ F_0(y_{(i)}; \hat{\alpha}; \hat{\beta}) - \frac{i-1}{n} \right\}; \\ D &= \max(D_n^+, D_n^-). \end{aligned} \quad (3.1)$$

2-The modified Cramer-Von Mises test

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^n \left[F_0(y_{(i)}; \hat{\alpha}; \hat{\beta}) - \frac{2i-1}{2n} \right]^2. \quad (3.2)$$

3-The modified Anderson-Darling statistics

$$A_n^2 = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\log \left(F_0(y_{(i)}; \hat{\alpha}; \hat{\beta}) \right) + \log \left(1 - F_0(y_{(n+1-i)}; \hat{\alpha}; \hat{\beta}) \right) \right]. \quad (3.3)$$

(3.2) Modified Goodness of Fit Tests for Type-II Right Censored Samples with Unknown Parameters

Consider a Type-II right censoring scheme $y_{(1)}, y_{(2)}, \dots, y_{(k)}$ from inverse flexible Weibull distribution with cumulative distribution function given in (1.5), and the null hypothesis is

H_0 : the censored sample $y_{(1)} < y_{(2)} < \dots < y_{(k)}$ comes from the inverse flexible Weibull distribution with unknown parameters, then

1- The modified Kolmogorov-Smirnov test

$$\begin{aligned} D_{n,k}^+ &= \max_{1 \leq i \leq k} \left\{ \frac{i}{n} - F_0(y_{(i)}; \hat{\alpha}; \hat{\beta}) \right\}; \\ D_{n,k}^- &= \max_{1 \leq i \leq k} \left\{ F_0(y_{(i)}; \hat{\alpha}; \hat{\beta}) - \frac{i-1}{n} \right\}; \\ D_{n,k} &= \max(D_{n,k}^+, D_{n,k}^-). \end{aligned} \quad (3.4)$$

2-The modified Cramer-Von Mises test

$$W_{n,k}^2 = \sum_{i=1}^k \left[F_0(y_{(i)}; \hat{\alpha}; \hat{\beta}) - \frac{2i-1}{2n} \right]^2 + \frac{k}{12n^2} - \frac{n}{3} \left[\frac{k}{n} - F_0(y_{(k)}; \hat{\alpha}; \hat{\beta}) \right]^3. \quad (3.5)$$

3-The modified Anderson-Darling statistics

$$\begin{aligned} A_{n,k}^2 &= -\frac{1}{n} \sum_{i=1}^k (2i-1) \left[\log(F_0(y_{(i)}; \hat{\alpha}; \hat{\beta})) - \log(1 - F_0(y_{(i)}; \hat{\alpha}; \hat{\beta})) \right] \\ &\quad - \frac{1}{n} \left[(k-n)^2 \log(1 - F_0(y_{(k)}; \hat{\alpha}; \hat{\beta})) - k^2 \log(F_0(y_{(k)}; \hat{\alpha}; \hat{\beta})) + n^2 F_0(y_{(k)}; \hat{\alpha}; \hat{\beta}) \right] \\ &\quad - 2 \sum_{i=1}^k \log[1 - F_0(y_{(i)}; \hat{\alpha}; \hat{\beta})]. \end{aligned} \quad (3.6)$$

(3.3) Goodness of Fit Tests with Data Transformations for Type-II Right Censored Samples

Consider a Type-II right censoring scheme $y_{(1)}, y_{(2)}, \dots, y_{(k)}$ from a random sample of size n where, $k < n$. The null hypothesis under consideration is:

H_0 : the censored sample $y_{(1)} < y_{(2)} < \dots < y_{(k)}$ comes from the inverse flexible Weibull distribution with unknown parameters. In this subsection, we introduce KS, CVM, and AD tests based on data transformations with Type-II right censored samples which have been introduced by Goldmann et al. (2015), see also Lin et al. (2008) as in the following steps:

Step(1) From the observed Type-II right censoring scheme $y_{(1)}, y_{(2)}, \dots, y_{(k)}$ estimate α and β by solving (2.4) and (2.5) using numerical iteration.

Step(2) Calculate $\hat{U}_{(i)} = F_0(y_{(i)}; \hat{\alpha}; \hat{\beta})$, $i = 1, 2, \dots, k$, where $\hat{U}_{(1)}, \hat{U}_{(2)}, \dots, \hat{U}_{(k)}$ is the censored set of order statistics in a random sample of size n .

Step(3) Transform $\hat{U}_{(i)}$ to $u_{(i)}$ using the following transformation (see Lin et al. (2008)).

$$u_{(i)} = \left[\frac{1 - \hat{U}_{(i)}}{1 - \hat{U}_{(i-1)}} \right]^{n-i+1}, i = 1, 2, \dots, k$$

where, $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ is the complete set of order statistics in a sample of size k .

One can use several other transformations. For more information see Goldmann et al. (2015).

Step(4) Calculate $x_i = \Phi^{-1}(u_{(i)})$, where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard normal.

Step(5) Compute $z_i = (x_i - \bar{x})/s_x$ where $\bar{x} = k^{-1} \sum_{i=1}^k x_i$, and $s_x^2 = (k-1)^{-1} \sum_{i=1}^k (x_i - \bar{x})^2$

Step(6) Apply a goodness of fit tests to the values z_i , $i = 1, 2, \dots, k$, then

1- The Kolmogorov-Smirnov test

$$\begin{aligned} D^+ &= \max_{1 \leq i \leq k} \left\{ \frac{i}{k} - \Phi(z_i) \right\}; \\ D^- &= \max_{1 \leq i \leq k} \left\{ \Phi(z_i) - \frac{i-1}{k} \right\}; \\ D &= \max(D^+, D^-). \end{aligned} \quad (3.7)$$

2-The Cramer-Von Mises test

$$W = \frac{1}{12k} + \sum_{i=1}^k \left[\Phi(z_i) - \frac{2i-1}{2k} \right]^2. \quad (3.8)$$

3-The Anderson-Darling statistics

$$A^2 = -k - \frac{1}{k} \sum_{i=1}^k \left[(2i-1) \log \Phi(z_i) + (2k+1-2i) \log (1 - \Phi(z_i)) \right] \quad (3.9)$$

(3.4) Grane' Goodness of Fit Test for Type-II Right Censoring Scheme

Grane' (2012) suggested exact goodness of fit tests which can be used in case of right, left, and doubly censored samples. In this study, we focus on Type-II right censoring scheme. Suppose that, $y_{(1)}, y_{(2)}, \dots, y_{(k)}$, are observed from a sample of size n where, $k < n$. This censoring scheme is defined as Type-II right censoring scheme. Simply, one can transform these available data set to m_i values ($m_i = F_0(y_i)$) is then $m_{(1)} < m_{(2)} < \dots < m_{(k)}$. Under the null hypothesis of uniformity, Grane' (2012) proposed the modified goodness of fit test statistic for Type-II right censoring scheme which is defined as:

$$Q_{kn} = \sum_{i=1}^k a_i m_{(i)} \quad (3.10)$$

where $a_i = \frac{6[(2i-1)k - n^2]}{(n^2 k)}$, for $1 \leq i \leq k-1$ and $a_k = \frac{6(k-1)(n^2 - k(k-1))}{n^2 k}$

Lower and upper tail critical values at the sample size 10, 20, 30 for 5% and 2.5% significance levels and different proportions $\left(p = \frac{k}{n}\right)$ of observed data in the sample can be found in Grane' (2012).

4. Critical Values for the Test Statistics

The Monte Carlo technique to produce the critical values can be depicted in the following steps:

Step(1) Generate a random sample $y_1, y_2, \dots, y_n$ from inverse flexible Weibull distribution with parameters α and β , n takes the values 10, 20, 30, 40, 50 and 100 and the censoring fraction $p = k/n = 0.8$.

Step(2) Arrange the random samples in an ascending order to get the order statistics as defined in section 2.

Step (3) Use these generated random samples to estimate α and β by solving (2.1) and (2.2) in case of complete samples and (2.4) and (2.5) when the data are Type-II censoring using numerical iteration.

Step(4) Use the maximum likelihood estimates of the parameters α and β , say $\hat{\alpha}$ and $\hat{\beta}$ to determine the hypothesized cumulative distribution function, $F_0(y_{(i)}; \hat{\alpha}; \hat{\beta})$ of the inverse flexible Weibull distribution. The suitable test statistic was calculated for the given values of n .

Step(5) The test statistics were modified by replacing the inverse flexible Weibull parameters by their maximum likelihood estimators as in (3.1), (3.2), and (3.3) in case of complete samples and as in (3.4), (3.5), and (3.6) when the data are Type-II censoring.

Step(6) The preceding procedure is repeated (steps 1-5) 5000 times to produce 5000 independent values of the test statistics for the KS, CvM, and AD tests.

Step(7) For every kind, the 5000 values of the test statistics are arranged, and the 80%, 85%, 90%, 95%, and 99% quantiles are obtained. These provided the critical values for particular significance levels γ of 0.20, 0.15, 0.10, 0.05, 0.01.

Similarly, on can get the critical values for Grane' test using (3.10) and for the goodness of fit tests based on data transformations using (3.7), (3.8) and (3.9) as the same manner.

Table 2: Critical values for modified KS, CvM, and AD tests for complete samples.

n	Statistics	Critical values				
		$\gamma = 0.20$	$\gamma = 0.15$	$\gamma = 0.10$	$\gamma = 0.05$	$\gamma = 0.01$
10	KS	0.226397	0.237784	0.252365	0.273854	0.311594
	CvM	0.086842	0.096079	0.110645	0.1318	0.186296

20	AD	0.516918	0.566212	0.636719	0.750417	1.030745
	KS	0.168307	0.176471	0.188326	0.205987	0.238067
	CvM	0.093341	0.103812	0.118911	0.145821	0.203503
30	AD	0.556807	0.612735	0.691534	0.826521	1.119335
	KS	0.139781	0.146362	0.154702	0.169582	0.197452
	CvM	0.094965	0.106035	0.12038	0.147922	0.211501
40	AD	0.570455	0.627454	0.705656	0.851066	1.124083
	KS	0.121727	0.128098	0.135453	0.147503	0.172522
	CvM	0.09287	0.103975	0.119647	0.148472	0.212195
50	AD	0.564712	0.618806	0.70462	0.849247	1.167251
	KS	0.109077	0.115015	0.12231	0.133462	0.155319
	CvM	0.094752	0.107718	0.12482	0.154016	0.241648
100	AD	0.569072	0.63485	0.725489	0.87846	1.294858
	KS	0.082289	0.086384	0.091853	0.101713	0.118475
	CvM	0.109002	0.122452	0.142884	0.178798	0.279881
	AD	0.648307	0.726229	0.823512	1.028783	1.536865

Table 3: Critical values for modified KS, CvM, and AD tests for censored samples , Grane' test and CvM test based on data transformation(W) where $k/n = 0.8$.

n	k	Statistics	Critical values				
			$\gamma = 0.20$	$\gamma = 0.15$	$\gamma = 0.10$	$\gamma = 0.05$	$\gamma = 0.01$
10	8	KS	0.211092	0.221637	0.235694	0.257608	0.293582
		CvM	0.064356	0.070612	0.080768	0.0977	0.13197
		AD	0.346871	0.382838	0.427832	0.507955	0.68865
		Grane'	1.443286	1.46936	1.500991	1.537656	1.596714
		W	0.096997	0.107231	0.123037	0.14951	0.217979
20	16	KS	0.156757	0.164569	0.174914	0.190293	0.222297
		CvM	0.068124	0.076127	0.086983	0.10263	0.145023
		AD	0.36665	0.407262	0.464916	0.54056	0.7484
		Grane'	1.423138	1.446304	1.473588	1.512674	1.571875
		W	0.098517	0.109399	0.124856	0.155207	0.219735
30	24	KS	0.129834	0.136498	0.144462	0.1577	0.184194
		CvM	0.067258	0.076109	0.087778	0.107326	0.150093
		AD	0.365588	0.407439	0.464559	0.557796	0.76719
		Grane'	1.417011	1.435398	1.459469	1.494556	1.544063
		W	0.0966	0.10904	0.123935	0.152097	0.217869
40	32	KS	0.113058	0.11825	0.125396	0.137632	0.160959
		CvM	0.067708	0.075003	0.085441	0.10501	0.148852
		AD	0.365933	0.403352	0.455967	0.554191	0.764026
		Grane'	1.406987	1.425353	1.44578	1.474079	1.519575
		W	0.097205	0.108767	0.124137	0.152136	0.218396
50	40	KS	0.101706	0.106998	0.113136	0.124357	0.146058
		CvM	0.069137	0.077374	0.089403	0.109544	0.160593
		AD	0.373399	0.414904	0.477775	0.576396	0.813605
		Grane'	1.402334	1.416306	1.433825	1.463122	1.515488
		W	0.095562	0.106602	0.12393	0.151077	0.22838
100	80	KS	0.074501	0.078441	0.083477	0.092071	0.108326
		CvM	0.07432	0.083953	0.098411	0.12269	0.185171
		AD	0.400541	0.451259	0.51924	0.645631	0.94671
		Grane'	1.384781	1.396286	1.409556	1.429826	1.469393
		W	0.097	0.108277	0.124775	0.150935	0.218757

From the preceding two tables it is clear that:

1. All critical values for the modified test statistics increase as the significant level γ decreases for fixed sample size n .
2. The critical values for the KS test statistics decrease as the sample size n increases for fixed significant level γ .
3. From Table 2 in all situations, for the specific sample size the critical values for AD test statistics have the highest values with corresponding to the critical values for the KS and CvM test statistics for all significant levels. In addition to, the critical values for CvM and AD test statistics increase as the sample size n increases for fixed significant level γ in most of time. Moreover, the critical values for KS test statistics are bigger than the critical values for CvM test statistics when $n=10, 20$ for all γ and for $n=30$ for all γ except $\gamma=0.01$ and for $n=40$ for all γ except $\gamma=0.05, 0.01$. Then as n increases and γ decreases the critical values for KS test statistics become greater than the critical values for CvM test statistics. Whereas, the critical values for CvM test statistics become greater than the critical values for KS test statistics when $n=50$ for all γ except $\gamma=0.20, 0.15$ and when $n=100$ for all γ .
4. From Table 3 the critical values for CvM, AD and the CvM based on data transformation (W) test statistics usually increase as the sample size n increases for fixed significant level γ . Whilst, the critical values for Grane' test statistics decrease as the sample size n increases for fixed significant level γ . Additionally, the critical values for Grane' test statistics are bigger than the critical values for KS, CvM, AD and W test statistics in all situations. Moreover, the critical values for W test statistics are bigger than the critical values for CvM test statistics in all cases. Also the critical values for KS test statistics are bigger than the critical values for CvM test statistics when $n=10, 20, 30, 40$ and 50 for all γ except when $n=50$ for $\gamma=0.01$. Whereas, the critical values for CvM test statistics are bigger than the critical values for KS when $n=100$ for all γ except $\gamma=0.20$.

5. Power Comparisons

In this section, a power study is executed to inspect the power of the modified test statistics, a goodness of fit tests based on data transformations with Type-II right censored samples which have been introduced by Goldmann et al. (2015) (we use only CvM statistics as example under this method) and the exact goodness of fit test with Type-II right censored samples which has been suggested by Grane' (2012). In this study the null hypothesis is that the random sample comes from the inverse flexible Weibull distribution with unknown parameters and the alternatives considered for tests of the inverse flexible Weibull distribution included the exponential, normal and chi-square distributions with one degree of freedom. The power was determined by generating 5000 random samples of size n . Tables 4-9 provide the power of each test statistics of size $n=10, 20, 30, 40, 50$ and 100 for $\gamma=0.20, 0.15, 0.10, 0.05, 0.01$. Tables 4-6, provide the power for complete samples when the alternative distribution is exponential, normal, and chi-square respectively. In additional, Tables 7-9, provide the power for Type-II censored samples where censoring levels

$k/n = 0.8$ and when the alternative distribution is exponential, normal, and chi-square respectively.

Table 4: Power of tests for inverse flexible Weibull distribution when the alternative distribution is Exp (1) based on complete samples.

n	Statistics	Power of the test				
		$\gamma = 0.20$	$\gamma = 0.15$	$\gamma = 0.10$	$\gamma = 0.05$	$\gamma = 0.01$
10	KS	0.6016	0.5374	0.4646	0.3664	0.233
	CvM	0.653	0.6012	0.535	0.4548	0.3008
	AD	0.7196	0.6708	0.6018	0.5096	0.343
20	KS	0.562	0.5002	0.419	0.3176	0.1746
	CvM	0.6226	0.5664	0.4964	0.398	0.2562
	AD	0.681	0.6268	0.5574	0.455	0.29
30	KS	0.5516	0.4958	0.4302	0.314	0.1624
	CvM	0.6038	0.5522	0.4926	0.395	0.2438
	AD	0.6752	0.621	0.5512	0.446	0.2968
40	KS	0.548	0.4812	0.413	0.316	0.1628
	CvM	0.6246	0.5672	0.5004	0.3904	0.2436
	AD	0.6798	0.6264	0.5534	0.4438	0.2808
50	KS	0.5596	0.4954	0.4166	0.3148	0.1644
	CvM	0.6148	0.553	0.4838	0.3832	0.2052
	AD	0.6764	0.6166	0.543	0.4336	0.2436
100	KS	0.4796	0.4238	0.3528	0.2404	0.1098
	CvM	0.5426	0.484	0.412	0.3222	0.157
	AD	0.6024	0.5364	0.4668	0.346	0.1726

Table 5: Power of tests for inverse flexible Weibull distribution when the alternative distribution is N (0,1) based on complete samples.

n	Statistics	Power of the test				
		$\gamma = 0.20$	$\gamma = 0.15$	$\gamma = 0.10$	$\gamma = 0.05$	$\gamma = 0.01$
10	KS	0.6042	0.5474	0.4626	0.3622	0.2242
	CvM	0.6588	0.61	0.5404	0.4488	0.297
	AD	0.724	0.6748	0.6038	0.5086	0.3392
20	KS	0.5736	0.5112	0.4278	0.318	0.1734
	CvM	0.6338	0.5808	0.5086	0.4188	0.258
	AD	0.6956	0.646	0.577	0.4772	0.3002
30	KS	0.556	0.4994	0.4238	0.3168	0.1688
	CvM	0.612	0.5596	0.4972	0.3974	0.2428
	AD	0.6658	0.6138	0.5472	0.4378	0.2984
40	KS	0.5624	0.5002	0.4316	0.32	0.1602
	CvM	0.63	0.5764	0.506	0.405	0.2402
	AD	0.6896	0.6364	0.5624	0.4506	0.2776
50	KS	0.5464	0.4818	0.4048	0.3046	0.1576
	CvM	0.61	0.5458	0.4728	0.372	0.196
	AD	0.676	0.6176	0.5332	0.42	0.2286
100	KS	0.4816	0.4202	0.3508	0.2382	0.1096
	CvM	0.5446	0.49	0.4128	0.3106	0.1548
	AD	0.6034	0.5354	0.4632	0.3418	0.173

Table 6: Power of tests for inverse flexible Weibull distribution when the alternative distribution is chi-square with one degree of freedom based on complete samples.

n	Statistics	Power of the test				
		$\gamma = 0.20$	$\gamma = 0.15$	$\gamma = 0.10$	$\gamma = 0.05$	$\gamma = 0.01$
10	KS	0.6104	0.544	0.4734	0.3662	0.2244
	CvM	0.668	0.6244	0.549	0.4546	0.2924
	AD	0.7334	0.685	0.6156	0.5132	0.3296
20	KS	0.5664	0.5066	0.425	0.3178	0.179
	CvM	0.6212	0.568	0.5014	0.4098	0.2654
	AD	0.6862	0.6338	0.5668	0.4682	0.3054
30	KS	0.5456	0.4874	0.4178	0.313	0.1648
	CvM	0.6088	0.5482	0.4874	0.3922	0.2442
	AD	0.6704	0.615	0.5432	0.4376	0.2924
40	KS	0.5392	0.4796	0.4098	0.312	0.1594
	CvM	0.6162	0.5642	0.4922	0.395	0.2422
	AD	0.6784	0.6222	0.5482	0.4436	0.2794
50	KS	0.5632	0.49	0.4146	0.312	0.167
	CvM	0.6182	0.561	0.4834	0.3814	0.2066
	AD	0.68	0.6232	0.5452	0.4276	0.2358
100	KS	0.4782	0.4226	0.3486	0.2332	0.1072
	CvM	0.5418	0.4834	0.408	0.3084	0.151
	AD	0.597	0.5286	0.4536	0.3372	0.1656

It is clear from Table 4 up to Table 6 that:

1. In all situations, the AD test statistic has the highest power in most of the time.
2. The KS test statistics has the lowest power in all cases.
3. The exponential distribution is a best alternative distribution when (n=100) in most of time.
4. The normal distribution is a best alternative distribution when (n=20, 30, 40) in most of time.
5. The chi-square distribution is a best alternative when (n=10) in most of time.

Table 7: Power of tests for inverse flexible Weibull distribution when the alternative distribution is Exp (1) based on Type-II censoring and $k/n = 0.8$.

n	k	Statistics	Power of the test				
			$\gamma = 0.20$	$\gamma = 0.15$	$\gamma = 0.10$	$\gamma = 0.05$	$\gamma = 0.01$
10	8	KS	0.6434	0.5882	0.5128	0.408	0.266
		CvM	0.6342	0.5906	0.5264	0.4388	0.3166
		AD	0.6846	0.638	0.5822	0.4982	0.3624
		Grane' test	0.8614	0.8562	0.8454	0.8342	0.8152
		W	0.621	0.5742	0.5054	0.4118	0.24
20	16	KS	0.6126	0.556	0.4802	0.381	0.2154
		CvM	0.6474	0.5976	0.5316	0.4588	0.3106
		AD	0.6956	0.6448	0.5776	0.5032	0.343
		Grane' test	0.9788	0.975	0.9718	0.9656	0.9564
		W	0.6018	0.5494	0.4844	0.3758	0.234
30	24	KS	0.6228	0.5566	0.4836	0.384	0.214

40	32	CvM	0.6594	0.608	0.5444	0.4544	0.3148
		AD	0.7084	0.6568	0.598	0.5034	0.351
		Grane' test	0.996	0.9954	0.9948	0.993	0.99
		W	0.619	0.5544	0.4914	0.394	0.239
		KS	0.6098	0.552	0.4816	0.372	0.2108
50	40	CvM	0.6558	0.6128	0.557	0.4636	0.3196
		AD	0.7044	0.6602	0.602	0.5034	0.346
		Grane' test	0.9994	0.9994	0.9994	0.9982	0.9976
		W	0.5992	0.5436	0.4842	0.3908	0.2334
		KS	0.6232	0.5592	0.4892	0.3746	0.2116
100	80	CvM	0.6628	0.6152	0.548	0.4538	0.3046
		AD	0.7062	0.6574	0.5932	0.4952	0.3364
		Grane' test	0.9998	0.9998	0.9998	0.9998	0.9996
		W	0.6158	0.5562	0.482	0.3842	0.2134
		KS	0.5872	0.5224	0.4546	0.3432	0.1774
		CvM	0.636	0.5808	0.516	0.4182	0.2604
		AD	0.6818	0.6224	0.5526	0.4538	0.2814
		Grane' test	1	1	1	1	1
		W	0.6012	0.5388	0.4682	0.378	0.2316
		KS	0.6232	0.5592	0.4892	0.3746	0.2116

Table 8: Power of tests for inverse flexible Weibull distribution when the alternative distribution is $N(0,1)$ based on Type-II censoring and $k/n = 0.8$.

n	k	Statistics	Power of the test				
			$\gamma = 0.20$	$\gamma = 0.15$	$\gamma = 0.10$	$\gamma = 0.05$	$\gamma = 0.01$
10	8	KS	0.6494	0.5888	0.5072	0.3998	0.2598
		CvM	0.6304	0.593	0.5298	0.4408	0.318
		AD	0.6826	0.6396	0.5858	0.4998	0.3592
		Grane' test	0.9928	0.9922	0.9902	0.9888	0.9854
		W	0.6258	0.5738	0.4994	0.3978	0.24
20	16	KS	0.6224	0.5658	0.492	0.386	0.2188
		CvM	0.6566	0.6082	0.5488	0.4682	0.3182
		AD	0.7058	0.653	0.5936	0.5172	0.3572
		Grane' test	1	1	1	1	0.9996
		W	0.614	0.5628	0.4956	0.3936	0.2466
30	24	KS	0.6016	0.5398	0.4748	0.367	0.2076
		CvM	0.6564	0.597	0.5324	0.4402	0.3024
		AD	0.7058	0.6528	0.5896	0.4882	0.34
		Grane' test	1	1	1	1	1
		W	0.6098	0.5484	0.4814	0.3816	0.229
40	32	KS	0.6154	0.5656	0.4908	0.3748	0.2112
		CvM	0.6614	0.6158	0.5602	0.4702	0.3194
		AD	0.7118	0.6686	0.606	0.5086	0.3568
		Grane' test	1	1	1	1	1
		W	0.6052	0.549	0.48	0.3794	0.2358
50	40	KS	0.6206	0.56	0.4938	0.3798	0.2108
		CvM	0.668	0.619	0.5524	0.4584	0.2948
		AD	0.713	0.6652	0.5878	0.495	0.325
		Grane' test	1	1	1	1	1
		W	0.6132	0.561	0.4938	0.402	0.2242
100	80	KS	0.5804	0.5252	0.4444	0.3276	0.17
		CvM	0.637	0.5828	0.5064	0.4128	0.2484

	AD	0.6864	0.6258	0.5564	0.442	0.268
	Grane' test	1	1	1	1	1
	W	0.5994	0.5454	0.4768	0.385	0.231

Table 9: Power of tests for inverse flexible Weibull distribution when the alternative distribution is chi-square with one degree of freedom based on Type-II censoring and $k/n = 0.8$.

n	k	Statistics	Power of the test				
			$\gamma = 0.20$	$\gamma = 0.15$	$\gamma = 0.10$	$\gamma = 0.05$	$\gamma = 0.01$
10	8	KS	0.6484	0.589	0.5194	0.4056	0.2592
		CvM	0.6348	0.5956	0.5326	0.4436	0.3158
		AD	0.6876	0.639	0.5846	0.5	0.3652
		Grane' test	0.8324	0.8246	0.8154	0.8056	0.788
		W	0.608	0.5586	0.4922	0.4014	0.2382
20	16	KS	0.6234	0.561	0.4824	0.3812	0.2184
		CvM	0.6534	0.6034	0.5392	0.4612	0.312
		AD	0.703	0.651	0.584	0.511	0.344
		Grane' test	0.959	0.9568	0.9514	0.9476	0.9424
		W	0.5974	0.5432	0.4762	0.3812	0.2398
30	24	KS	0.62	0.5664	0.4916	0.3878	0.226
		CvM	0.669	0.6158	0.5494	0.4562	0.3134
		AD	0.7148	0.663	0.5972	0.505	0.3536
		Grane' test	0.9856	0.9848	0.9838	0.9808	0.9774
		W	0.6002	0.5366	0.471	0.3776	0.2236
40	32	KS	0.6184	0.5642	0.492	0.3756	0.211
		CvM	0.6634	0.6232	0.5654	0.4698	0.3162
		AD	0.7094	0.6642	0.604	0.5072	0.3472
		Grane' test	0.9952	0.995	0.9944	0.9934	0.991
		W	0.5922	0.5324	0.4722	0.3802	0.2402
50	40	KS	0.6248	0.5624	0.4922	0.3758	0.202
		CvM	0.6712	0.6184	0.5512	0.456	0.2976
		AD	0.7172	0.6662	0.5968	0.5002	0.3322
		Grane' test	0.999	0.999	0.9986	0.9982	0.9976
		W	0.61	0.5596	0.4904	0.3958	0.2172
100	80	KS	0.6036	0.5384	0.4624	0.3476	0.1816
		CvM	0.6492	0.5968	0.525	0.4284	0.2616
		AD	0.6898	0.6326	0.5596	0.4574	0.2832
		Grane' test	1	1	1	1	1
		W	0.6098	0.555	0.4866	0.3964	0.2382

It is obvious from Table 7 up to Table 9 that:

1. In all situations, the Grane' test has the highest power in all cases.
2. When the exponential is the alternative distribution the W test statistics has the lowest power when $n=10, 20, 30, 40$ and 50 at $\gamma=0.20, 0.15$. Whereas, the KS test statistics has the lowest power when $n=100$ for all γ and for the same previous sample sizes at $\gamma=0.10, 0.05, 0.01$ except when $n=10$ at $\gamma=0.10, 0.01$ and when $n=50$ at $\gamma=0.10$ and when $n=20$ at $\gamma=0.05$.
3. When the normal is the alternative distribution the W test statistics has the lowest power when $n=10$ for all γ and when $n=40$ at $\gamma=0.20, 0.15, 0.10$. While, the KS test statistics has the lowest power when $n=20, 30, 50, 100$ for

all γ except when $n=20$ at $\gamma=0.20, 0.15$ and when $n=30$ at $\gamma=0.01$ and when $n=50$ at $\gamma=0.20$.

4. When the chi-square is the alternative distribution the W test statistics has the lowest power when $n=10, 20, 30, 40, 50$ for all γ except when $n=20, 50$ at $\gamma=0.05, 0.01$ and when $n=40$ at $\gamma=0.01$. However, when $n=100$ for all γ , the KS test statistics has the lowest power.
5. The power of CvM test statistics is greater than W test statistics in all time.

6. Real Data Example

In this section, one can use real data set as example to illustrate how the introduced tests can be used to test the goodness of fit of a hypothesized continuous distribution in case of complete and a Type-II censored sample. The data set in Table 10, gives the lifetimes of 50 devices that were provided by Aarset (1987) cited in EL-Gohary et al. (2015).

Table 10: Life time of 50 devices, see EL-Gohary et al (2015).

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

The maximum likelihood estimators of unknown parameters, the KS, CvM, and AD test statistics for complete sample, the KS, CvM, AD, Grane' and W test statistics for Type-II right censoring scheme (when $k=20$) and the critical values for each test which are generated based on 5000 Monte Carlo runs for sample size $n=50$ and significance level $\gamma=0.05$ in Table 11 and Table 12 respectively.

Table 11: The maximum likelihood estimators of unknown parameters and the KS, CvM, and AD test statistics for complete sample.

The model	The MLEs of parameters		Test statistics			Critical values		
	$\hat{\alpha}$	$\hat{\beta}$	KS	CvM	AD	KS	CvM	AD
Inverse flexible Weibull distribution	0.165	0.0238	0.2761	1.653	8.4002	0.133804	0.155595	0.902218

Table 12: The maximum likelihood estimators of unknown parameters and the KS, CvM, AD, Grane' and W test statistics for Type-II right censoring scheme.

The model	The MLEs of parameters		Test statistics					Critical values				
	$\hat{\alpha}$	$\hat{\beta}$	KS	CvM	AD	Grane' test	W test	KS	CvM	AD	Grane' test	W test
Inverse flexible Weibull distribution	0.1693	0.0154	0.2715	0.6854	3.4475	2.9335	1.66667	0.09305	0.034	0.25299	1.21659	0.14737

it is obvious that, the all test statistics in two cases complete and Type-II censored samples reach the same conclusion of admission that, the inverse flexible Weibull distribution is not well fitted model to this data set at a significance level of 0.05.

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Discriminating between Gamma and log-logistic distributions

Elsayed A. Elsherpieny¹, Hiba Z. Muhammed², Noha U. M. M. Radwan³

Abstract

Gamma and log-logistic distributions are two popular distributions for analyzing lifetime data. In this paper we consider the problem of discriminating between the two distribution functions. It is assumed that the data are coming either from gamma or log-logistic distributions. The ratio of the maximized likelihood test and the ratio of Minimized Kullback-Leibler Divergence (RMKLD) are used to discriminate between them. The asymptotic distributions of the logarithm of the ratio of the maximized likelihood are obtained. These asymptotic results can be used to estimate the probability of correct selection. Some simulation experiments are performed to see how the probability of correct selection (PCS) under each model work for small sample sizes. Two real life data are analyzed to see how the proposed method works in practice.

Keywords: Asymptotic distributions; Gamma distribution; Log-logistic distribution; Goodness of fit tests; Likelihood ratio statistic; the ratio of Minimized Kullback-Leibler Divergence.

1 Introduction

Choosing the correct or best-fitting distribution for a given data set is an important issue. Most of the times distribution functions may provide a similar data fit but still it is desirable to select the correct or more nearly correct model. The problem of choosing the correct model has been attempted by many researchers. Cox (1961) discussed the effect of choosing the wrong model. Cox (1962) tackled the problem of discriminating between the log-normal and the exponential distributions, based on the likelihood function, and derived the asymptotic distribution of the likelihood ratio statistic. Wiens (1999) discussed the effect of choosing the wrong model through a real data example and by using log-normal and gamma models. Also Pascual (2005) discussed the effect of misspecification on the maximum likelihood estimates between the log-normal and gamma distribution functions.

Special attention has been given to discriminate some specific distribution functions, due to the increase of their applications. Cox (1961, 1962) suggested tests of separate families of hypotheses and applied his test to discriminate between lognormal and exponential distribution. Atkinson (1970) combined Cox's two hypotheses in a general model and applied his test to discriminate between lognormal and exponential distribution. Dumonceaux et al. (1973)

^{1,2} Institute of Statistical Studies and Research, Cairo University, Egypt.

³ Ph.D. degrees from Institute of Statistical Studies and Research, Cairo University, Egypt.

applied the ratio of the maximized likelihoods (RML) statistic on discriminating between normal and Cauchy distributions, and between normal and exponential distributions. Dumonceaux and Antle (1973) developed the application of this approach to discriminate between lognormal and Weibull distributions. Pereira (1977) developed another two tests to discriminate between lognormal and Weibull distributions. While Bain and Engelhardt (1980) covered the case Weibull versus gamma. They evaluated the performance of a discrimination procedure based on likelihood ratio in terms of the probability of correct selection. Kappenman (1982) studied the PCS for the pairs Weibull versus lognormal, Weibull versus gamma, and gamma versus lognormal. Firth (1988), discussed the problem of discriminating between the lognormal and gamma distributions. The ratio of the maximized likelihood procedure has been applied in discriminating between distributions by many authors as Pandey et al. (1991) , Fearn and Nebenzahl (1991) , Gupta and Kundu (2003), Gupta and Kundu (2004), Kundu and Manglick (2005), Kundu and Raqab (2007), Radwan (2009), Dey and Kundu (2010), Ashkar and Aucion (2012), Raqab (2013), Elsherpieny et al. (2013), Rao and Kantam (2014).

The RML is mostly used in the literature to choose suitable models among others. But in some cases it is not the best test statistic and definitely it relays on sample size. For instance, to discriminate between Weibull and log-normal, RML is not a suitable selection criteria due to lack of inclusion of the location parameter of log-normal (μ) and on the other hand, for small sample size it has low power (Dumonceaux and Antle,1973). Mohd Saat et al (2008) compared RML with Vuong's closeness test to discriminate between gamma and Weibull, in which they found both methods relatively similar. Despite of significant amount of work on discrimination by different method, there is no much work to use the Kullback-Leibler Divergence (KLD), Bromideh (2012) considered the KLD in discriminating between Weibull and log-normal distributions, and Bromideh and Valizadeh (2013) considered the ratio of Minimized Kullback-Leibler Divergence (RMKLD) in discriminating between gamma and log-normal distributions. This paper aims to fill this gap by examining selection between gamma and log-logistic distributions.

In this paper the problem of discriminating between gamma and log-logistic distribution functions is considered. The maximized likelihood ratio test is used to discriminate between them in Section 2. The asymptotic distribution of the logarithm of the ratio of the maximized likelihood is obtained in Section 3. These asymptotic results are used to estimate the probability of correct selection and to calculate the minimum sample size needed to discriminate between the two distribution functions for a user specified probability of correct selection and a given tolerance limit in Section 4. A simulation experiment is given in Section 5. The PCS calculated from RMKLD is given in section 6 and two real data life are analyzed to see how the proposed method works in practice in Section 7. Finally the paper is concluded in Section 8.

2 Likelihood Ratio Test

Suppose $X_1, \dots, X_n$ are independent and identically distributed (i.i.d) random variables. The probability density function (p.d.f.) of gamma distribution, denoted by $GA(\lambda, \alpha)$, with scale parameter $\lambda > 0$ and shape parameter $\alpha > 0$ is given by

$$f_{GA}(x; \lambda, \alpha) = \frac{1}{\lambda^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\lambda}} \quad \lambda, \alpha > 0 \quad x > 0. \quad (1)$$

The probability density function (p.d.f.) of log-logistic distribution, denoted by $LL(\varepsilon, \sigma)$, with scale parameter $\varepsilon > 0$ and shape parameter $\sigma > 0$ is given by

$$f_{LL}(x; \varepsilon, \sigma) = \frac{(\sigma/\varepsilon)(x/\varepsilon)^{\sigma-1}}{[1 + (x/\varepsilon)^\sigma]^2}, \quad \varepsilon, \sigma > 0 \quad x > 0. \quad (2)$$

Let us define the likelihood functions assuming that the data are coming from a $GA(\lambda, \alpha)$ or from $LL(\varepsilon, \sigma)$ as

$$\ln L_{GA}(\lambda, \alpha) = -n \ln(\Gamma(\alpha)) - n \alpha \ln(\lambda) + (\alpha - 1) \sum_{i=1}^n \ln(x_i) - \frac{n \bar{x}}{\lambda}$$

$$\ln L_{LL}(\varepsilon, \sigma) = n \ln \sigma - n \sigma \ln \varepsilon + (\sigma - 1) \sum_{i=1}^n \ln x_i - 2 \sum_{i=1}^n \ln [1 + (x_i / \varepsilon)^\sigma]$$

$\hat{\alpha}$ and $\hat{\lambda}$ can be obtained as a solutions of these two equations

$$\hat{\lambda} - \frac{\bar{x}}{\hat{\alpha}} = 0$$

$$-n \psi(\hat{\alpha}) - n \ln(\hat{\lambda}) + \sum_{i=1}^n \ln(x_i) = 0, \quad \text{Where } \psi(\hat{\alpha}) = \frac{\partial \ln(\Gamma \alpha)}{\partial \alpha}$$

Similarly, $\hat{\varepsilon}$ and $\hat{\sigma}$ can be obtained as a solutions of these two equations

$$n - 2 \sum_{i=1}^n \frac{(x_i / \hat{\varepsilon})^{\hat{\sigma}}}{[1 + (x_i / \hat{\varepsilon})^{\hat{\sigma}}]} = 0.$$

$$\frac{n}{\hat{\sigma}} + \sum_{i=1}^n \ln x_i - 2 \sum_{i=1}^n \frac{(x_i / \hat{\varepsilon})^{\hat{\sigma}} \ln x_i}{[1 + (x_i / \hat{\varepsilon})^{\hat{\sigma}}]} = 0.$$

The logarithm of RML can be obtained as follows

$$T = \ln \frac{L_{GA}(\hat{\lambda}, \hat{\alpha})}{L_{LL}(\hat{\varepsilon}, \hat{\sigma})} = \ln L_{GA}(\hat{\lambda}, \hat{\alpha}) - \ln L_{LL}(\hat{\varepsilon}, \hat{\sigma}) \quad (3)$$

Here $(\hat{\lambda}, \hat{\alpha})$ and $(\hat{\varepsilon}, \hat{\sigma})$ are the maximum likelihood estimators of (λ, α) and (ε, σ) respectively.

In this discrimination procedure, choose gamma distribution if

$$T > 0 \quad \text{i.e., if } \ln L_{GA}(\lambda, \alpha) > \ln L_{LL}(\hat{\varepsilon}, \hat{\sigma}) \quad \text{Otherwise choose Log-logistic.}$$

3 Asymptotic Properties of The RML Under Null Hypotheses

In this Section, the asymptotic distributions of the RML statistics will be obtained under null hypotheses in two different cases. From now on the almost sure convergence will be denoted by a.s.

For any Borel measurable function $h(\cdot)$, $E_{GA}(h(U))$ and $V_{GA}(h(U))$ denote mean and variance of $h(U)$ under the assumption that U follows gamma distribution. Similarly define $E_{LL}(h(U))$ and $V_{LL}(h(U))$ as mean and variance of $h(U)$ under the assumption that U follows log-logistic distribution. Also if $g(\cdot)$ and $h(\cdot)$ are two Borel measurable functions, define along the same line :

$\text{Cov}_{GA}(g(U), h(U)) = E_{GA}(g(U).h(U)) - E_{GA}(g(U)).E_{GA}(h(U))$, and similarly

$\text{Cov}_{LL}(g(U), h(U)) = E_{LL}(g(U).h(U)) - E_{LL}(g(U)).E_{LL}(h(U))$,

Lemma 1. Under the assumption that the data are from $GA(\alpha, \lambda)$, we have the following results as $n \longrightarrow \infty$

a- $\hat{\alpha} \longrightarrow \alpha$, $\hat{\lambda} \longrightarrow \lambda$ where

$$E_{GA}(\ln f_{GA}(x; \lambda, \alpha)) = \max_{\bar{\alpha}, \bar{\lambda}} E_{GA}(\ln f_{GA}(x; \bar{\lambda}, \bar{\alpha}))$$

b- $\hat{\sigma} \longrightarrow \tilde{\sigma}$, $\hat{\varepsilon} \longrightarrow \tilde{\varepsilon}$, where

$$E_{GA}(\ln f_{LL}(x; \tilde{\varepsilon}, \tilde{\sigma})) = \max_{\sigma, \varepsilon} E_{GA}(\ln f_{LL}(x; \varepsilon, \sigma))$$

Denote

$$T^* = \ln \left[\frac{L_{GA}(\lambda, \alpha)}{L_{LL}(\tilde{\varepsilon}, \tilde{\sigma})} \right]$$

c- $\frac{1}{\sqrt{n}} [T - E_{GA}(T)]$ Asymptotically equivalent to $\frac{1}{\sqrt{n}} [T^* - E_{GA}(T^*)]$

Proof of Lemma 1. The proof follows using similar arguments as of White (1982.theorem 1) and therefore it is omitted.

Lemma 2. Under the assumption that the data are from $LL(\varepsilon, \sigma)$, we have the following results as $n \longrightarrow \infty$

a- $\hat{\sigma} \longrightarrow \sigma$, $\hat{\varepsilon} \longrightarrow \varepsilon$ where

$$E_{LL}(\ln f_{LL}(x; \varepsilon, \sigma)) = \max_{\bar{\sigma}, \bar{\varepsilon}} E_{LL}(\ln f_{LL}(x; \bar{\varepsilon}, \bar{\sigma}))$$

b- $\hat{\alpha} \longrightarrow \tilde{\alpha}$, $\hat{\lambda} \longrightarrow \tilde{\lambda}$, a.s, where

$$E_{LL}(\ln f_{GA}(x; \tilde{\lambda}, \tilde{\alpha})) = \max_{\alpha, \lambda} E_{LL}(\ln f_{GA}(x; \lambda, \alpha))$$

Denote

$$T^* = \ln \left[\frac{L_{GA}(\tilde{\lambda}, \tilde{\alpha})}{L_{LL}(\tilde{\varepsilon}, \tilde{\sigma})} \right]$$

c- $\frac{1}{\sqrt{n}} [T - E_{LL}(T)]$ Asymptotically equivalent to $\frac{1}{\sqrt{n}} [T^* - E_{LL}(T^*)]$

Proof of Lemma 2. The proof follows using similar arguments as of White (1982.theorem 1) and therefore it is omitted.

Theorem 1. Under the assumption that the data are from a gamma distribution, the distribution of T is approximately normally distributed with mean $E_{GA}(T)$ and variance $V_{GA}(T)$.

Proof: Using the central limit theorem and using (c) of Lemma 1, one can easily shows that $\frac{1}{\sqrt{n}} [T^* - E_{GA}(T^*)]$ is asymptotically normally distributed.

Theorem 2. Under the assumption that the data are from log-logistic distribution, the distribution of T is approximately normally distributed with mean $E_{LL}(T)$ and variance $V_{LL}(T)$. The proof follows the same line as of Theorem 1.

Case (1): The data are coming from a gamma distribution and the alternative is a log-logistic distribution.

Assumed that n data points $x_1, x_2, \dots, x_n$ are obtained from $GA(\lambda, \alpha)$ with scale parameter λ and shape parameter α .

Now to obtain $\tilde{\varepsilon}$ and $\tilde{\sigma}$ as defined in Lemma 1, let us define

$$\begin{aligned} h(\varepsilon, \sigma) &= E_{GA} [\ln(f_{LL}(x; \varepsilon, \sigma))] \\ &= E_{GA} \left[\ln \sigma - \ln \varepsilon + (\sigma - 1) [\ln x - \ln \varepsilon] - 2 \ln \left[1 + \left(\frac{x}{\varepsilon} \right)^\sigma \right] \right] \end{aligned}$$

Let $x = \lambda y$, then

$$h(\varepsilon, \sigma) = \ln(\sigma) + (\sigma - 1) [E_{GA}(\ln(y) + \ln(\lambda))] - \sigma \ln \varepsilon - 2 E_{GA} \ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right] \quad (4)$$

Where $y \sim$ gamma distribution $(1, \alpha)$

Differentiating Equation (4) with respect to ε , σ and equating to zero .

Therefore, $\tilde{\varepsilon}$ and $\tilde{\sigma}$ can be obtained as solutions of the following equations

$$\tilde{\varepsilon} + \frac{\tilde{\sigma}}{2 \left[\frac{\partial E_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right] \right)}{\partial \varepsilon} \right]} = 0 \quad (5)$$

$$\frac{1}{\sigma} + [E_{GA} \ln(y) + \ln(\lambda)] - \ln \tilde{\varepsilon} - 2 \frac{\partial E_{GA} \ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right]}{\partial \varepsilon} = 0 \quad (6)$$

From these equations it is clear that $\tilde{\varepsilon}$ and $\tilde{\sigma}$ are both functions of λ and α .

To obtain $E_{GA}(T)$ and $V_{GA}(T)$ denote,

$$AM_{GA} = \lim_{n \rightarrow \infty} \frac{1}{n} E_{GA}(T) \quad \text{and} \quad AV_{GA} = \lim_{n \rightarrow \infty} \frac{1}{n} V_{GA}(T) \quad ,$$

Therefore for large n we have,

$$\lim_{n \rightarrow \infty} \frac{1}{n} E_{GA}(T) = AM_{GA} = E_{GA} \left[\ln f_{GA}(x; \lambda, \alpha) - \ln f_{LL}(x; \tilde{\varepsilon}, \tilde{\sigma}) \right]$$

$$AM_{GA} = -\alpha \ln \lambda - \ln \Gamma(\alpha) + (\alpha - \sigma)(E_{GA} \ln(y) + \ln(\lambda)) - \frac{\lambda \alpha}{\lambda} - \ln(\sigma) - \sigma \ln \varepsilon - 2E_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right] \right) \quad (7)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} V_{GA}(T) = AV_{GA} = V_{GA} \left[\ln f_{GA}(x; \lambda, \alpha) - \ln f_{LL}(x; \tilde{\varepsilon}, \tilde{\sigma}) \right]$$

$$AV_{GA} = (\alpha - \sigma)^2 \psi_1(\alpha) + \alpha + 4V_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right] \right) - 2(\alpha - \sigma) \text{cov}(\ln(\lambda y), y)$$

$$+ 4(\alpha - \sigma) \text{cov}(\ln(\lambda y), \ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right]) - 4 \text{cov}(y, \ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right]) \quad (8)$$

For more details see Appendix (1).

Case (2): The data are coming from a log-logistic distribution and the alternative is a gamma distribution.

Assumed that n data points $x_1, x_2, \dots, x_n$ are obtained from LL(ε, σ) with scale parameter ε and shape parameter σ . Now to obtain $\tilde{\lambda}, \tilde{\alpha}$ let us define

$$g(\lambda, \alpha) = E_{LL} [\ln f_{GA}(x; \lambda, \alpha)]$$

$$= -\alpha \ln \lambda - \ln \Gamma(\alpha) + (\alpha - 1)E_{LL}(\ln(x)) - \frac{1}{\lambda} E_{LL}(x)$$

$$g(\lambda, \alpha) = -\alpha \ln \lambda - \ln \Gamma(\alpha) + (\alpha - 1)(E(\ln y) + \ln \varepsilon) - \frac{1}{\lambda} \frac{\varepsilon \pi / \sigma}{\sin(\pi / \sigma)} \quad (9)$$

If $\sigma > 1$ else undefined, Where $Y \sim LL(1,1)$

Differentiating Equation (9) with respect to α, λ and equating to zero. Therefore, $\tilde{\lambda}$ and $\tilde{\alpha}$ can be obtained as solutions of the following equations

$$-\ln \tilde{\lambda} - \psi(\alpha) + E_{LL}(y) + \ln \varepsilon = 0 \quad (10)$$

$$\frac{\tilde{\alpha}}{\tilde{\lambda}} + \frac{1}{\tilde{\lambda}^2} \frac{\varepsilon \pi / \sigma}{\sin(\pi / \sigma)} = 0 \quad (11)$$

Where $y \sim LL(1, \sigma)$

From these equations it is clear that $\tilde{\lambda}$ and $\tilde{\alpha}$ are both functions of ε and σ .
Now to obtain $E_{LL}(T)$ and $V_{LL}(T)$, denote

$$\begin{aligned} AM_{LL} &= \lim_{n \rightarrow \infty} \frac{1}{n} E_{LL}(T) \text{ and } AV_{LL} = \lim_{n \rightarrow \infty} \frac{1}{n} V_{LL}(T) \quad , \\ \lim_{n \rightarrow \infty} \frac{1}{n} E_{LL}(T) &= AM_{LL} = E_{LL} \left[\ln f_{GA}(X; \tilde{\lambda}, \tilde{\alpha}) - \ln f_{LL}(X; \varepsilon, \sigma) \right] \\ &= E_{LL} \left(\ln f_{GA}(X; \tilde{\lambda}, \tilde{\alpha}) \right) - E_{LL} \left[\ln f_{LL}(X; \varepsilon, \sigma) \right] \\ AM_{LL} &= -\alpha \ln \lambda - \ln \Gamma \alpha - \frac{1}{\lambda} \frac{\varepsilon \pi / \sigma}{\sin(\pi / \sigma)} + (\alpha - \sigma)(E(\ln y) + \ln \varepsilon) - \ln(\sigma) \\ &\quad + \sigma \ln \varepsilon + 2E_{LL}(\ln(1+z)) \quad \text{If } \sigma > 1 \text{ else undefined} \quad (12) \end{aligned}$$

Where $y \sim LL(1, \sigma)$ and $z \sim LL(1, 1)$

Moreover;

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} V_{LL}(T) &= AV_{LL} = V_{LL} \left[\ln f_{GA}(X; \tilde{\lambda}, \tilde{\alpha}) - \ln f_{LL}(X; \varepsilon, \sigma) \right] \\ AV_{LL} &= (\alpha - \sigma)^2 V_{LL}(\ln y) + \frac{1}{\lambda^2} (\varepsilon^2 (2b / \sin 2b - b^2 / \sin b^2)) + 4V_{LL}([1 + y^\sigma]) \\ &\quad - \frac{2\varepsilon(\alpha - \sigma)}{\lambda} \text{cov}((\ln y), y) + 4(\alpha - \sigma) \text{cov}((\ln y), \ln[1 + y^\sigma]) - \frac{4\varepsilon}{\lambda} \text{cov}(\ln(1 + y^\sigma), y) \quad (13) \end{aligned}$$

Where $y \sim LL(1, \sigma)$

For more details see Appendix (2).

Different values of $\tilde{\sigma}$, $\tilde{\varepsilon}$, $AM_{GA}(\lambda, \alpha)$, $AV_{GA}(\lambda, \alpha)$, $\tilde{\alpha}$ and $\tilde{\lambda}$, $AM_{LL}(\varepsilon, \sigma)$, $AV_{LL}(\varepsilon, \sigma)$ are computed numerically by using Equations (5), (6), (7), (8), (10), (11), (12), (13) and reported in Table 1, Table 2

Table 1. Different Values of $AM_{GA}(\alpha)$, $AV_{GA}(\alpha)$, $\tilde{\sigma}$ and $\tilde{\varepsilon}$ for different α when $\lambda=1$

α	AM_{GA}	AV_{GA}	$\tilde{\sigma}$	$\tilde{\varepsilon}$
1.2	-0.179	0.476	0.161	375.432
1.5	-0.102	0.472	0.168	394.131
2.2	0.076	0.428	0.181	429.509
2.5	0.076	0.428	0.186	431.661
3	0.076	0.428	0.192	457.002

Table 2. Different Values of $AM_{LL}(\sigma)$, $AV_{LL}(\sigma)$, $\tilde{\lambda}$ and $\tilde{\alpha}$ for different σ when $\varepsilon=1$

σ	AM_{LL}	AV_{LL}	$\tilde{\lambda}$	$\tilde{\alpha}$
2.2	-0.109	5.397	0.956	1.509
2.5	-0.087	2.326	0.679	1.945
3	-0.064	2.058	0.434	2.788
3.5	-0.051	2.681	0.304	3.78
4	-0.042	3.635	0.226	4.922

4 Determination of Sample Size

In this Section, we propose a method to determine the minimum sample size needed to discriminate between gamma and log-logistic distributions, for a given user specified probability of correct selection (PCS). Intuitively, it is clear that if two distributions are very close, one needs a very large sample size to discriminate between them for a given probability of correct selection. On the other hand if two distribution functions are quite different, then one may not need very large sample size to discriminate between two distribution functions. It is also true that if two distribution functions are very close to each other, then one may not need to differentiate the two distributions.

It is expected that the user will specify beforehand the PCS and tolerance limit in terms of the distance between two distribution functions in terms of the distance between two distribution functions. Based on the PCS, the required minimum sample size can be determined. The tolerance limit simply indicates that the user does not want to make the distinction between two distribution functions if their distance is less than the tolerance limit. Based on the PCS and the tolerance limit, the required minimum sample size can be determined.

In Section 3 the RML statistics follow normal distribution approximately for large n will be observed. Now it will be used with the help of Kolmogorov-Smirnov (K-S) distance to determine the required sample size n .

Using Theorem 1 and since it is assumed that the data are coming from $GA(\lambda, \alpha)$. In this case the probability of correct selection PCS_{GA} is given by

$$PCS_{GA}(\lambda, \alpha) = \Pr(T > 0)$$

$$\approx 1 - \Phi\left(-\frac{E_{GA}(T)}{\sqrt{V_{GA}(T)}}\right)$$

Since $AM_{GA} = \lim_{n \rightarrow \infty} \frac{1}{n} E_{GA}(T)$ and $AV_{GA} = \lim_{n \rightarrow \infty} \frac{1}{n} V_{GA}(T)$, then

$$PCS_{GA}(\lambda, \alpha) = \Phi\left(\frac{n \times AM_{GA}(\lambda, \alpha)}{\sqrt{n \times AV_{GA}(\lambda, \alpha)}}\right).$$

Also using Theorem 2 and since it is assumed that the data are coming from $LL(\varepsilon, \sigma)$. In this case, the PCS_{LL} is given by

$$PCS_{LL}(\varepsilon, \sigma) = \Pr(T < 0) \\ \approx \Phi\left(\frac{-E_{LL}(T)}{\sqrt{V_{LL}(T)}}\right)$$

Since $AM_{LL} = \lim_{n \rightarrow \infty} \frac{1}{n} E_{LL}(T)$ and $AV_{LL} = \lim_{n \rightarrow \infty} \frac{1}{n} V_{LL}(T)$, then

$$PCS_{LL} = \Phi\left(-\frac{n \times AM_{LL}(\varepsilon, \sigma)}{\sqrt{n \times AV_{LL}(\varepsilon, \sigma)}}\right).$$

Where AM and AV denote the asymptotic mean and variance respectively, and Φ is the distribution function of the standard normal random variable.

Therefore, to determine the minimum sample size required to achieve at least α^* protection level, equate $PCS_{GA}(\lambda, \alpha)$ by $PCS_{LL}(\varepsilon, \sigma)$ i.e., Let:

$$\Phi\left(\frac{n_1 \times AM_{GA}(\lambda, \alpha)}{\sqrt{n_1 \times AV_{GA}(\lambda, \alpha)}}\right) = \alpha^* \quad \text{and} \quad \Phi\left(-\frac{n_2 \times AM_{LL}(\varepsilon, \sigma)}{\sqrt{n_2 \times AV_{LL}(\varepsilon, \sigma)}}\right) = \alpha^*$$

From previous equations give n_1 and n_2 as

$$n_1 = \frac{Z_{\alpha^*}^2 \times AV_{GA}(\lambda, \alpha)}{(AM_{GA}(\lambda, \alpha))^2} \quad (14), \quad n_2 = \frac{Z_{\alpha^*}^2 \times AV_{LL}(\varepsilon, \sigma)}{(AM_{LL}(\varepsilon, \sigma))^2} \quad (15)$$

Here z_{α^*} is the α -th percentile point of a standard normal distribution.

Reported n_1 , with the help of Table(1) for different α when the protection level $\alpha^*=0.7$, in Table(3) and reported n_2 , with the help of Table(2) for different σ when the protection level $\alpha^*=0.7$, in Table(4).

Therefore, the minimum sample size n required to discriminate between gamma and log-logistic distribution can be taken as $\max(n_1; n_2)$.

The distance between two distribution functions is defined by the K-S distance. The K-S distance between two distribution functions, say $F(x)$ and $G(x)$ is defined as

$$\sup_x |F(x) - G(x)|.$$

The K-S distance between $GA(1, \alpha)$ and $LL(\tilde{\varepsilon}, \tilde{\sigma})$ reported with the help of Table(1) for different values of α , in Table(3). Similarly K-S distance between $LL(1, \sigma)$ and $GA(\tilde{\lambda}, \tilde{\alpha})$ reported with the help of Table(2) for different values of σ , in Table(4).

Table 3. The minimum sample size n for $\alpha^*=0.7$, when the null distribution is gamma distribution is presented. The K-S distance between $GA(1, \alpha)$ and $LL(\tilde{\varepsilon}, \tilde{\sigma})$ for different values of α is reported.

α	1.2	1.5	2.2	2.5	3
n	5	125	125	125	125
K-S	0.217	0.230	0.227	0.227	0.227

Table 4. The minimum sample size n for $\alpha^*=0.7$, when the null distribution is log-logistic distribution is presented. The K-S distance between $LL(1, \sigma)$ and $GA(\tilde{\lambda}, \tilde{\alpha})$ for different values of σ is reported.

σ	2.2	2.5	3	3.5	4
n	125	85	137	283	565
K-S	0.351	0.310	0.265	0.232	0.220

If one knows the range of the shape parameter of the null distribution and for a given PCS that achieves a certain protection level α^* , then the minimum sample size can be obtained by taking the maximum n obtained from Equations (14) and (15).

But unfortunately in practice the shape parameter may be completely unknown; therefore, the K-S distances can replace the unknown parameters in taking the decision. That is, for a given protection level α^* and a given pre-specified tolerance limit D^* , the minimum sample size can be obtained by taking the maximum n obtained from Equations (14) and (15). For example, suppose that for a given $\alpha^*=0.7$ and for $\alpha=2.2$ and $\sigma=2.2$, then from Tables 3 and 4 the minimum sample size required to discriminate between Weibull and log-logistic distributions is $\max(21, 125)=125$.

On the other hand if α and σ are unknown and suppose that the practitioner wants to discriminate between gamma and log-logistic distribution functions only when the distance between them is greater than or equal 0.217, i.e., $D^* \geq 0.217$ and with $\alpha^*=0.7$. Then from Tables 3 and 4, it is clear that, $D^* \geq 0.217$ if $\alpha \geq 1.2$ and $\sigma \geq 2.2$. Also, when the null distribution is gamma, then for the tolerance limit $D^* \geq 0.217$, one needs $n=5$ to meet the PCS, $\alpha^*=0.7$. Similarly when the null distribution is log-logistic then one needs $n=125$ to meet the same protection level. Finally, the minimum sample size required to discriminate between gamma and log-logistic distributions with $\alpha^*=0.7$ and $D^* \geq 0.217$ is $\max(5, 125)=125$.

Notice that, Tables 3 and 4 are obtained for the protection level $\alpha^*=0.7$ but for other protection levels the tables can be easily modified. For example, if we need a sample size corresponding to protection level $\alpha^*=0.9$, then all the entries corresponding to the row of n, must be multiplied by $Z_{0.9}^2 / Z_{0.7}^2$.

5 Numerical Experiments

In this section some experimental results is presented to examine how the asymptotic results derived in Section 3 behave for finite sample sizes. Moreover when the sample size is not sufficiently large, the PCS's involved in the discrimination between the gamma and the log-logistic distributions based on likelihood ratio can be determined with more accuracy through simulations sample. The PCSs obtained using simulations and based on the asymptotic results derived in Section 3 are compared.

Different sample sizes and different shape parameters of the null distributions are considered. The details are explained below. First case when the null distribution is gamma and the alternative is log-logistic. In this case consider $n=20, 40, 60, 80, 100$ and $\alpha=1.2, 1.5, 2.2, 2.5$ and 3 .

Computation of the PCS is performed as follows:

1. For a sample size n , a random sample $\{x_1, x_2, \dots, x_n\}$ is generated from a $GA(1, \alpha)$ distribution.
2. By maximum likelihood, both the gamma and the log-logistic distributions are fitted to the sample $\{x_1, x_2, \dots, x_n\}$, and a realization, t , of the statistic $T = \ln \frac{L_{GA}}{L_{LL}}$, is calculated and stored.
3. Steps 1 and 2 are repeated many times (in this study, the repetition was done 10,000 times).
4. The approximate PCS under the assumption that the true distribution is gamma,

is: $PCS_{GA} = \Pr[T > 0]$

$$\approx (\text{number of } t \text{ values in step 2 } > 0) / 10,000 .$$

Also the PCS's obtained by using the asymptotic results as given in section 3 are computed. The results are reported in Table 5.

Similarly, the results when the null distribution is log-logistic and the alternative is gamma are obtained. In this case considered the same set of n and $\sigma= 2.2, 2.5, 3, 3.5$ and 4 . The results are reported in Table 6.

6 PCS of RMKLD

In probability and information theory, the Kullback-Leibler divergence (also information discrepancy, information gain, relative entropy, or KLD) is a non-symmetric measure of the difference (dissimilarity) between two probability distributions $f_1(x)$ and $f_2(x)$ it denotes the "information lost when $f_2(x)$ is used to approximate $f_1(x)$ or the distance from $f_1(x)$ to $f_2(x)$ ". In other words, KLD is a measure of inefficiency of assuming that the distribution is $f_2(x)$

when the true distribution is $f_1(x)$. Since the measure from $f_1(x)$ to $f_2(x)$ is not the same as the measure from $f_2(x)$ to $f_1(x)$, then it can be conceptualized as a "directed/oriented distance" between the two models.

The KLD is a natural distance function between models. It is usually used as a logical basis for model selection in conjunction with likelihood inference. Values of KLD are not based on only the mean and variance of the distributions; rather, the distributions in their entirety are the subject of comparison. The later is regarded an advantage of the KLD as a test statistic. The KLD is defined as follow

$$\begin{aligned} KLD(f_1(x), f_2(x)) &= \int_0^{\infty} f_1(x) \ln\left(\frac{f_1(x)}{f_2(x)}\right) dx \\ &= \int_0^{\infty} f_1(x) \ln(f_1(x)) dx - \int_0^{\infty} f_1(x) \ln(f_2(x)) dx \end{aligned}$$

It is well known that $KLD(f_1(x), f_2(x)) \neq KLD(f_2(x), f_1(x))$ and $KLD(f_1(x), f_2(x)) \geq 0$ and the equality holds if and only if $f_1(x) = f_2(x)$. The smaller $KLD(f_1(x), f_2(x))$ means that " $f_1(x)$ " is preferred and large values of KLD favor " $f_2(x)$ ". However, the KLD (based test statistic) is considered as a ruler to measure the similarity between the two hypotheses / distributions. The test statistic is defined as the natural logarithm of two ratios of KLDs. The idea is similar to RML, in which our interested to select a model maximizing the likelihood. But in KLD method our interested to select a model minimizing the KLD, That's why it named as ratio of minimized KLD (RMKLD) which is defined as follow

$$RMKLD = \ln\left(\frac{KLD(f_1(x), f_2(x))}{KLD(f_2(x), f_1(x))}\right).$$

In this Section how the proposed RMKLD test statistic work for different parameters and sample sizes is presented.

First considered case 1 when the null distribution is gamma. The RMKLD testing procedure is introduced as follows:

- 1- For a sample size n , a random sample $\{x_1, x_2, \dots, x_n\}$ is generated from a $GA(1, \alpha)$ distribution.
- 2- Calculate $KLD(f_1(x), f_2(x))$ and $KLD(f_2(x), f_1(x))$.

$$RMKLD = \ln\left(\frac{KLD(f_1(x), f_2(x))}{KLD(f_2(x), f_1(x))}\right) \text{ is calculated and stored.}$$

- 3- Select $f_1(x)$ if $KLD < 0$.

4- Steps 2 and 3 are repeated many times (in this study, the repetition was done 10,000 times).

5- Compute the percentage of the times if $RMKLD < 0$

$$= (\text{number of RMKLD values in step 2} < 0) / 10,000$$

as the PCS of gamma distribution. The results are reported in Table 5. Similarly, the results when the null distribution is log-logistic and the alternative is gamma are obtained. The results are reported in Table 6.

RESULTS

Table 5: The Probability of Correct Selection Based on Simulations and also Based on Asymptotic Results When the Null Distribution is gamma distribution.

α	n					
	PCS	20	40	60	80	100
1.2	RMLMC	0.77	0.83	0.89	0.91	0.92
	RMLAR	(0.75)	(0.77)	(0.78)	(0.86)	(0.89)
	RMKLD	0.83	0.86	0.92	0.94	0.95
1.5	RMLMC	0.79	0.84	0.88	0.90	0.93
	RMLAR	(0.76)	(0.80)	(0.83)	(0.85)	(0.90)
	RMKLD	0.85	0.87	0.92	0.93	0.96
2.2	RMLMC	0.82	0.86	0.90	0.91	0.94
	RMLAR	(0.78)	(0.82)	(0.84)	(0.88)	(0.93)
	RMKLD	0.86	0.89	0.94	0.96	0.97
2.5	RMLMC	0.83	0.86	0.91	0.93	0.94
	RMLAR	(0.80)	(0.83)	(0.86)	(0.92)	(0.93)
	RMKLD	0.88	0.90	0.95	0.96	0.98
3.00	RMLMC	0.85	0.89	0.93	0.94	0.95
	RMLAR	(0.83)	(0.86)	(0.90)	(0.91)	(0.94)
	RMKLD	0.90	0.92	0.96	0.97	0.98

Note: In Table 5 the element in the first row in each box represents the results based on Simulations (10,000 replications) in case of RML functions denoted by RMLMC and the number in bracket immediately below represents the result obtained by using asymptotic results in case of RML functions denoted by RMLAC. The element in third row represents the RMKLD.

Table 6: The Probability of Correct Selection Based on Simulations and also Based on Asymptotic Results When the Null Distribution is log-logistic distribution.

σ	n					
	PCS	20	40	60	80	100
2.2	RMLMC	0.51	0.56	0.64	0.67	0.70
	RMLAR	(0.47)	(0.53)	(0.57)	(0.62)	(0.66)
	RMKLD	0.57	0.62	0.68	0.73	0.75
2.5	RMLMC	0.53	0.58	0.65	0.69	0.72
	RMLAR	(0.48)	(0.53)	(0.58)	(0.63)	(0.68)
	RMKLD	0.58	0.64	0.68	0.74	0.77
3.0	RMLMC	0.54	0.59	0.66	0.70	0.74
	RMLAR	(0.51)	(0.55)	(0.58)	(0.64)	(0.69)
	RMKLD	0.61	0.64	0.70	0.73	0.78
3.5	RMLMC	0.57	0.64	0.70	0.76	0.77
	RMLAR	(0.57)	(0.62)	(0.66)	(0.69)	(0.75)
	RMKLD	0.65	0.70	0.74	0.79	0.82
4.00	RMLMC	0.59	0.66	0.73	0.78	0.81
	RMLAR	(0.61)	(0.63)	(0.67)	(0.71)	(0.78)
	RMKLD	0.69	0.71	0.77	0.81	0.83

Note: In Table 6 the element in the first row in each box represents the results based on Simulations (10,000 replications) in case of RML functions denoted by RMLMC and the number in bracket immediately below represents the result obtained by using asymptotic results in case of RML functions denoted by RMLAC. The element in third row represents the RMKLD.

It is quite clear from Tables 5 and 6 that as the sample size increases the PCS increases as expected. It is also clear that as the shape parameter moves away from 1, the PCS increases. Even when the sample size is 20.

Interestingly, when the null distribution is gamma distribution, then the PCS based on MC simulation, AR and RMKLD is found to be significantly higher than the other case particularly for small sample sizes. For example, when the sample size is 20, and null distribution is gamma, the PCS for MC, AR and RMKLD are 0.77, 0.75 and 0.83 respectively. But when the null distribution is log-logistic, for the same sample size the PCS for MC, AR and RMKLD are 0.51, 0.47 and 0.57 respectively.

The asymptotic results work reasonable well for both the distributions and for all possible ranges of the parameters. From the simulation study it is recommended that the asymptotic results can be used quite effectively even when the sample size is as small as 20 for all possible choices of the shape parameters.

Also from Tables 5 and 6 notice that, the RMKLD works better than RML, because it generates higher PCS (about 3-9 %) in particular for small sample size. In other words, the error type I for RMKLD is remarkably less, compared to the RML one. For instance, consider in table 5 the case at $\alpha = 1.2$. The PCS for RMKLD is 83% and 77% for RML which indicates the error type I equals to 17% for RMKLD and 23% for RML. also consider in table 6 the case $\sigma = 2.2$ The PCS for RMKLD is 57% and 51% for RML which indicates the error type I equals to 43% for RMKLD and 49% for RML. It also found that both methods behave similarly, for example, as sample size increases the PCS capture higher values, as expected. Moreover, as shape parameter increases, the PCS increases.

7 Data Analysis

For illustrative purposes two data sets will be analyzed to discriminate between the gamma and log-logistic distribution functions using RML method and some goodness of fit tests.

Data Set1: (Lawless, 1982, p. 228) represents the number of revolution before failure of each of 23 ball bearings in the life tests and they are as follows: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

In this case the two fitted distributions have the following MLEs :log-likelihood values are as follows. Gamma: $\hat{\lambda} = 3.7138$, $\hat{\alpha} = 19.449$, $LL_{GA}(\hat{\lambda}, \hat{\alpha}) = -113.0274$, Log-Logistic: $\hat{\sigma} = 3.0591$, $\hat{\varepsilon} = 60.639$, $LL_{LL}(\hat{\varepsilon}, \hat{\sigma}) = -113.3662$.

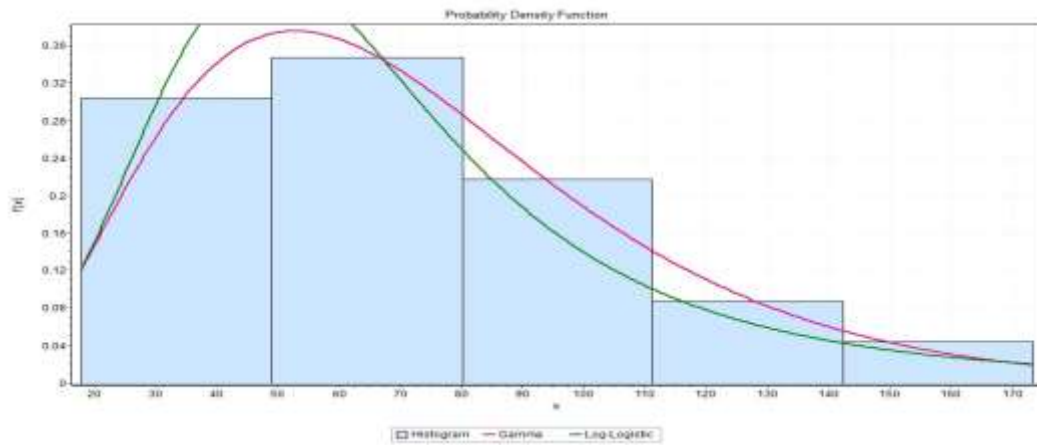


Figure (1) Lawless data, probability density function of gamma and log-logistic distribution

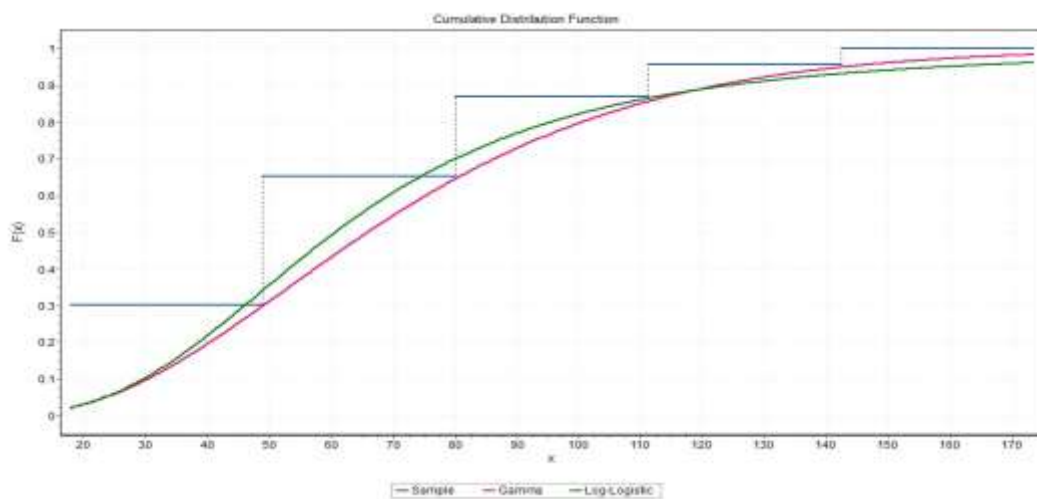


Figure (2) Lawless data, cumulative distribution function of gamma and log-logistic distribution

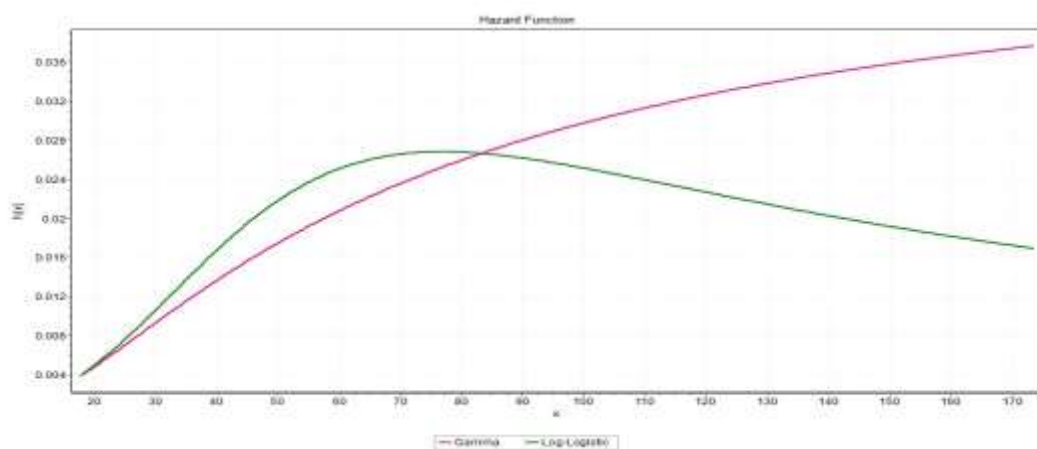


Figure (3) Lawless data, hazard function of gamma and log-logistic distribution

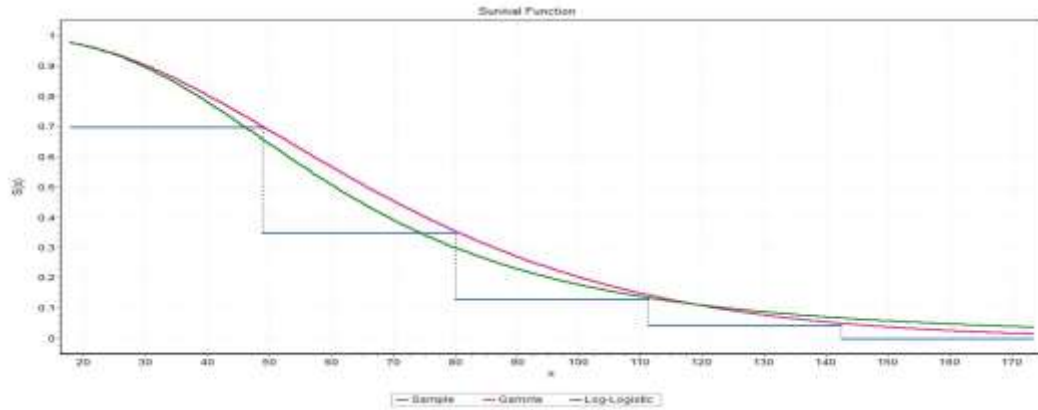


Figure (4) Lawless data, survival function of gamma and log-logistic distribution.

From these Figures of fitting data set 1, the closeness of the two distribution functions can be easily visualized. Therefore, if the data are coming from any one of them, it can be easily modeled by the other one.

Therefore, based on the log-likelihood values clearly, and by using the method of the Maximized Likelihood Procedure:

$$T = \ln L_{GA} - \ln L_{LL}$$

$$T = -113.0274 + 113.3662 = 0.3388 > 0$$

Then gamma distribution is the preferred one. The closeness of results may be caused because the sample size is not very large. Also, by computing The Kolmogorov-Smirnov distances and the empirical distribution function and the fitted gamma: 0.11878 (P-value = 0.864), Log-logistic: 0.10847 (P-value = 0.922). Also by computing Anderson darling for gamma = 0.20569, log-logistic = 0.30294, and by computing chi-squared of gamma: 0.67705 (P-value = 0.7128), log-logistic: 0.93098 (P-value = 0.6278) Therefore, it is nearly shows that the fitted gamma is much closer to the empirical distribution function than the log-logistic distribution.

Data Set 2: (Linhart and Zucchini, 1986, p. 69) represents the failure times of 30 air conditions of an airplane (in hours): 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

In this case the two fitted distributions have the following MLEs :log-likelihood values are as follows. Gamma: $\hat{\lambda} = 0.68741$, $\hat{\alpha} = 86.702$, $LL_{GA}(\hat{\lambda}, \hat{\alpha}) = -151.706$. Log-Logistic: $\hat{\sigma} = 1.2024$, $\hat{\varepsilon} = 26.629$, $LL_{LL}(\hat{\varepsilon}, \hat{\sigma}) = -152.3468$.

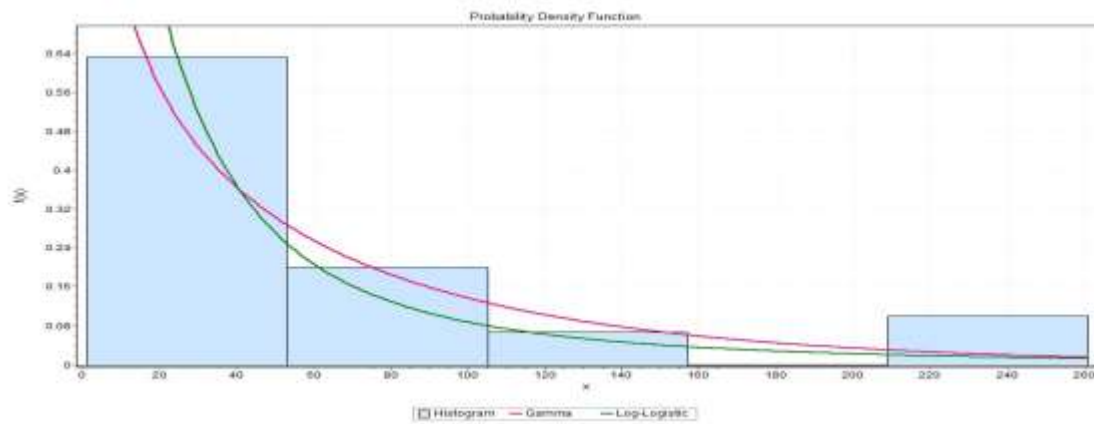


Figure (5) Linhart and Zucchini data , probability density function of gamma and log-logistic distribution

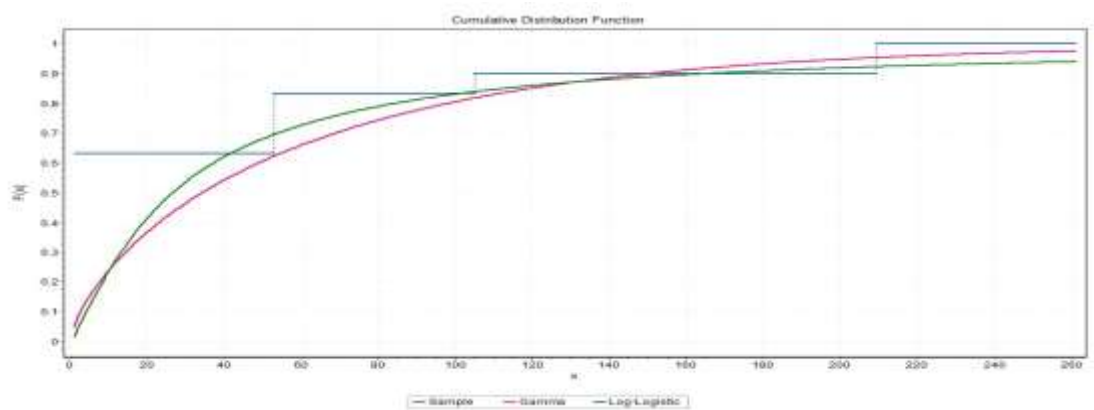


Figure (6) Linhart and Zucchini data, cumulative distribution function of gamma and log-logistic distribution

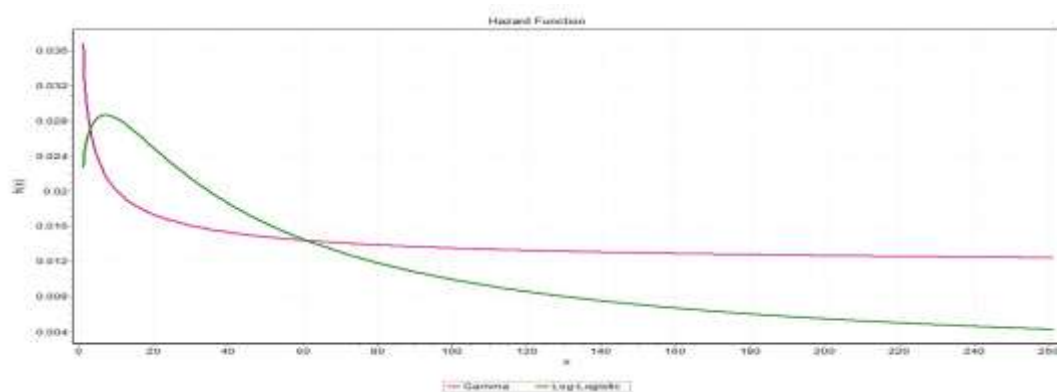


Figure (7) Linhart and Zucchini data, hazard function of gamma and log-logistic distribution

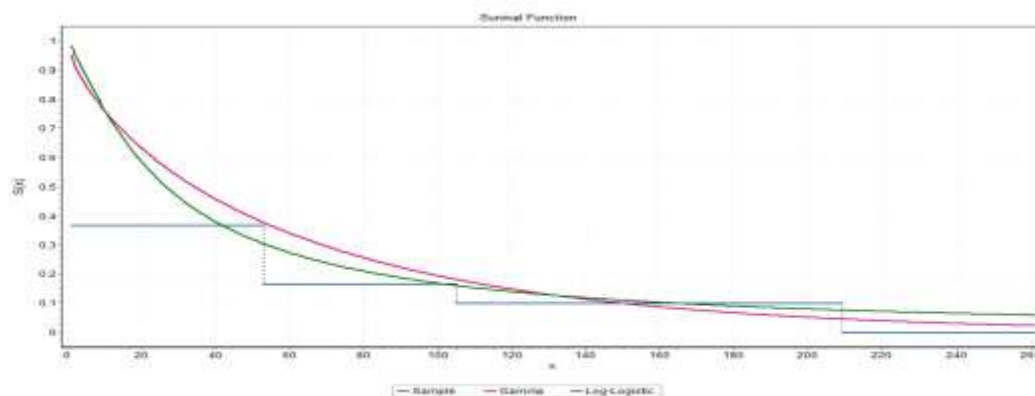


Figure (8) Linhart and Zucchini data, survival function of gamma and log-logistic distribution.

From these Figures of fitting data set 2, the closeness of the two distribution functions can be easily visualized. Therefore, if the data are coming from any one of them, it can be easily modeled by the other one.

Therefore, based on the log-likelihood values clearly, and by using the method of the Maximized Likelihood Procedure:

$$T_n = \ln L_{GA} - \ln L_{LL}$$

$$T_n = -151.406 + 152.3468 = 0.9408 > 0$$

Then gamma distribution is the preferred one.

Moreover by computing The Kolmogorov-Smirnov distances and the empirical distribution function and the fitted gamma: 0.13432 (P-value = 0.603), Log-logistic: 0.1234 (P-value = 0.705). also by computing Anderson darling for gamma = 0.57009, log-logistic = 0.49039, and by computing chi-squared of gamma: 1.7764 (P-value = 0.411), log-logistic: 2.6152 (P-value = 0.270). Therefore, it is nearly shows that the fitted logistic is much closer to the empirical distribution function than the gamma distribution.

8 Conclusion

Gamma and log-logistic distributions are two popular distributions for analysing lifetime data. In this paper, the problem of discriminating between these two distribution functions is considered. A hypothesis testing method is used in which it is assumed that a data are coming either from gamma or log-logistic distribution. Then the ratio of the maximized likelihood test and the ratio of Minimized Kullback-Leibler Divergence (RMKLD) are used to discriminate between them. The asymptotic results are used to estimate the probability of correct selection which is used calculate the minimum sample size required for discriminating between two distributions. The probability of correct selection using simulations with the asymptotic results and RMKLD is compared and it is observed that even when the sample size is very small the asymptotic results work quite well for a wide range of the parameter space. Therefore, the asymptotic results can be used to estimate the probability of correct selection. Also it noticed that when the null distribution is gamma distribution, then the PCS is found to be significantly higher than the other case particularly for small sample sizes. Two Real data life are analyzed to see how the proposed method works in practice.

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Appendix (1) Asymptotic Properties Proofs of Theorems in Case 1

Case 1. The data are coming from a gamma distribution and the alternative is a log-logistic distribution.

Let us define

$$\begin{aligned}
 h(\varepsilon, \sigma) &= E_{GA} [\ln(f_{LL}(X; \varepsilon, \sigma))] \\
 &= E_{GA} \left[\ln \sigma - \ln \varepsilon + (\sigma - 1) [\ln X - \ln \varepsilon] - 2 \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right] \right] \\
 &= \ln \sigma + (\sigma - 1) E_{GA}(\ln X) - \sigma \ln \varepsilon - 2 E_{GA} \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right].
 \end{aligned}$$

$$\text{let } \frac{x}{\lambda} = y \quad x = \lambda y \quad dx = \lambda dy$$

$$\begin{aligned} E_{GA}(\ln x) &= \int_0^{\infty} \ln(y) \frac{1}{\Gamma(\alpha)} (y)^{\alpha-1} e^{-(y)} dy + \ln(\lambda) \\ &= E_{GA} \ln(y) + \ln(\lambda) \end{aligned}$$

$$E_{GA}(\ln x) = \int_0^{\infty} \ln \left[1 + \left(\frac{x}{\varepsilon} \right)^{\sigma} \right] \frac{1}{\lambda^{\alpha} \Gamma(\alpha)} (x)^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)} dx$$

$$E_{GA} \left(\ln \left[1 + \left(\frac{x}{\varepsilon} \right)^{\sigma} \right] \right) = \int_0^{\infty} \ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^{\sigma} \right] \frac{1}{\lambda^{\alpha} \Gamma(\alpha)} (\lambda y)^{\alpha-1} e^{-(y)} \lambda dy$$

$$\begin{aligned} E_{GA} \left(\ln \left[1 + \left(\frac{x}{\varepsilon} \right)^{\sigma} \right] \right) &= \int_0^{\infty} \ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^{\sigma} \right] \frac{1}{\Gamma(\alpha)} (y)^{\alpha-1} e^{-(y)} dy \\ &= E_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^{\sigma} \right] \right) \end{aligned}$$

Where $y \sim$ gamma distribution $(1, \alpha)$

$$h(\varepsilon, \sigma) = \ln(\sigma) + (\sigma - 1) [E_{GA} \ln(y) + \ln(\lambda)] - \sigma \ln \varepsilon - 2 E_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^{\sigma} \right] \right)$$

Now to compute $E_{GA}(T)$ and $V_{GA}(T)$ denote

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} E_{GA}(T) &= AM_{GA} = E_{GA} \left[\ln f_{GA}(X; \lambda, \alpha) - \ln f_{LL}(X; \tilde{\varepsilon}, \tilde{\sigma}) \right] \\ &= E_{GA} [\ln f_{GA}(X; \lambda, \alpha)] - E_{GA} [\ln f_{LL}(X; \tilde{\varepsilon}, \tilde{\sigma})] \end{aligned}$$

$$E_{GA} [\ln f_{LL}(X; \tilde{\varepsilon}, \tilde{\sigma})] = \ln(\sigma) + (\sigma - 1) [E_{GA} \ln(y) + \ln(\lambda)] - \sigma \ln \varepsilon - 2 E_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^{\sigma} \right] \right)$$

$$E_{GA} [\ln(f_{GA}(x; \lambda, \alpha))] = -\alpha \ln \lambda - \ln \Gamma(\alpha) + (\alpha - 1) E_{GA}(\ln(x)) - E_{GA} \left(\frac{x}{\lambda} \right)$$

$$\begin{aligned} AM_{GA} &= -\alpha \ln \lambda - \ln \Gamma(\alpha) + (\alpha - 1) E_{GA}(\ln(x)) - E_{GA} \left(\frac{x}{\lambda} \right) \\ &\quad - \ln(\sigma) + (\sigma - 1) [E_{GA} \ln(y) + \ln(\lambda)] - \sigma \ln \varepsilon - 2 E_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^{\sigma} \right] \right) \end{aligned}$$

$$E_{GA}(\ln x) = E_{GA} \ln(y) + \ln(\lambda)$$

$$AM_{GA} = -\alpha \ln \lambda - \ln \Gamma(\alpha) + (\alpha - \sigma) E_{GA}(\ln(x)) - \left(\frac{E_{GA}(x)}{\lambda} \right) - \ln(\sigma) + -\sigma \ln \varepsilon - 2E_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right] \right)$$

$$AM_{GA} = -\alpha(\ln \lambda + 1) - (\ln \Gamma(\alpha) + \ln(\sigma) + \sigma \ln \varepsilon) + (\alpha - \sigma)(E_{GA} \ln(y) + \ln(\lambda)) - 2E_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right] \right)$$

Moreover,

$$\begin{aligned} AV_{GA} &= V_{GA} \left[-\alpha \ln \lambda - \ln \Gamma \alpha + (\alpha - 1) \ln(x) - \frac{x}{\lambda} + \ln \sigma - (\sigma - 1) \ln X + \sigma \ln \varepsilon + 2 \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right] \right] \\ &= (\alpha - \sigma)^2 V_{GA}(\ln(x)) + \frac{1}{\lambda^2} V_{GA}(x) + 4V_{GA} \left(\ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right] \right) - \frac{2(\alpha - \sigma)}{\lambda} \text{cov}(\ln(x), x) \\ &\quad + 4(\alpha - \sigma) \text{cov}(\ln(x), \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right]) - \frac{4}{\lambda} \text{cov}(x, \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right]) \end{aligned}$$

Where $\psi_1(\alpha) = \text{var}(\ln(x))$, Also $v(x) = \alpha \lambda^2$

let $\frac{x}{\lambda} = y$ $x = \lambda y$ $dx = \lambda dy$ Where $y \sim$ gamma distribution $(1, \alpha)$

$$\begin{aligned} AV_{GA} &= (\alpha - \sigma)^2 \psi_1(\alpha) + \alpha + 4V_{GA} \left(\ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right] \right) - 2(\alpha - \sigma) \text{cov}(\ln(\lambda y), y) \\ &\quad + 4(\alpha - \sigma) \text{cov}(\ln(\lambda y), \ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right]) - 4 \text{cov}(y, \ln \left[1 + \left(\frac{\lambda y}{\varepsilon} \right)^\sigma \right]) \end{aligned}$$

Appendix (2) Asymptotic Properties Proofs of Theorems in Case 2

Case 2. The data are coming from a log-logistic distribution and the alternative is a gamma distribution.

$$\begin{aligned} g(\alpha, \lambda) &= E_{LL} [\ln f_{GA}(x; \lambda, \alpha)] \\ &= -\alpha \ln \lambda - \ln \Gamma \alpha + (\alpha - 1) E_{LL}(\ln(x)) - \frac{1}{\lambda} E_{LL}(x) \end{aligned}$$

$$\begin{aligned} E_{LL}(x) &= \int_0^\infty x \frac{(\sigma / \varepsilon)(x / \varepsilon)^{\sigma-1}}{[1 + (x / \varepsilon)^\sigma]^\sigma} dx \\ &= \frac{\varepsilon \pi / \sigma}{\sin(\pi / \sigma)} \end{aligned}$$

If $\sigma > 1$ else undefined.

$$\begin{aligned} E_{LL} \ln(X) &= \int_0^\infty \ln x f_{LL}(X; \varepsilon, \sigma) \\ &= \int_0^\infty \ln x \frac{(\sigma / \varepsilon)(x / \varepsilon)^{\sigma-1}}{[1 + (x / \varepsilon)^\sigma]^\sigma} dx \end{aligned}$$

$$\text{let } (x/\varepsilon)^\sigma = z \quad \therefore x/\varepsilon = z^{\frac{1}{\sigma}} \quad \therefore x = \varepsilon z^{\frac{1}{\sigma}} \quad \therefore dx = \frac{\varepsilon}{\sigma} z^{\frac{1}{\sigma}-1} dz$$

$$\begin{aligned} E_{LL} \ln(X) &= \int_0^\infty \ln(\varepsilon z^{\frac{1}{\sigma}}) \frac{(\sigma/\varepsilon)(z^{\frac{1}{\sigma}})^{\sigma-1}}{[1+z]^2} \cdot \frac{\varepsilon}{\sigma} z^{\frac{1}{\sigma}-1} dz \\ &= \int_0^\infty \ln(z^{\frac{1}{\sigma}}) \frac{1}{[1+z]^2} dz + \varepsilon \int_0^\infty \frac{1}{[1+z]^2} dz \\ &\quad \frac{1}{[1+z]^2} \end{aligned}$$

where $\frac{1}{[1+z]^2}$ is the p.d.f of log-logistic distribution when $\sigma=1$

$$\text{then } \int_0^\infty \frac{1}{[1+z]^2} dz = 1$$

$$E_{LL} \ln(X) = \int_0^\infty \ln(z^{\frac{1}{\sigma}}) \frac{1}{[1+z]^2} dz + \ln \varepsilon \cdot 1$$

$$\text{let } z^{\frac{1}{\sigma}} = y \quad \therefore z = y^\sigma \quad \therefore dz = \sigma y^{\sigma-1} dy$$

$$E_{LL} \ln(X) = \int_0^\infty \ln(y) \frac{\sigma y^{\sigma-1}}{[1+y^\sigma]^2} dy + \ln \varepsilon$$

$$= E(\ln y) + \ln \varepsilon \quad \text{where } y \sim LL(1, \sigma)$$

$$\therefore g(\alpha, \lambda) = -\alpha \ln \lambda - \ln \Gamma \alpha + (\alpha-1)(E(\ln y) + \ln \varepsilon) - \frac{1}{\lambda} \frac{\varepsilon \pi / \sigma}{\sin(\pi / \sigma)}$$

Moreover

$$\lim_{n \rightarrow \infty} \frac{1}{n} E_{LL}(T) = AM_{LL} = E_{LL} \left[\ln f_{GA}(X; \tilde{\lambda}, \tilde{\alpha}) - \ln f_{LL}(X; \varepsilon, \sigma) \right]$$

$$E_{LL} \left(\ln f_{GA}(X; \tilde{\lambda}, \tilde{\alpha}) \right) = -\alpha \ln \lambda - \ln \Gamma \alpha + (\alpha-1)(E(\ln y) + \ln \varepsilon) - \frac{1}{\lambda} \frac{\varepsilon \pi / \sigma}{\sin(\pi / \sigma)}$$

$$E_{LL} \left[\ln f_{LL}(X; \varepsilon, \sigma) \right] = \ln(\sigma) + (\sigma-1)E_{LL}(\ln x) - \sigma \ln \varepsilon - 2E_{LL} \ln \left[1 + \left(\frac{x}{\varepsilon} \right)^\sigma \right]$$

$$AM_{LL} = -\alpha \ln \lambda - \ln \Gamma \alpha + (\alpha-1)(E \ln y + \ln \varepsilon) - \frac{1}{\lambda} \frac{\varepsilon \pi / \sigma}{\sin(\pi / \sigma)} - \ln(\sigma) - (\sigma-1)E_{LL}(\ln x) + \sigma \ln \varepsilon + 2E_{LL} \ln \left[1 + \left(\frac{x}{\varepsilon} \right)^\sigma \right]$$

$$E_{LL} \ln(X) = E(\ln y) + \ln \varepsilon$$

$$E_{LL} \ln \left[1 + \left(\frac{x}{\varepsilon} \right)^\sigma \right] = \int_0^\infty \ln \left[1 + \left(\frac{x}{\varepsilon} \right)^\sigma \right] \frac{(\sigma/\varepsilon)(x/\varepsilon)^{\sigma-1}}{[1+(x/\varepsilon)^\sigma]^2} dx$$

$$\text{let } (x/\varepsilon)^\sigma = z \quad \therefore x/\varepsilon = z^{\frac{1}{\sigma}} \quad \therefore x = \varepsilon z^{\frac{1}{\sigma}} \quad \therefore dx = \frac{\varepsilon}{\sigma} z^{\frac{1}{\sigma}-1} dz$$

$$\begin{aligned} E_{LL} \ln \left[1 + \left(\frac{x}{\varepsilon} \right)^\sigma \right] &= \int_0^\infty \ln(1+z) \frac{1}{[1+z]^2} dz \\ &= E_{LL}(\ln(1+z)) \quad \text{.....where } z \sim LL(1,1) \end{aligned}$$

$$AM_{LL} = -\alpha \ln \lambda - \ln \Gamma \alpha - \frac{1}{\lambda} \frac{\varepsilon \pi / \sigma}{\sin(\pi / \sigma)} + (\alpha - \sigma)(E(\ln y) + \ln \varepsilon) - \ln(\sigma) + \sigma \ln \varepsilon + 2E_{LL}(\ln(1+z))$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} V_{LL}(T) = AV_{LL} = V_{LL} \left[\ln f_{GA}(X; \lambda, \alpha) - \ln f_{LL}(X; \tilde{\varepsilon}, \tilde{\sigma}) \right]$$

$$\begin{aligned} AV_{LL} &= V_{LL} \left[-\alpha \ln \lambda - \ln \Gamma \alpha + (\alpha - 1) \ln(x) - \frac{x}{\lambda} + \ln \sigma - (\sigma - 1) \ln X + \sigma \ln \varepsilon + 2 \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right] \right] \\ &= V_{LL} \left[-\alpha \ln \lambda - \ln \Gamma \alpha + (\alpha - \sigma) \ln(x) - \frac{x}{\lambda} + \ln \sigma + \sigma \ln \varepsilon + 2 \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right] \right] \\ &= (\alpha - \sigma)^2 V_{LL}(\ln(x)) + \frac{1}{\lambda^2} V_{LL}(x) + 4V_{LL} \left(\ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right] \right) - \frac{2(\alpha - \sigma)}{\lambda} \text{cov}(\ln(x), x) \\ &\quad + 4(\alpha - \sigma) \text{cov}(\ln(x), \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right]) - \frac{4}{\lambda} \text{cov}(x, \ln \left[1 + \left(\frac{X}{\varepsilon} \right)^\sigma \right]) \end{aligned}$$

we know that : $V_{LL}(x) = \varepsilon^2 (2b / \sin 2b - b^2 / \sin b^2) \dots$ if $\sigma > 2$, and where $b = \pi / \sigma$

$$\begin{aligned} AV_{LL} &= (\alpha - \sigma)^2 V_{LL}(\ln y) + \frac{1}{\lambda^2} (\varepsilon^2 (2b / \sin 2b - b^2 / \sin b^2)) + 4V_{LL}([1+z]) \\ &\quad - \frac{2(\alpha - \sigma)}{\lambda} \text{cov}((\ln y), x) + 4(\alpha - \sigma) \text{cov}((\ln y), \ln[1+z]) - \frac{4}{\lambda} \text{cov}(\ln(1+z), x) \end{aligned}$$

where $y \sim LL(1, \sigma)$ and $z \sim LL(1, 1)$

Since $x = \varepsilon y$

$$\begin{aligned} AV_{LL} &= (\alpha - \sigma)^2 V_{LL}(\ln y) + \frac{1}{\lambda^2} (\varepsilon^2 (2b / \sin 2b - b^2 / \sin b^2)) + 4V_{LL}([1+z]) \\ &\quad - \frac{2(\alpha - \sigma)}{\lambda} \text{cov}((\ln y), \varepsilon y) + 4(\alpha - \sigma) \text{cov}((\ln y), \ln[1+z]) - \frac{4}{\lambda} \text{cov}(\ln(1+z), \varepsilon y) \end{aligned}$$

$$\begin{aligned} AV_{LL} &= (\alpha - \sigma)^2 V_{LL}(\ln y) + \frac{1}{\lambda^2} (\varepsilon^2 (2b / \sin 2b - b^2 / \sin b^2)) + 4V_{LL}([1+z]) \\ &\quad - \frac{2\varepsilon(\alpha - \sigma)}{\lambda} \text{cov}((\ln y), y) + 4(\alpha - \sigma) \text{cov}((\ln y), \ln[1+z]) - \frac{4\varepsilon}{\lambda} \text{cov}(\ln(1+z), y) \end{aligned}$$

Let $z = y^\sigma$

$$\begin{aligned} AV_{LL} &= (\alpha - \sigma)^2 V_{LL}(\ln y) + \frac{1}{\lambda^2} (\varepsilon^2 (2b / \sin 2b - b^2 / \sin b^2)) + 4V_{LL}([1+y^\sigma]) \\ &\quad - \frac{2\varepsilon(\alpha - \sigma)}{\lambda} \text{cov}((\ln y), y) + 4(\alpha - \sigma) \text{cov}((\ln y), \ln[1+y^\sigma]) - \frac{4\varepsilon}{\lambda} \text{cov}(\ln(1+y^\sigma), y) \end{aligned}$$

Where $y \sim LL(1, \sigma)$

Estimation of the Bivariate Kumaraswamy Lifetime Distribution under Progressive Type I Censoring

Hanan. M. Aly . Hiba. Z. Muhammed . Ola. A. Abuelamayem

Abstract

Analyzing time to event data arises in a number of fields such as Biology and Engineering. A common feature of this data is that, the exact failure time for all units may not be observable. Accordingly, several types of censoring were presented. Progressive censoring allows units to be randomly removed before the terminal point of the experiment. Marshall-Olkin bivariate lifetime distribution was first introduced in 1967 using the exponential distribution. Recently, bivariate Marshall-Olkin Kumaraswamy lifetime distribution was derived. This paper derives the Likelihood function under progressive type I censoring for the bivariate Marshall-Olkin family in general and applies it on the bivariate Kumaraswamy lifetime distribution. Maximum likelihood estimators of model parameters were derived. Simulation study and a real data set are presented to illustrate the proposed procedure. Absolute bias, mean square error, asymptotic confidence intervals, confidence width and coverage probability are obtained. Simulation results indicate that the mean square error is smaller and confidence width is narrower and more precise when number of removals gets smaller. Also, increasing the terminal point of the experiment results in reducing the mean square error and confidence width.

Key Words

Bivariate lifetime distributions. Bivariate Kumaraswamy distribution. Bivariate Marshall-Olkin. Progressive censoring. Maximum likelihood Estimation.

Hanan. M. Aly

Dept. of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt
han_m.hasan@yahoo.com

Hiba. Z. Muhammed

Dept. of Math. Statistics, Institute of Statistical Studies and Research, Cairo University, Egypt
hiba_stat@yahoo.com

Ola. A. Abuelamayem

Dept. of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt
ola.abuelamayem@feps.edu.eg

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Abstract

Analyzing time to event data arises in a number of fields such as Biology and Engineering. A common feature of this data is that, the exact failure time for all units may not be observable. Accordingly, several types of censoring were presented. Progressive censoring allows units to be randomly removed before the terminal point of the experiment. Marshall-Olkin bivariate lifetime distribution was first introduced in 1967 using the exponential distribution. Recently, bivariate Marshall-Olkin Kumaraswamy lifetime distribution was derived. This paper derives the Likelihood function under progressive type I censoring for the bivariate Marshall-Olkin family in general and applies it on the bivariate Kumaraswamy lifetime distribution. Maximum likelihood estimators of model parameters were derived. Simulation study and a real data set are presented to illustrate the proposed procedure. Absolute bias, mean square error, asymptotic confidence intervals, confidence width and coverage probability are obtained. Simulation results indicate that the mean square error is smaller and confidence width is narrower and more precise when number of removals gets smaller. Also, increasing the terminal point of the experiment results in reducing the mean square error and confidence width.

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Hanan. M. Aly

Dept. of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt
han_m.hasan@yahoo.com

Hiba. Z. Muhammed

Dept. of Math. Statistics, Institute of Statistical Studies and Research, Cairo University, Egypt
hiba_stat@yahoo.com

Ola. A. Abuelamayem

Dept. of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt
ola.abuelamayem@feps.edu.eg

1 Introduction

In many real life situations, there is a great need to understand and study the distribution of two variables interacting together and not each one separately. There are several examples in Biology such as; the study of twins, study of blindness in the left and right eye, failure time of the left and right kidney, the age at death of parent and child in a genetic study, studying the relation between heights and weights for a children, and studying the relation between blood pressure and body weight for a patient. Also, in reliability engineering to model the lifetime of a twine-engine plane, warranty polices based on failure time and warranty servicing time, as well as, different lifetime models, for example, shock model, maintenance model, bivariate longevity model.

Different bivariate family distributions were constructed. See for example, Marshall and Olkin (1967), Sankaran and Gleeja (2006), Regoli (2009), Mirhosseini *et al* (2015). Great review for the bivariate distributions can be found in Lai and Balakrishnan (2009).

Marshall and Olkin (1967) presented a bivariate exponential distribution with exponential marginals and loss of memory property. This bivariate distribution was used to represent the shock model. Several distributions were constructed by the same way, for example, Muhammed(2015) introduced the bivariate inverse Weibull distribution. Different extensions for the Marshall - Olkin family were presented, see for example, Sarhan and Balakrishnan (2007), Jose *et al* (2011), Li and Pellerey (2011), Ozkut and Bayramoglu (2014), and Davarzani *et al* (2015).

Kumaraswamy (1976) showed that the classical distributions such as normal, lognormal, gamma, beta and others do not provide a good fit for hydrological data such as daily rainfall, daily stream flow, etc. Accordingly, the sinepower probability density function (pdf) was introduced to fit data which are bounded on both the lower and upper sides and which have a single mode within its range. Kumaraswamy (1978) extended the sinepower pdf to include cases of minimum values having a non-zero probability. Kumaraswamy (1980) relaxed the condition of having a single mode and introduced a generalized double bounded pdf which can be applied not only for hydrological data but also for other double bounded random process, such as, scores obtained in a quiz, ground water levels, atmospheric temperatures, etc.

Souza and Lemonte (2013) introduced the bivariate Kumaraswamy (BK) distribution, which can be applied in several reliability models like shock model, maintenance model and stress model. Ozel (2015) showed that the BK model fits the earthquakes well.

Censoring is a common phenomenon in analyzing time to event data. There are several types of censoring that were applied in the bivariate case; type I, censored data with concomitant of order statistics, random and hybrid random. In type I censoring, the termination time of the experiment is pre-specified, and the number of failures is random, for more details see Angali *et al* (2014). Lu (1997) introduced a censoring scheme using concomitant of order statistics. Suppose $\{(x_{11}, x_{21}), \dots, (x_{1n}, x_{2n})\}$ is a random sample of size N . Let $X_{1(i)}$ be the i^{th} order statistic and $X_{2[i]}$ be its concomitant of X_2 , the experiment is terminated at $X_{1(r)}$ and $X_{2[i]}$ are randomly censored at $X_{1(r)}$. Several work were done using this type of censoring, see for example (Chen *et al* (2000), Kavitha and Saiva (2013)). In random censoring, the censoring time varies from individual to individual and is unknown in advance. Hanagal (1992) applied the idea of random censoring in the bivariate case, and other authors have

dealt with this type of censoring, for example, Hanagal and Ahmadi (2008), Hanagal and Sharma (2015). Hybrid random censoring was first introduced in the bivariate case for the Marshall- Olkin family by Hanagal (1997), the experiment is terminated at the random time $T_i = \min(X_m, T_i)$, where X_m is the m^{th} order statistics and T_i is the censoring time for the i^{th} observation.

Type I, censored data with concomitant of order statistics, random and hybrid random censoring do not allow any unit to be randomly removed at a point different from the terminal point of the experiment. Progressive censoring deals with this disadvantage by allowing units to be randomly removed during the experiment. Progressive censoring was first introduced in the univariate case by Cohen (1963). The censoring here occurs progressively in m stages at pre-specified times t_j , such that $t_j > t_{j-1}$, $j = 1, 2, \dots, m$. Suppose N units are observed until the first censoring time t_1 when the failed n_1 units are counted, R_1 units are randomly removed from the experiment. The remaining $N - n_1 - R_1$ units are observed till the second censoring time t_2 when the failed n_2 units are counted, and R_2 units are randomly removed from the experiment and so on. At time t_m , the failed n_m units are counted and all the remaining survived units are removed (i.e. $R_m = N - \sum_{j=1}^m n_j - \sum_{j=1}^{m-1} R_j$) and the experiment terminates. For more details, see Balakrishnan and Aggarwala (2000). For the best of our knowledge, the likelihood function in the bivariate case under progressive censoring was not derived yet. This paper, considers bivariate Marshall - Olkin family and derives the likelihood under progressive type-I censoring in general. The derived likelihood function is applied on the bivariate Kumaraswamy distribution.

The paper is organized as follows. The model is described in Section 2. The derivation of the likelihood function is done in Section 3. The maximum likelihood estimators of the model parameters and fisher information matrix are presented in Section 4. In Section 5, numerical analysis including both simulated data and a real data set is done. Finally, the paper is concluded in Section 6.

2 The Model under Type I Censoring

Marshall-Olkin (1967) was the first to introduce a bivariate exponential distribution with exponential marginals, loss of memory property and with applicability in real life. Using the same procedure introduced by Marsall-Olkin (1967) several distributions were derived. Recently, Souza and Lemonte (2013) derived the BK distribution as follows.

Let Z_1 , Z_2 and Z_3 be independent random variables such that Z_i follows Kumraswamy (α_i, β) , with probability density function given by

$$f_k(z; \alpha, \beta) = \alpha \beta z^{\beta-1} (1 - z^\beta)^\alpha,$$

where $\alpha_i > 0$, $i = 1, 2, 3$ and $\beta > 0$ are shape parameters.

Assume that $X_1 = \min(Z_1, Z_3)$ and $X_2 = \min(Z_2, Z_3)$, then the bivariate vector (X_1, X_2) follows BK with the joint probability density function as follows

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & 0 < x_1 < x_2 < 1 \\ f_2(x_1, x_2), & 0 < x_2 < x_1 < 1 \\ f_3(x), & 0 < x_1 = x_2 = x < 1 \end{cases},$$

where

$$\begin{aligned} f_1(x_1, x_2) &= \alpha_1(\alpha_2 + \alpha_3)\beta^2 x_1^{\beta-1}(1-x_1^\beta)^{\alpha_1-1}x_2^{\beta-1}(1-x_2^\beta)^{(\alpha_2+\alpha_3-1)}, \\ f_2(x_1, x_2) &= \alpha_2(\alpha_1 + \alpha_3)\beta^2 x_1^{\beta-1}(1-x_1^\beta)^{\alpha_1+\alpha_3-1}x_2^{\beta-1}(1-x_2^\beta)^{(\alpha_2-1)}, \\ f_3(x) &= \alpha_3\beta x^{\beta-1}(1-x^\beta)^{\alpha_1+\alpha_2+\alpha_3-1}. \end{aligned}$$

Hangal (1992) presented type I censoring in the bivariate case. let $\{(x_{11}, x_{21}), \dots, (x_{1N}, x_{2N})\}$ be a random sample of size N , and t be a pre-specified time at which the experiment is terminated. The lifetimes associated with the i^{th} pair are given by

$$(x_{1i}, x_{2i}) = \begin{cases} (x_{1i}, x_{2i}) & \text{if } \max(x_{1i}, x_{2i}) < t \\ (x_{1i}, t) & \text{if } x_{1i} < t < x_{2i} \\ (t, x_{2i}) & \text{if } x_{2i} < t < x_{1i} \\ (t, t) & \text{if } \min(x_{1i}, x_{2i}) > t \end{cases}.$$

The Likelihood function has the following form

$$L = \prod_{i=1}^{n_1} f_1(x_{1i}, x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i}, x_{2i}) \prod_{i=1}^{n_3} f_3(x_i) \prod_{i=1}^{n_4} f_4(x_{1i}, t) \prod_{i=1}^{n_5} f_5(t, x_{2i}) \{[S(t, t)]\}^{n_6}, \quad (1)$$

where $f_1(x_{1i}, x_{2i})$ corresponds to the case where $0 < x_{1i} < x_{2i} \leq t$,

$f_2(x_{1i}, x_{2i})$ corresponds to the case where $0 < x_{2i} < x_{1i} \leq t$,

$f_3(x_i)$ corresponds to the case where $0 < x_{1i} = x_{2i} = x_i \leq t$,

$$f_4(x_{1i}, t) = \lim_{\delta x_{1i} \rightarrow 0} \frac{P(x_{1i} < X_{1i} < x_{1i} + \delta x_{1i} | X_{2i} > t) P(X_{2i} > t)}{\delta x_{1i}},$$

$$f_5(t, x_{2i}) = \lim_{\delta x_{2i} \rightarrow 0} \frac{P(x_{2i} < X_{2i} < x_{2i} + \delta x_{2i} | X_{1i} > t) P(X_{1i} > t)}{\delta x_{2i}},$$

$$S(t, t) = P(X_{1i} > t, X_{2i} > t),$$

n_i are the number of observations falling in the range corresponding to f_i , $i=1, 2, \dots, 5$, respectively, and n_6 is the number corresponding to S .

3 Derivation of the Likelihood Function under Progressive Censoring

In this Section we derive the likelihood function for the Marshall-Olkin family under progressive type I censoring in general.

Theorem

Suppose $\{(x_{11}, x_{21}), \dots, (x_{1N}, x_{2N})\}$ is a random sample of size N , and t_j be a pre-specified times, such that (x_1, x_2) is independent from t_j , $j=1, \dots, m$. The censoring here occurs progressively in m stages at pre-specified times t_j , such that $t_j > t_{j-1}$.

In the first stage. N units are observed until the first censoring time t_1 when the failed n_1 units are counted. From the survived units (i.e. $x_1 > t_1$ and $x_2 > t_1$) R_1 pairs of units are randomly removed from the experiment. The remaining $N - n_1 - R_1$ units entered the second stage. Similarly, in the second stage $N - n_1 - R_1$ are observed till the second censoring time t_2 when the failed n_2 units are counted. From the survived units (i.e. $x_1 > t_2$ and $x_2 > t_2$) R_2 pairs of units are randomly removed from the experiment and so on. At time t_m , the failed n_m units are counted. All the remaining survived pairs of units are removed (i.e. $R_m = N - \sum_{j=1}^m n_j - \sum_{j=1}^{m-1} R_j$) and the experiment terminates.

The failed units in each stage are divided according to 5 different cases illustrated as follows

$$\begin{cases} x_{1ij} < x_{2ij} \leq t_j \\ x_{2ij} < x_{1ij} \leq t_j \\ x_{1ij} = x_{2ij} = x_{ij} \leq t_j \\ x_{1ij} \leq t_j < x_{2ij} \\ x_{2ij} \leq t_j < x_{1ij} \end{cases}$$

To reflect this division we define 5 indicator functions as follows

$$\begin{aligned} \delta_{1ij} &= \begin{cases} 1 & \text{if } x_{1ij} < x_{2ij} \leq t_j \\ 0 & \text{Otherwise} \end{cases}, & \delta_{2ij} &= \begin{cases} 1 & \text{if } x_{2ij} < x_{1ij} \leq t_j \\ 0 & \text{Otherwise} \end{cases}, \\ \delta_{3ij} &= \begin{cases} 1 & \text{if } x_{1ij} = x_{2ij} \leq t_j \\ 0 & \text{Otherwise} \end{cases}, & \delta_{4ij} &= \begin{cases} 1 & \text{if } x_{1ij} \leq t_j < x_{2ij} \\ 0 & \text{Otherwise} \end{cases}, \\ \delta_{5ij} &= \begin{cases} 1 & \text{if } x_{2ij} \leq t_j < x_{1ij} \\ 0 & \text{Otherwise} \end{cases}. \end{aligned}$$

The likelihood function is derived to be as follows

$$L = \prod_{j=1}^m \prod_{i=1}^{n_j} \{ [f_1(x_{1ij}, x_{2ij})]^{\delta_{1ij}} [f_2(x_{1ij}, x_{2ij})]^{\delta_{2ij}} [f_3(x_{ij})]^{\delta_{3ij}} [f_4(x_{1ij}, t_j)]^{\delta_{4ij}} [f_5(t_j, x_{2ij})]^{\delta_{5ij}} \} [S(t_j, t_j)]^{R_j}, \quad (2)$$

where $f_1(x_{1ij}, x_{2ij})$ corresponds to the case where $0 < x_{1ij} < x_{2ij} \leq t_j$,

$f_2(x_{1ij}, x_{2ij})$ corresponds to the case where $0 < x_{2ij} < x_{1ij} \leq t_j$,

$f_3(x_{ij})$ corresponds to the case where $0 < x_{1ij} = x_{2ij} = x_{ij} \leq t_j$,

$f_4(x_{1ij}, t_j) = \lim_{\delta x_{1ij} \rightarrow 0} \frac{P(x_{1ij} < X_{1ij} < x_{1ij} + \delta x_{1ij} | X_{2ij} > t_j) P(X_{2ij} > t_j)}{\delta x_{1ij}}$,

$f_5(t_j, x_{2ij}) = \lim_{\delta x_{2ij} \rightarrow 0} \frac{P(x_{2ij} < X_{2ij} < x_{2ij} + \delta x_{2ij} | X_{1ij} > t_j) P(X_{1ij} > t_j)}{\delta x_{2ij}}$, and

$S(t_j, t_j) = P(X_{1ij} > t_j, X_{2ij} > t_j)$.

Applying on the bivariate kumaraswamy lifetime distribution, the likelihood function has

the following form

$$L = \prod_{j=1}^m \prod_{i=1}^{n_j} \{ \alpha_1^{\delta_{1ij} + \delta_{4ij}} \alpha_2^{\delta_{2ij} + \delta_{5ij}} \alpha_3^{\delta_{3ij}} \beta^{2\delta_{1ij} + 2\delta_{2ij} + \delta_{3ij} + \delta_{4ij} + \delta_{5ij}} (\alpha_2 + \alpha_3)^{\delta_{1ij}} (\alpha_1 + \alpha_3)^{\delta_{2ij}} x_{1ij}^{(\beta-1)(\delta_{1ij} + \delta_{2ij} + \delta_{4ij})} x_{2ij}^{(\beta-1)(\delta_{1ij} + \delta_{2ij} + \delta_{5ij})} x_{ij}^{(\beta-1)\delta_{3ij}} (1 - x_{1ij}^\beta)^{\delta_{1ij}(\alpha_1-1) + \delta_{2ij}(\alpha_1 + \alpha_3 - 1) + (\alpha_1-1)\delta_{4ij}} (1 - x_{2ij}^\beta)^{\delta_{2ij}(\alpha_2-1) + \delta_{1ij}(\alpha_2 + \alpha_3 - 1) + (\alpha_2-1)\delta_{5ij}} (1 - x_{ij}^\beta)^{\delta_{3ij}(\alpha_1 + \alpha_2 + \alpha_3 - 1)} (1 - t_j^\beta)^{(\alpha_2 + \alpha_3)\delta_{4ij} + (\alpha_1 + \alpha_3)\delta_{5ij}} \} (1 - t_j^\beta)^{(\alpha_1 + \alpha_2 + \alpha_3)R_j}.$$

Lemma

Suppose we only have one stage ($m=1$). The Likelihood function (2) will be reduced to type I censoring with the following form

$$L = \prod_{i=1}^{n_1} \{ [f_1(x_{1i1}, x_{2i1})]^{\delta_{1i1}} [f_2(x_{1i1}, x_{2i1})]^{\delta_{2i1}} [f_3(x_{i1})]^{\delta_{3i1}} [f_4(x_{1i1}, t_1)]^{\delta_{4i1}} [f_5(t_1, x_{2i1})]^{\delta_{5i1}} \} [S(t_1, t_1)]^{R_1}. \quad (3)$$

which is another way to express (1) as the indicators δ_{1i1} , δ_{2i1} , δ_{3i1} , δ_{4i1} , δ_{5i1} reflect n_1 , n_2 , n_3 , n_4 , n_5 , and R_1 reflects n_6 .

4 Maximum Likelihood Estimation

In this Section, the maximum likelihood estimators (MLEs) of the unknown parameters are obtained by maximizing the logarithm of the likelihood function derived in Section 3. Moreover, the observed fisher information matrix is obtained. The confidence intervals are constructed using the asymptotic properties of the MLEs.

The derivatives of the unknown parameters are as follows

$$\frac{\partial \log L}{\partial \alpha_1} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} (\delta_{1ij} + \delta_{4ij})}{\alpha_1} + \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{2ij}}{\alpha_1 + \alpha_3} + \sum_{j=1}^m \sum_{i=1}^{n_j} (\delta_{1ij} + \delta_{2ij} + \delta_{4ij}) \log(1 - x_{1ij}^\beta) + \sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{3ij} \log(1 - x_{ij}^\beta) + \sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{5ij} \log(1 - t_j^\beta) + \sum_{j=1}^m R_j \log(1 - t_j^\beta),$$

$$\frac{\partial \log L}{\partial \alpha_2} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} (\delta_{2ij} + \delta_{5ij})}{\alpha_2} + \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{1ij}}{\alpha_2 + \alpha_3} + \sum_{j=1}^m \sum_{i=1}^{n_j} (\delta_{1ij} + \delta_{2ij} + \delta_{5ij}) \log(1 - x_{2ij}^\beta) + \sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{3ij} \log(1 - x_{ij}^\beta) + \sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{4ij} \log(1 - t_j^\beta) + \sum_{j=1}^m R_j \log(1 - t_j^\beta),$$

$$\frac{\partial \log L}{\partial \alpha_3} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{1ij}}{\alpha_2 + \alpha_3} + \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{2ij}}{\alpha_1 + \alpha_3} + \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{3ij}}{\alpha_3} + \sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{2ij} \log(1 - x_{1ij}^\beta) + \sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{1ij} \log(1 - x_{2ij}^\beta) + \sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{3ij} \log(1 - x_{ij}^\beta) + \sum_{j=1}^m \sum_{i=1}^{n_j} (\delta_{4ij} + \delta_{5ij}) \log(1 - t_j^\beta) + \sum_{j=1}^m R_j \log(1 - t_j^\beta),$$

$$\frac{\partial \log L}{\partial \beta} = A_1 + B_1 + A_2 + B_2 + A_3 + B_3 + A_4 + C_1 + C_2,$$

where

$$\begin{aligned} A_1 &= \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} (2\delta_{1ij} + 2\delta_{2ij} + \delta_{3ij} + \delta_{4ij} + \delta_{5ij})}{\beta}, \\ A_2 &= -\sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\{(\alpha_1 - 1)\delta_{1ij} + (\alpha_1 + \alpha_3 - 1)\delta_{2ij} + (\alpha_1 - 1)\delta_{4ij}\} x_{1ij}^\beta \log x_{1ij}}{1 - x_{1ij}^\beta}, \\ A_3 &= -\sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\{(\alpha_2 + \alpha_3 - 1)\delta_{1ij} + (\alpha_2 - 1)\delta_{2ij} + (\alpha_2 - 1)\delta_{5ij}\} x_{2ij}^\beta \log x_{2ij}}{1 - x_{2ij}^\beta}, \\ A_4 &= -(\alpha_1 + \alpha_2 + \alpha_3 - 1) \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\delta_{3ij} x_{ij}^\beta \log x_{ij}}{1 - x_{ij}^\beta}, \quad B_1 = \sum_{j=1}^m \sum_{i=1}^{n_j} \{\delta_{1ij} + \delta_{2ij} + \delta_{4ij}\} \log x_{1ij}, \\ B_2 &= \sum_{j=1}^m \sum_{i=1}^{n_j} (\delta_{1ij} + \delta_{2ij} + \delta_{5ij}) \log x_{2ij}, \quad B_3 = \sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{3ij} \log x_{ij}, \\ C_1 &= -\sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\{(\alpha_2 + \alpha_3)\delta_{4ij} + (\alpha_1 + \alpha_3)\delta_{5ij}\} t_j^\beta \log t_j}{1 - t_j^\beta}, \text{ and} \\ C_2 &= -(\alpha_1 + \alpha_2 + \alpha_3) \sum_{j=1}^m \frac{R_j t_j^\beta \log t_j}{1 - t_j^\beta}. \end{aligned}$$

Let the fisher information matrix be denoted by

$$I(\alpha_1, \alpha_2, \alpha_3, \beta) = \begin{bmatrix} I_{11} & 0 & I_{13} & I_{14} \\ 0 & I_{22} & I_{23} & I_{24} \\ I_{13} & I_{23} & I_{33} & I_{34} \\ I_{14} & I_{24} & I_{34} & I_{44} \end{bmatrix},$$

where

$$\begin{aligned} I_{11} &= -\frac{\partial^2 \log L}{\partial \alpha_1^2} \Big|_{\hat{\alpha}_1, \hat{\alpha}_3} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} (\delta_{1ij} + \delta_{4ij})}{\hat{\alpha}_1^2} + \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{2ij}}{(\hat{\alpha}_1 + \hat{\alpha}_3)^2}, \\ I_{13} &= -\frac{\partial^2 \log L}{\partial \alpha_1 \partial \alpha_3} \Big|_{\hat{\alpha}_1, \hat{\alpha}_3} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{2ij}}{(\hat{\alpha}_1 + \hat{\alpha}_3)^2}, \\ I_{14} &= -\frac{\partial^2 \log L}{\partial \alpha_1 \partial \beta} \Big|_{\hat{\beta}} = \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{(\delta_{1ij} + \delta_{2ij} + \delta_{4ij}) x_{1ij}^{\hat{\beta}} \log x_{1ij}}{1 - x_{1ij}^{\hat{\beta}}} + \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\delta_{3ij} x_{ij}^{\hat{\beta}} \log x_{ij}}{1 - x_{ij}^{\hat{\beta}}} \\ &\quad + \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\delta_{5ij} t_j^{\hat{\beta}} \log t_j}{1 - t_j^{\hat{\beta}}} + \sum_{j=1}^m \frac{R_j t_j^{\hat{\beta}} \log t_j}{1 - t_j^{\hat{\beta}}}, \end{aligned}$$

$$\begin{aligned}
 I_{22} &= -\frac{\partial^2 \text{Log} L}{\partial \alpha_2^2} \Big|_{\hat{\alpha}_2, \hat{\alpha}_3} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{1ij}}{(\hat{\alpha}_2 + \hat{\alpha}_3)^2} + \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} (\delta_{2ij} + \delta_{5ij})}{\hat{\alpha}_2^2}, \\
 I_{23} &= -\frac{\partial^2 \text{Log} L}{\partial \alpha_2 \partial \alpha_3} \Big|_{\hat{\alpha}_2, \hat{\alpha}_3} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{1ij}}{(\hat{\alpha}_2 + \hat{\alpha}_3)^2}, \\
 I_{24} &= -\frac{\partial^2 \text{Log} L}{\partial \alpha_2 \partial \beta} \Big|_{\hat{\beta}} = \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{x_{ij}^{\hat{\beta}} \log x_{ij} \delta_{3ij}}{1 - x_{ij}^{\hat{\beta}}} + \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{(\delta_{1ij} + \delta_{2ij} + \delta_{5ij}) x_{2ij}^{\hat{\beta}} \log x_{2ij}}{1 - x_{2ij}^{\hat{\beta}}} \\
 &\quad + \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\delta_{4ij} t_j^{\hat{\beta}} \log t_j}{1 - t_j^{\hat{\beta}}} + \sum_{j=1}^m \frac{R_j t_j^{\hat{\beta}} \log t_j}{1 - t_j^{\hat{\beta}}}, \\
 I_{33} &= -\frac{\partial^2 \text{Log} L}{\partial \alpha_3^2} \Big|_{\hat{\alpha}_2, \hat{\alpha}_3} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{1ij}}{(\hat{\alpha}_2 + \hat{\alpha}_3)^2} + \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{2ij}}{(\hat{\alpha}_1 + \hat{\alpha}_3)^2} + \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} \delta_{3ij}}{\hat{\alpha}_3^2}, \\
 I_{34} &= -\frac{\partial^2 \text{Log} L}{\partial \alpha_3 \partial \beta} \Big|_{\hat{\beta}} = \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\delta_{2ij} x_{1ij}^{\hat{\beta}} \log x_{1ij}}{1 - x_{1ij}^{\hat{\beta}}} + \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\delta_{1ij} x_{2ij}^{\hat{\beta}} \log x_{2ij}}{1 - x_{2ij}^{\hat{\beta}}} \\
 &\quad + \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\delta_{3ij} x_{ij}^{\hat{\beta}} \log x_{ij}}{1 - x_{ij}^{\hat{\beta}}} + \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{(\delta_{4ij} + \delta_{5i}) t_j^{\hat{\beta}} \log t_j}{1 - t_j^{\hat{\beta}}} + \sum_{j=1}^m \frac{R_j t_j^{\hat{\beta}} \log t_j}{1 - t_j^{\hat{\beta}}}, \text{ and} \\
 I_{44} &= -\frac{\partial^2 \text{Log} L}{\partial \beta^2} \Big|_{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\beta}} = D_1 + D_2 + D_3 + D_4 + E_1 + E_2,
 \end{aligned}$$

where

$$\begin{aligned}
 D_1 &= \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} (2\delta_{1ij} + 2\delta_{2ij} + \delta_{3ij} + \delta_{4ij} + \delta_{5ij})}{\hat{\beta}^2}, \\
 D_2 &= \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\{(\hat{\alpha}_1 - 1)\delta_{1ij} + (\hat{\alpha}_1 + \hat{\alpha}_3 - 1)\delta_{2ij} + (\hat{\alpha}_1 - 1)\delta_{4ij}\} x_{1ij}^{\hat{\beta}} (\log x_{1ij})^2}{(1 - x_{1ij}^{\hat{\beta}})^2}, \\
 D_3 &= \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\{(\hat{\alpha}_2 + \hat{\alpha}_3 - 1)\delta_{1ij} + (\hat{\alpha}_2 - 1)\delta_{2ij} + (\hat{\alpha}_2 - 1)\delta_{5ij}\} x_{2ij}^{\hat{\beta}} (\log x_{2ij})^2}{(1 - x_{2ij}^{\hat{\beta}})^2}, \\
 D_4 &= (\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 - 1) \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\delta_{3ij} (\log x_{ij})^2 x_{ij}^{\hat{\beta}}}{(1 - x_{ij}^{\hat{\beta}})^2}, \\
 E_1 &= \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\{(\hat{\alpha}_2 + \hat{\alpha}_3)\delta_{4ij} + (\hat{\alpha}_1 + \hat{\alpha}_3)\delta_{5ij}\} (\log t_j)^2 t_j^{\hat{\beta}}}{(1 - t_j^{\hat{\beta}})^2}, \\
 E_2 &= (\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3) \sum_{j=1}^m \frac{R_j (\log t_j)^2 t_j^{\hat{\beta}}}{(1 - t_j^{\hat{\beta}})^2}.
 \end{aligned}$$

Using the asymptotic distribution of the MLEs, the confidence intervals can be obtained as follows;

$$\hat{\lambda} \pm z_{\frac{\gamma}{2}} \sqrt{\text{var}(\hat{\lambda})},$$

where $\hat{\lambda} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\beta})$, $var(\hat{\alpha})$ is the estimated variance, and $z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ th percentile of the standard normal table.

5 Numerical Results

In this Section, a simulation study is carried out to investigate the performance of the derived likelihood function under different schemes using different arbitrary chosen values of t , R and using different n , and also, using different populations. Moreover, a real data set is analyzed.

5.1 Simulation Results

Here, the following schemes are considered

- Scheme 1 : $n=40, m=2, R_1=3$;
- Scheme 2: $n=40, m=2, R_1=5$;
- Scheme 3: $n=60, m=2, R_1=12$;
- Scheme 4: $n=60, m=2, R_1=18$;
- Scheme 5: $n=50, m=3, R_1=R_2=2$;
- Scheme 6: $n=50, m=3, R_1=R_2=3$;
- Scheme 7: $n=70, m=3, R_1=R_2=5$;
- Scheme 8: $n=70, m=3, R_1=R_2=7$.

R program were used with 5000 replications, the results are presented in Tables (1) to (8) for schemes (1) to (8) respectively. Absolute Bias (ABias), Mean Square Error (MSE), Confidence Interval (CI), Confidence Width (CW) and Coverage Probability (CP) are obtained for each scheme. The results are analyzed by two different ways. First, consider the following division to study the effect of increasing the number of removals

- Set 1: schemes 1, 3, 5 and 7, and
- Set 2: schemes 2, 4, 6 and 8.

It can be seen that for the majority of the cases MSEs are smaller in set 1 than that in set 2. Also CWs are narrower for the majority of the cases in set 1 than that in set 2. ABias decreases for more than half of the cases in set 1 than that in set 2.

Second, studying the effect of increasing the value of the terminal point of the experiment as follows

- Set 1: $t_1=0.05$ and $t_2=0.4$ (i.e. Table 1(a), Table 2(a), Table 3(a), Table 4(a)),
- Set 2: $t_1=0.1$ and $t_2=0.6$ (i.e. Table 1(b), Table 2(b), Table 3(b), Table 4(b)),

- Set 3: $t_1=0.09$, $t_2=0.15$ and $t_3=0.4$ (i.e. Table 5(a), Table 6(a), Table 7(a), Table 8(a)), and
- Set 4: $t_1=0.05$, $t_2=0.25$ and $t_3=0.55$ (i.e. Table 5(b), Table 6(b), Table 7(b), Table 8(b)).

Increasing the terminal point of the experiment results in decreasing MSEs. Also, it leads to narrower and more precise CWs. Finally, by summarizing all tables, it is clear that CIs includes the true values of the proposed parameters. Also, for the majority of the cases the coverage probability is above 93%, β considers more satisfactory as its coverage probability is more close to 95%.

Table 1(a): The results under scheme 1 ($R_1=3$, $T_1 = 0.05$, $T_2 = 0.4$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.036	0.044	0.040	0.023
MSE	0.046	0.048	0.044	0.012
CI	(0.120,0.952)	(0.124,0.964)	(0.138,0.942)	(0.317,0.729)
CW	0.832	0.840	0.804	0.412
CP	93.36	93.84	94.00	94.98
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.039	0.034	0.032	0.027
MSE	0.046	0.034	0.032	0.014
CI	(0.128,0.950)	(0.078,0.790)	(0.087,0.777)	(0.304,0.750)
CW	0.822	0.712	0.690	0.446
CP	94.04	93.50	93.70	95.28
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.106	0.104	0.092	0.037
MSE	0.242	0.241	0.213	0.036
CI	(0.164,2.048)	(0.164,2.044)	(0.205,1.979)	(0.670,1.404)
CW	1.884	1.880	1.774	0.734
CP	93.56	93.84	93.38	95.04
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.109	0.119	0.108	0.032
MSE	0.311	0.316	0.278	0.031
CI	(0.237,2.381)	(0.242,2.396)	(0.297,2.319)	(0.693,1.371)
CW	2.144	2.154	2.022	0.678
CP	93.80	93.58	94.30	95.50

Table 1(b): The results under scheme 1 ($R_1=3$, $T_1 = 0.1$, $T_2 = 0.6$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.034	0.040	0.033	0.024
MSE	0.036	0.038	0.033	0.011
CI	(0.167,0.901)	(0.168,0.912)	(0.182,0.884)	(0.328,0.720)
CW	0.734	0.744	0.702	0.392
CP	94.32	94.22	94.76	94.96

Table 1(b): The results under scheme 1 ($R_1=3, T_1 = 0.1, T_2 = 0.6$) (Cont'd)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.035	0.027	0.023	0.023
MSE	0.034	0.025	0.023	0.012
CI	(0.179,0.891)	(0.123,0.731)	(0.132,0.714)	(0.317,0.729)
CW	0.712	0.608	0.582	0.412
CP	94.60	94.22	93.36	95.12
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.088	0.083	0.081	0.036
MSE	0.161	0.158	0.138	0.028
CI	(0.321,1.855)	(0.321,1.845)	(0.372,1.790)	(0.714,1.358)
CW	1.534	1.524	1.418	0.644
CP	94.42	94.34	94.66	94.58
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.101	0.105	0.099	0.033
MSE	0.228	0.230	0.196	0.025
CI	(0.386,2.216)	(0.388,2.222)	(0.454,2.144)	(0.729,1.337)
CW	1.830	1.834	1.690	0.608
CP	94.14	94.42	94.52	94.78

Table 2 (a): The results under scheme 2 ($R_1=5, T_1 = 0.05, T_2 = 0.4$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.047	0.042	0.036	0.023
MSE	0.051	0.051	0.046	0.013
CI	(0.113, 0.981)	(0.108,0.976)	(0.120,0.952)	(0.308,0.738)
CW	0.868	0.868	0.832	0.430
CP	94.18	93.72	93.34	95.56
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.044	0.036	0.027	0.026
MSE	0.050	0.036	0.033	0.014
CI	(0.115,0.973)	(0.069,0.803)	(0.076,0.778)	(0.303,0.749)
CW	0.858	0.734	0.702	0.446
CP	93.84	93.14	92.68	94.78
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.116	0.111	0.113	0.046
MSE	0.265	0.261	0.238	0.038
CI	(0.132,2.100)	(0.133,2.089)	(0.183,2.043)	(0.674,1.418)
CW	1.968	1.956	1.860	0.744
CP	93.30	93.34	94.08	95.14
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.128	0.132	0.121	0.036
MSE	0.342	0.346	0.304	0.032
CI	(0.209,2.447)	(0.208,2.456)	(0.267,2.375)	(0.691,1.381)
CW	2.238	2.248	2.108	0.690
CP	93.86	93.62	94.10	94.78

Table 2(b): The results under scheme 2 ($R_1=5, T_1 = 0.1, T_2 = 0.6$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.038	0.038	0.037	0.023
MSE	0.040	0.039	0.036	0.011
CI	(0.151,0.925)	(0.156,0.920)	(0.170,0.904)	(0.327,0.719)
CW	0.774	0.764	0.734	0.392
CP	93.20	93.84	94.28	95.06
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.038	0.030	0.029	0.024
MSE	0.037	0.027	0.025	0.012
CI	(0.166,0.910)	(0.114,0.746)	(0.125,0.733)	(0.318,0.730)
CW	0.744	0.632	0.608	0.412
CP	94.48	93.86	94.22	95.08
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.085	0.094	0.082	0.034
MSE	0.170	0.173	0.148	0.028
CI	(0.294,1.876)	(0.300,1.888)	(0.346,1.818)	(0.712,1.356)
CW	1.582	1.588	1.472	0.644
CP	94.24	94.50	94.82	94.94
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.112	0.108	0.102	0.037
MSE	0.250	0.247	0.211	0.026
CI	(0.358,2.266)	(0.358,2.258)	(0.423,2.181)	(0.727,1.347)
CW	1.908	1.900	1.758	0.620
CP	94.54	94.32	94.20	94.84

Table 3 (a): The results under scheme 3 ($R_1=12, T_1 = 0.05, T_2 = 0.4$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.034	0.031	0.028	0.019
MSE	0.035	0.034	0.032	0.008
CI	(0.173, 0.895)	(0.175, 0.887)	(0.183,0.873)	(0.344,0.694)
CW	0.722	0.712	0.690	0.350
CP	93.46	94.38	94.26	94.54
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.036	0.032	0.027	0.022
MSE	0.034	0.025	0.024	0.009
CI	(0.180,0.892)	(0.128,0.736)	(0.130,0.724)	(0.336,0.708)
CW	0.712	0.608	0.594	0.372
CP	94.22	94.16	93.88	94.84
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.082	0.086	0.070	0.028
MSE	0.168	0.169	0.148	0.025
CI	(0.296,1.868)	(0.297,1.875)	(0.329,1.811)	(0.724,1.332)
CW	1.572	1.578	1.482	0.608
CP	93.90	94.32	94.12	94.86

Table 3 (a): The results under scheme 3 ($R_1=12, T_1 = 0.05, T_2 = 0.4$)(Cont'd)

	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.088	0.084	0.082	0.022
MSE	0.219	0.217	0.194	0.021
CI	(0.388,2.188)	(0.386,2.182)	(0.434,2.130)	(0.738,1.306)
CW	1.800	1.796	1.696	0.568
CP	94.40	94.60	94.64	95.28

Table 3(b): The results under scheme 3 ($R_1=12, T_1 = 0.1, T_2 = 0.6$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.028	0.030	0.029	0.017
MSE	0.028	0.028	0.026	0.007
CI	(0.206,0.850)	(0.208,0.852)	(0.219,0.839)	(0.353,0.681)
CW	0.644	0.644	0.620	0.328
CP	94.36	94.26	94.38	95.02

	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.029	0.021	0.021	0.018
MSE	0.026	0.018	0.017	0.008
CI	(0.219,0.839)	(0.158,0.684)	(0.165,0.677)	(0.343,0.693)
CW	0.620	0.526	0.512	0.350
CP	94.54	94.40	94.42	95.36

	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.063	0.070	0.061	0.027
MSE	0.115	0.117	0.101	0.020
CI	(0.410,1.716)	(0.414,1.726)	(0.451,1.671)	(0.757,1.297)
CW	1.306	1.312	1.220	0.540
CP	94.12	94.32	94.54	94.64

	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.090	0.080	0.086	0.027
MSE	0.171	0.168	0.147	0.018
CI	(0.499,2.081)	(0.491,2.069)	(0.553,2.019)	(0.771,1.283)
CW	1.582	1.578	1.466	0.512
CP	94.96	94.32	95.60	94.98

Table 4 (a): The results under scheme 4 ($R_1=18, T_1 = 0.05, T_2 = 0.4$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.031	0.035	0.034	0.017
MSE	0.040	0.040	0.038	0.008
CI	(0.144, 0.918)	(0.148, 0.922)	(0.157,0.911)	(0.342,0.692)
CW	0.774	0.774	0.754	0.350
CP	93.72	93.96	93.18	94.60

Table 4 (a): The results under scheme 4 ($R_1=18, T_1 = 0.05, T_2 = 0.4$) (Cont'd)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.038	0.032	0.029	0.020
MSE	0.039	0.029	0.027	0.009
CI	(0.156,0.920)	(0.104,0.760)	(0.113,0.745)	(0.334,0.706)
CW	0.764	0.656	0.632	0.372
CP	94.02	93.50	94.40	95.00
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.087	0.097	0.085	0.031
MSE	0.195	0.198	0.176	0.026
CI	(0.239,1.935)	(0.245,1.949)	(0.279,1.891)	(0.721,1.341)
CW	1.696	1.704	1.612	0.620
CP	92.96	94.22	94.08	95.10
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.087	0.094	0.090	0.025
MSE	0.251	0.255	0.226	0.023
CI	(0.321,2.253)	(0.322,2.266)	(0.375,2.205)	(0.734,1.316)
CW	1.932	1.944	1.830	0.582
CP	93.28	93.34	93.74	95.10

Table 4(b): The results under scheme 4 ($R_1=18, T_1 = 0.1, T_2 = 0.6$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.039	0.043	0.037	0.019
MSE	0.036	0.036	0.032	0.007
CI	(0.178,0.900)	(0.182,0.904)	(0.192,0.882)	(0.355,0.683)
CW	0.722	0.722	0.690	0.328
CP	94.62	94.24	94.10	95.06
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.035	0.026	0.025	0.018
MSE	0.031	0.023	0.021	0.008
CI	(0.196,0.874)	(0.135,0.717)	(0.148,0.702)	(0.343,0.693)
CW	0.678	0.582	0.554	0.350
CP	94.62	94.46	94.06	95.18
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.065	0.073	0.059	0.026
MSE	0.135	0.137	0.116	0.021
CI	(0.356,1.774)	(0.361,1.785)	(0.400,1.718)	(0.749,1.303)
CW	1.418	1.424	1.318	0.554
CP	93.74	94.20	94.36	94.16
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.097	0.110	0.093	0.031
MSE	0.206	0.212	0.179	0.019
CI	(0.427,2.167)	(0.433,2.187)	(0.485,2.101)	(0.768,1.294)
CW	1.740	1.754	1.616	0.526
CP	94.42	94.80	94.96	94.50

Table 5(a): The results under scheme 5 ($R_1=R_2=2, T_1 = 0.09, T_2 = 0.15, T_3=0.4$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.019	0.021	0.023	0.024
MSE	0.034	0.034	0.034	0.010
CI	(0.158,0.880)	(0.160,0.882)	(0.167,0.879)	(0.338,0.710)
CW	0.722	0.722	0.712	0.372
CP	93.00	93.62	92.90	94.02
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.027	0.020	0.022	0.028
MSE	0.035	0.025	0.024	0.012
CI	(0.166, 0.888)	(0.110, 0.730)	(0.118, 0.726)	(0.322, 0.734)
CW	0.722	0.620	0.608	0.412
CP	93.54	93.16	93.48	94.98
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.085	0.082	0.091	0.043
MSE	0.189	0.188	0.180	0.031
CI	(0.249,1.921)	(0.248, 1.916)	(0.278, 1.904)	(0.709,1.377)
CW	1.672	1.668	1.626	0.668
CP	94.26	93.44	93.90	95.12
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.098	0.096	0.094	0.038
MSE	0.251	0.250	0.233	0.026
CI	(0.336,2.260)	(0.334, 2.258)	(0.366,2.222)	(0.728,1.348)
CW	1.924	1.924	1.856	0.620
CP	93.74	93.66	94.10	94.64

Table 5(b): The results under scheme 5 ($R_1=R_2=2, T_1 = 0.05, T_2 = 0.25, T_3=0.55$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.019	0.016	0.014	0.019
MSE	0.028	0.028	0.026	0.008
CI	(0.191, 0.847)	(0.188, 0.844)	(0.198,0.830)	(0.344,0.694)
CW	0.656	0.656	0.632	0.350
CP	93.16	93.38	93.46	94.50
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.019	0.016	0.013	0.021
MSE	0.026	0.020	0.019	0.009
CI	(0.203, 0.835)	(0.139,0.693)	(0.143, 0.683)	(0.335, 0.707)
CW	0.632	0.554	0.540	0.372
CP	93.68	93.08	93.28	95.06

Table 5(b): The results under scheme 5 ($R_1=R_2=2$, $T_1 = 0.05$, $T_2 = 0.25$, $T_3=0.55$)(Cont'd)

	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.064	0.062	0.063	0.036
MSE	0.135	0.134	0.124	0.025
CI	(0.355,1.773)	(0.355,1.769)	(0.384,1.742)	(0.732,1.340)
CW	1.418	1.414	1.358	0.608
CP	94.30	94.70	94.46	95.34
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.078	0.074	0.063	0.031
MSE	0.193	0.191	0.172	0.022
CI	(0.430,2.126)	(0.429,2.119)	(0.460, 2.066)	(0.747,1.315)
CW	1.696	1.690	1.606	0.568
CP	93.88	93.90	94.32	94.90

Table 6 (a): The results under scheme 6 ($R_1=R_2=3$, $T_1 = 0.09$, $T_2 = 0.15$, $T_3=0.4$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.019	0.017	0.017	0.025
MSE	0.035	0.035	0.033	0.010
CI	(0.152,0.886)	(0.150,0.884)	(0.161,0.873)	(0.339,0.711)
CW	0.734	0.734	0.712	0.372
CP	92.92	92.60	92.90	94.30
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.017	0.015	0.014	0.026
MSE	0.033	0.025	0.024	0.012
CI	(0.161, 0.873)	(0.105, 0.725)	(0.110, 0.718)	(0.320, 0.732)
CW	0.712	0.620	0.608	0.412
CP	93.22	92.58	92.98	94.40
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.098	0.084	0.076	0.049
MSE	0.200	0.193	0.179	0.031
CI	(0.244, 1.952)	(0.239, 1.929)	(0.261, 1.891)	(0.715,1.383)
CW	1.709	1.690	1.630	0.668
CP	93.40	93.20	93.72	94.88
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.091	0.101	0.093	0.045
MSE	0.255	0.261	0.241	0.027
CI	(0.317, 2.265)	(0.319, 2.283)	(0.349,2.237)	(0.735,1.355)
CW	1.948	1.964	1.888	0.620
CP	93.90	93.88	93.66	94.46

Table 6(b): The results under scheme 6 ($R_1=R_2=3$, $T_1 = 0.05$, $T_2 = 0.25$, $T_3=0.55$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.012	0.015	0.015	0.018
MSE	0.028	0.029	0.027	0.008
CI	(0.184, 0.840)	(0.181, 0.849)	(0.193,0.837)	(0.343,0.693)
CW	0.656	0.668	0.644	0.350
CP	93.84	94.16	93.36	95.66
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.015	0.011	0.011	0.024
MSE	0.027	0.020	0.019	0.010
CI	(0.193,0.837)	(0.134,0.688)	(0.141,0.681)	(0.338, 0.710)
CW	0.644	0.554	0.540	0.372
CP	93.38	92.82	93.54	94.72
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.063	0.060	0.056	0.038
MSE	0.141	0.141	0.128	0.025
CI	(0.338, 1.788)	(0.335, 1.785)	(0.363, 1.749)	(0.734,1.342)
CW	1.450	1.450	1.386	0.608
CP	93.80	93.48	93.66	94.48
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.063	0.070	0.065	0.033
MSE	0.197	0.200	0.181	0.023
CI	(0.402,2.124)	(0.404,2.136)	(0.440,2.090)	(0.742,1.324)
CW	1.722	1.732	1.650	0.582
CP	93.10	93.62	93.62	94.36

Table 7 (a): The results under scheme 7 ($R_1=R_2=5$, $T_1 = 0.09$, $T_2 = 0.15$, $T_3=0.4$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.006	0.005	0.003	0.021
MSE	0.023	0.023	0.022	0.007
CI	(0.209,0.803)	(0.208,0.802)	(0.212,0.794)	(0.357,0.685)
CW	0.594	0.594	0.582	0.328
CP	93.32	92.78	92.72	94.94
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.010	0.007	0.006	0.024
MSE	0.023	0.017	0.016	0.009
CI	(0.213, 0.807)	(0.151, 0.663)	(0.158, 0.654)	(0.349, 0.699)
CW	0.594	0.512	0.496	0.350
CP	93.40	92.56	93.26	94.96

Table 7 (a): The results under scheme 7 ($R_1=R_2=5$, $T_1 = 0.09$, $T_2 = 0.15$, $T_3=0.4$)(Cont'd)

	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.059	0.064	0.054	0.044
MSE	0.127	0.129	0.119	0.023
CI	(0.369, 1.749)	(0.371, 1.757)	(0.386, 1.722)	(0.760,1.328)
CW	1.380	1.386	1.336	0.568
CP	93.92	93.88	94.10	94.38
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.081	0.078	0.070	0.046
MSE	0.177	0.175	0.162	0.020
CI	(0.473, 2.089)	(0.472, 2.084)	(0.493,2.047)	(0.783,1.309)
CW	1.616	1.612	1.554	0.526
CP	94.52	94.60	94.64	94.48

Table 7(b): The results under scheme 7 ($R_1=R_2=5$, $T_1 = 0.05$, $T_2 = 0.25$, $T_3=0.55$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.003	0.006	0.005	0.017
MSE	0.020	0.020	0.019	0.006
CI	(0.226, 0.780)	(0.229, 0.783)	(0.235,0.775)	(0.365,0.669)
CW	0.554	0.554	0.540	0.304
CP	93.16	93.24	93.82	95.04
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.001	0.003	0.003	0.017
MSE	0.018	0.014	0.013	0.007
CI	(0.238,0.764)	(0.171,0.635)	(0.180,0.626)	(0.353, 0.681)
CW	0.526	0.464	0.446	0.328
CP	93.66	93.00	93.64	94.84
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.035	0.039	0.039	0.031
MSE	0.093	0.094	0.087	0.018
CI	(0.441, 1.629)	(0.445, 1.633)	(0.468, 1.610)	(0.775,1.287)
CW	1.188	1.188	1.142	0.512
CP	93.98	94.00	94.80	94.84
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.040	0.042	0.042	0.029
MSE	0.133	0.133	0.122	0.017
CI	(0.531,1.949)	(0.533,1.951)	(0.563,1.921)	(0.781,1.277)
CW	1.418	1.418	1.358	0.496
CP	93.72	93.78	93.82	94.74

Table 8 (a): The results under scheme 8 ($R_1=R_2=7$, $T_1 = 0.09$, $T_2 = 0.15$, $T_3=0.4$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.002	0.004	0.003	0.022
MSE	0.024	0.023	0.022	0.007
CI	(0.198,0.806)	(0.199,0.793)	(0.206,0.788)	(0.358,0.686)
CW	0.608	0.594	0.582	0.328
CP	92.26	91.90	92.76	94.06
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.005	0.001	0.002	0.027
MSE	0.023	0.017	0.017	0.009
CI	(0.208, 0.802)	(0.145, 0.657)	(0.146, 0.658)	(0.352, 0.702)
CW	0.594	0.512	0.512	0.350
CP	92.88	92.70	92.90	94.20
	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.059	0.061	0.063	0.052
MSE	0.134	0.135	0.127	0.024
CI	(0.350,1.768)	(0.352, 1.770)	(0.376, 1.750)	(0.768,1.336)
CW	1.418	1.418	1.374	0.568
CP	93.40	93.78	94.28	94.32
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.061	0.071	0.064	0.048
MSE	0.178	0.181	0.168	0.020
CI	(0.443, 2.079)	(0.449, 2.093)	(0.470,2.058)	(0.785,1.311)
CW	1.636	1.644	1.588	0.526
CP	93.58	93.64	93.84	93.66

Table 8(b): The results under scheme 8 ($R_1=R_2=7$, $T_1 = 0.05$, $T_2 = 0.25$, $T_3=0.55$)

	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$	$\alpha_3 = 0.5$	$\beta=0.5$
ABias	0.006	0.009	0.005	0.014
MSE	0.020	0.020	0.019	0.006
CI	(0.217,0.771)	(0.214,0.768)	(0.225,0.765)	(0.362,0.666)
CW	0.554	0.554	0.540	0.304
CP	92.52	91.92	92.48	95.04
	$\alpha_1 = 0.5$	$\alpha_2 = 0.4$	$\alpha_3 = 0.4$	$\beta=0.5$
ABias	0.004	0.005	0.002	0.018
MSE	0.019	0.014	0.013	0.007
CI	(0.226, 0.766)	(0.163, 0.627)	(0.175, 0.621)	(0.354, 0.682)
CW	0.540	0.464	0.446	0.328
CP	92.48	92.50	92.62	94.68

Table 8(b): The results under scheme 8 ($R_1=R_2=7$, $T_1 = 0.05$, $T_2 = 0.25$, $T_3=0.55$) (Cont'd)

	$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	$\beta=1$
ABias	0.034	0.034	0.032	0.040
MSE	0.098	0.098	0.091	0.020
CI	(0.424,1.644)	(0.424, 1.644)	(0.444, 1.620)	(0.777,1.303)
CW	1.220	1.220	1.176	0.526
CP	92.90	93.86	93.10	93.82
	$\alpha_1 = 1.2$	$\alpha_2 = 1.2$	$\alpha_3 = 1.2$	$\beta=1$
ABias	0.040	0.042	0.037	0.033
MSE	0.142	0.142	0.129	0.017
CI	(0.507, 1.973)	(0.509, 1.975)	(0.536,1.938)	(0.785,1.281)
CW	1.466	1.466	1.402	0.496
CP	93.98	93.56	94.06	94.72

5.2 A Real Life Data Set

A real data set is obtained from Lemonte and Souza (2013), it corresponds to football (soccer) data such that at least one goal scored by the home team and at least one goal scored directly from a penalty kick, a foul kick or any other direct kick by any team have been considered. Let Z_1 corresponds to the time in minutes of the first kick goal scored by any team, and Z_2 corresponds to the first goal of any type scored by the home team. All the data points have been divided by 90. Accordingly, we analyze the proportion of time that the home team and any other team scored the first kick goal. Lemonte and Souza (2013) computed the modified Cramer-vonMises and Anderson-Darling statistics. The values of the two statistics and their corresponding p-values showed that the BK model fits the data well.

We will try to apply the data they used to illustrate our procedure and we consider the following two schemes

- Scheme 1: $n=37$, $m=2$, $t_1= 0.25$ and $t_2=0.55$, $R_1=3$ and $R_2=5$, and
- Scheme 2: $n=37$, $m=2$, $t_1= 0.1$ and $t_2=0.4$, $R_1=3$ and $R_2=11$.

CIs, CWs and MLEs are presented in Table 9 and it is observed that scheme 1 is more precise than scheme 2. which is a logical result due to increasing the number of the analyzed failed units. Also, it supports the conclusion obtained from studying the effect of increasing the terminal point of the experiment in the simulation study.

Table 9: Estimated parameters and CWs for the BK distribution.

$T_1 = 0.25, T_2 = 0.55$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\beta}$
estimate	0.396	1.394	1.067	1.284
CI	(-0.038, 0.830)	(0.395,2.393)	(0.273,1.861)	(0.816,1.752)
CW	0.868	1.998	1.588	0.936

Table 9: Estimated parameters and CWs for the BK distribution.(Cont'd)

$T_1 = 0.1, T_2 = 0.4$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\beta}$
estimate	0.585	1.394	1.088	1.293
CI	(-0.088,1.258)	(0.151,2.637)	(0.096,2.080)	(0.771,1.815)
CW	1.346	2.486	1.984	1.044

6 Conclusion

In this paper, the likelihood function was derived for the Marshall-Olkin family under progressive type I censoring. The derived likelihood function was applied on the Marshall -Olkin bivariate Kumaraswamy lifetime distribution. Maximum likelihood estimation are obtained to estimate the unknown parameters. Moreover, asymptotic confidence intervals are constructed using observed fisher information matrix. Simulation results indicated that for most of the cases MSEs are smaller, CWs are narrower and more precise when numbers of removals are less. Also, increasing the terminal point of the experiment results in reducing the MSEs and CWs. Moreover, a real data set is analyzed using different values of the terminal point which supports the last conclusion.

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Seasonal Adjustment of Sudan Airways Passengers Time Series

Mohammed Ahmed Ibrahim

Dept. of Math., Faculty of Sc. & Tech., Omdurman Islamic University, Khartoum, Sudan,

E mail: abushouck@gmail.com

ABSTRACT

In this paper, one of advanced and electronic method of seasonal adjustment of time series, namely, X-12-ARIMA as well as classical decomposition method are applied to a monthly time series of the Sudan Airways passengers in period (1993-2006).

It is shown that, the X-12-ARIMA method led to smoother trend/cycle series, more flexible seasonal factor series and more isolated irregular variations than the simple classical decomposition method. Use of the two methods in forecasting number of passengers in 2007 also showed that the sum of squares residuals for the X-12-ARIMA method forecasts is less than that for the classical decomposition method.

1. Introduction

The fluctuations seen in a time series can be classified as repeatable or non-repeatable. Seasonality can be defined as a pattern of a time series, which repeats at regular intervals every year. Seasonal adjustment is a statistical technique which eliminates the influences of weather, holidays, and other recurring seasonal events from economic time series. This permits easier observation and analysis of trend/cycle, and other non-seasonal movements in the data. By eliminating seasonal fluctuations, the series becomes smoother and it is easier to compare data from month to month.

In seasonal adjusted data, the impact of the regular within-a-year seasonal pattern, the influences of moving holidays and the number of working/trading days and the weekday composition in each period (the trading day effect, for short) are removed. By removing the repeated impact of these effects, seasonal adjusted data highlight the underlying trends and short-run movements in time series.

The main reasons for adjusting a time series seasonally may be summarised in the following:-

- 1- Helping to carry out short-term prediction.

- 2- Link the time-series under study with other variables.
- 3- Being able to compare the value of the time series from month to month.

2. Methodolgy

In this paper X-12-ARIMA can be used for adjusted time series seasonally, X-12-ARIMA is the Census Bureau's new method in the X-11 family of seasonal adjustment programs. X-12-ARIMA uses signal-to-noise ratios to choose between a fixed set of moving-average filters, often called X-11-type filters. X-12-ARIMA is based on the well-known X-11 electronic method (Shiskin, et. al., 1967) and Statistics Canada's X-11-ARIMA and X-11-ARIMA/88 (Dagum, 1980, 1988). Major improvements in X-12-ARIMA over X-11-ARIMA/88 are discussed in Findley, et. al.,(1998). (See also U.S. Census Bureau, 2002, 2013).

2.1 Description of the methods used in the X-11 family

The moving average method that is applied in the X-11 family can be described in main features by the following iterative process;

The first task is to include pre-adjustments if applicable.

The next step is the estimation of the trend-cycle of the pre-adjusted series.

This is conducted by a calculation of centred moving averages. When the figures are quarterly then the length of the moving averages applied is four quarters, while the length is twelve for monthly series.

The trend-cycle is removed from the original series by division/subtraction (correspondence to the type of model, either multiplicative or additive) of the estimated trend-cycle in/from the original series. The series that have the trend-cycle removed are denoted the SI – rate (Seasonal and Irregular).

Extreme values of these SI – rate are substituted by less extreme values.

These modified values of the SI – rate are then smoothened by applying moving averages. The seasonal factors that are calculated in this manner are then divided up into or subtracted from the original series, and hence a first estimate of the seasonally adjusted series is obtained.

Using this initial estimate of the seasonally adjusted series as a starting point, the first steps of calculations (1.-5.) are repeated three times. The aim is an improvement, i.e. a smoothing of the trend-cycle estimate, and hence a better identification of extreme values.

2.2 The X-12-ARIMA Method

The X-12-ARIMA method is a seasonal adjustment electronic method of the US Census Bureau extracts the different components (mainly: seasonal component, trend component, outlier component and irregular component) of a monthly or quarterly time series.

The structure in X-12-ARIMA is given in the following figure (Linde, 2005).

Fig (2.1): X-12-ARIMA structure

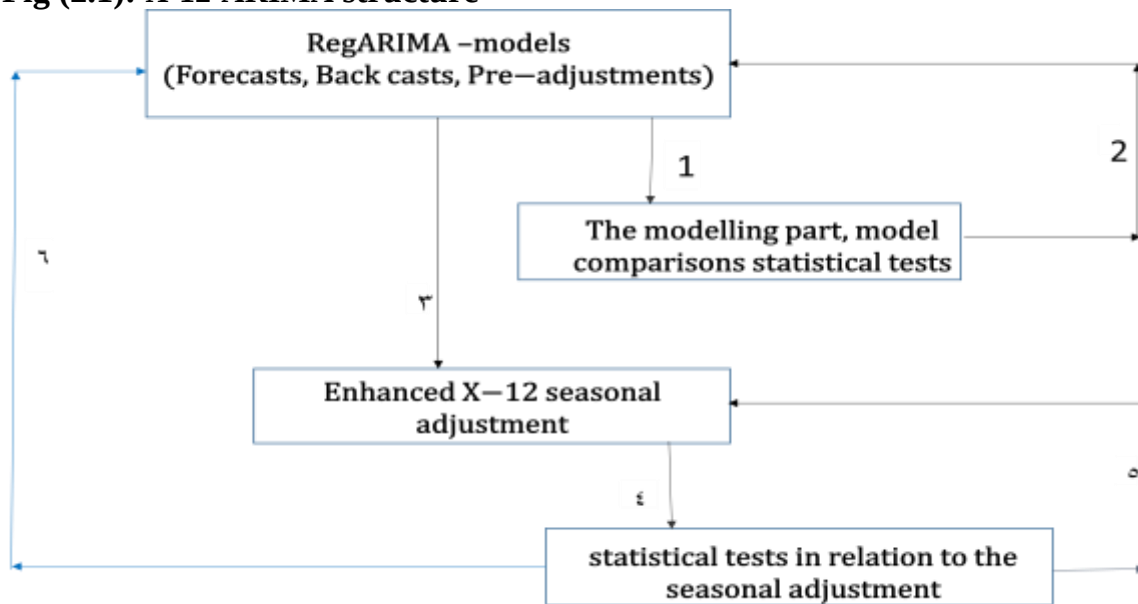


Figure (2. 1), read as follows:

The Reg-ARIMA-models are used as a starting point, then the next item is the model comparisons, and thereafter the aim is again the Regalia-part.

Has the modelling part turned out successfully in the sense that a suitable model has been chosen then the next step is to perform the seasonal adjustment. If no suitable

model has been chosen, then one returns to the starting point until a suitable model has been found.

When the seasonal adjustment has taken place then the test statistics calculated in the seasonal adjustment procedure are considered. If the test statistics are acceptable, then the process is terminated. If this is not the case, another seasonal adjustment is performed.

2.2.1 The Reg-ARIMA-model

A Reg-ARIMA-model is an extension of the class ARIMA-models. These are characterized by a linear regression equation for the process x_t in the following equation

$$x_t = \sum_i \beta_i y_{it} + z_t .$$

Where the y's are regression variables observed simultaneously with x, and the β 's are the regression parameters. The error term z is assumed given by a general ARIMA(p, d, q) $\times$ ARIMA(P, D, Q)^s model.

This archetypical regression model is denoted a Reg-ARIMA-model. In this model, it is implicitly assumed that the explanatory variables only affect the dependent variable x through present values, there are no historical values of the y's in this model formulation.

The extension in this model compared to the standard ARIMA-model is obvious. In this model is both the possibility of a standard regression equation in the mean of the process, and furthermore an extension of the standard regression model so that the error terms can be assumed to follow a general ARIMA-process. This set of models is fairly wide.

The specification of a Reg-ARIMA-model demands that both the regression variables (the y_{it} 's) and the ARIMA-model for the regression error terms z_t are defined.

2.2.2 Standard regression variables in X-12-ARIMA

Several standard regression variables have been included in X-12-ARIMA. (See the X-12-ARIMA Reference Manual from the US Bureau of the Census for further explanations).

The simplest regression model is simply a constant term. When no differences are present in the Reg-ARIMA-model, then the constant term is simply the level in the regression. If differences are present in the model definition then X-12 ARIMA uses a trend as regression variable. The trend is defined in such a way that a differencing of this term according to the Reg-ARIMA-model yields a column, which is a constant term.

Other types of standard regression variables are the standard seasonal effects, such as trading day effects, holiday effects and several dummies used for modelling temporary movements of the series.

2.2.3 Monthly series

In a monthly series the standard seasonal effects are modelled by using 12 indicator variables, one for each month. Furthermore, if one also wants to include a constant term, then these 13 variables together are an over parameterization of the model. This is the reason for choosing 11 indicator variables and the constant term.

Trading day effects are present in a series when the series is affected by the number of different days in a calendar month. Trading day effects are modelled by variables given by (The number of Mondays),...,(The number of Sundays).

Only six of these dummies are used since a constant term also is included in the model.

A working day effect is another approach to be used when the aim is to model differences in the days of the week. A working day correction is used on series where there is an effect on the series of the number of work days in a month, for example production series. A working day effect is modelled by the two variables (The working days) and (The non-working days). Only one of these is used, since a constant term is also included in the model.

Holiday effects can be seen in monthly series and the effect is due to holiday periods where the dates vary over time. The Easter effect is the most frequently found holiday covered by this concept.

X-12-ARIMA can also take sudden level changes of a series into account, they can be either temporary or permanent. The following four types of regression variables are used in X-12-ARIMA, if necessary.

- AO (Additive Outliers): A constant is added at one given point of time.
- LS (Level Shift): A constant is added from a given point of time.
- TC (Temporary Change): A level shift with exponential growth is used.
- Ramp: A linear function in t (over a limited period of time) is added. The function equals -1 until the starting level. Then a linear function in t is used, until the end level, and thereafter the function equals 0.

3. Case study

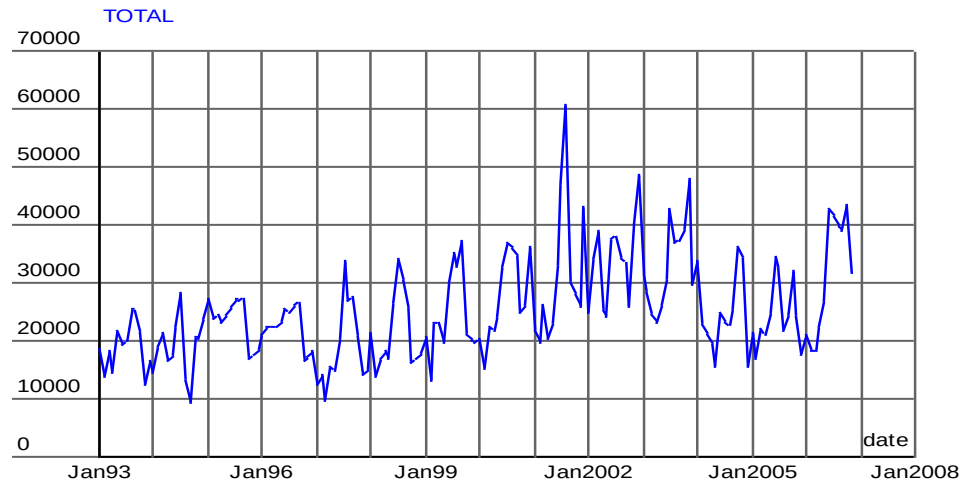
In this section, we apply the X-11-ARIMA, which was explained in Section 2 to the time series data of the total number of passenger that used Sudan Airways during the period from January 1993 to December 2006.

3.1 Application of the X-12-ARIMA Method:-

The reason for using this method is that a computer program, namely DEMETRA program, -issued by the European Union-which is available for its execution.

Before applying the method, we should first take a look to the plot of the original time series.

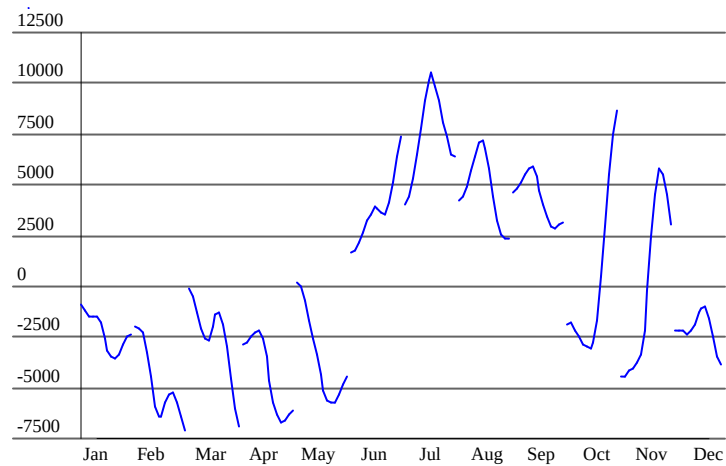
Fig (3. 1): Original Time Series



Fig(3.1) shows the curve of the original time series, it is clear that, the series contains both seasonal as well as trend component.

To illustrate the existence of seasonal changes in the data, Fig(3.2) shows the curve of the seasonal factor in the time series partitioned separately according to months. There are obvious differences between the curves of the different months.

Fig (3. 2): Seasonal Factors per Month



The application of the X-12-ARIMA method to the given time series data ran as follows, first, an appropriate ARIMA model was fitted to the data. That led to the model $ARIMA(1,1,0)(0,1,1)$ as the appropriate model based on the average absolute error. It is found that the average absolute percentage error in within-sample forecasts that took

place in the last year, is less than 15%, which also supports the suitability of the model. To test the existence of autocorrelation among the residuals obtained after fitting of the model, we used Ljung-Box test. The P-value was more than 0.05, which means that the residuals are independent. There is also no evidence that the residuals do not follow a normal curve. Other properties of the model can be seen in Table (3. 1). For example, as far as outliers are concerned, the results show one value (No. 168 in December 2006) as an outlier. There was no evidence of existence of additive outliers or outliers of moving or transitory changes.

Table (3. 1): Model Definition (X-12-ARIMA Method)

Transformation: No transformation	
Regression Model (fixed): Constant	
ARIMA Model: (1 1 0)(0 1 1)	
MODEL ESTIMATION/EVALUATION	
Exact ARMA likelihood estimation	
Max total ARMA iterations	1500
Convergence tolerance	1.00E-05
Likelihood Statistics	
Number of observations (nobs)	168
Effective number of observations (nefobs)	155
Number of parameters estimated (np)	1
Log likelihood (L)	-1608.0970
AIC	3218.1940
AICC (F-Corrected-AIC)	3218.2202
BIC	3221.2375
Existence of the Residual Autocorrelation	
Ljung-Box Test	p-value 0.086
OUTLIER DETECTION	
From 2006.Dec to 2006.Dec	
Observations	1
Types AO and TC	
Method add one	
Critical t for AO outliers	3.65
Critical t for TC outliers	3.65
No AO or TC outliers identified	
Normality Statistics for RegARIMA Model Residuals:	
Number of residuals : 155	
Skewness coefficient : -0.3516	
Geary's a : 0.7030	
Kurtosis : 5.3980	
Average absolute percentage error in within-sample forecasts:	
Last year: 0.20 Last-1 year: 0.12 Last-2 year: 0.10	
Last three years: 0.14	

Having found that the fitted ARIMA model fits the data reasonably, the next step was to use the fitted model to extend the series at both ends to compensate of data loss when moving averages are applied to it later.

Method X-12-ARIMA is then applied to the extended series to adjust it seasonally. But before doing that, we should first verify whether the extended (original) series contains a seasonal component or not, by using the following three tests as follows:-

1- Stable Seasonality Test: which is used to determine whether the series reflect stable seasonality. The test is based on a one-way analysis of variance in which the between month variation (seasonal variation) is compared to residual (irratic) variation (i.e SI – ratio). The larger value of the observed ratio supports the presence of seasonality. Table(3. 2) shows an observed F-value of 9.955, where as using F-Table we find that the 1% critical value is less than 1.85. This means that there is an evidence of the presence of stable seasonality.

Table (3. 2): Test for the Presence of Seasonality Assuming Stability.

Source	Sum of Sguares	Degrees of Freedom	Mean of Squares	F-value
Between months	263464507.25	11	239513318.84	9.955
Residual	3753381500.22	156	24060009.617	
Total	6388008007.48	167		

2 – Moving Seasonality Test: It is used to determine whether the series reflects a moving seasonality. The test is also analysis of variance test in which between year variation (after eliminating within-years monthly variations) are compared to residual variations. A large value of the observed ratio supports the presence of moving seasonality. Table(3.3) shows the observed F-value as 2.234. The critical 5% and 1% values are about 1.80 and 2.28 respectively. Hence we can conclude that there is a significant moving seasonality at the 5% level but not the 1% level.

Table (3. 3): Moving Seasonality Test.

Source	Sum of Squares	Degrees of Freedom	Mean of Squares	F-value
Between years	336065803.9	13	25851215.68	2.234
Error	16546582620	143	11571036.80	

3 – Identifiable Seasonality Test: It is used to test whether the series reflects an identifiable seasonality, through the integration of both stable seasonality test and moving seasonality, A pplication of this test should that identifiable seasonality is present.

After verifying the existence of seasonal component in the series, other steps of X-12-ARIMA method were applied, having split the series and got the values of the various components. Fig(3.3) to Fig(3.5) showed the graphs of the trend/cycle, irregular and seasonal component, respectively.

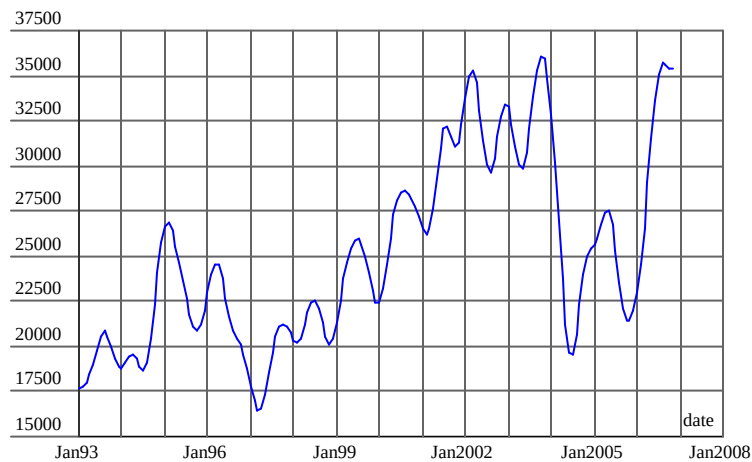
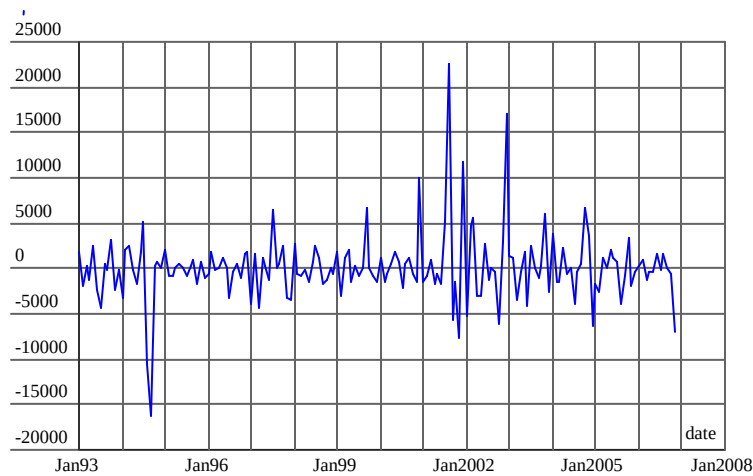
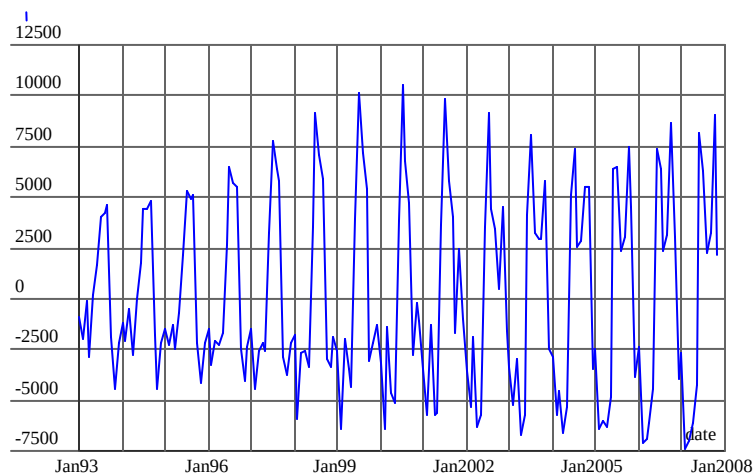
Fig (3. 3): Final trend-Cycle (X-12-ARIMA Method)

Fig (3. 4): Final Irregular Component (X-12-ARIMA Method)



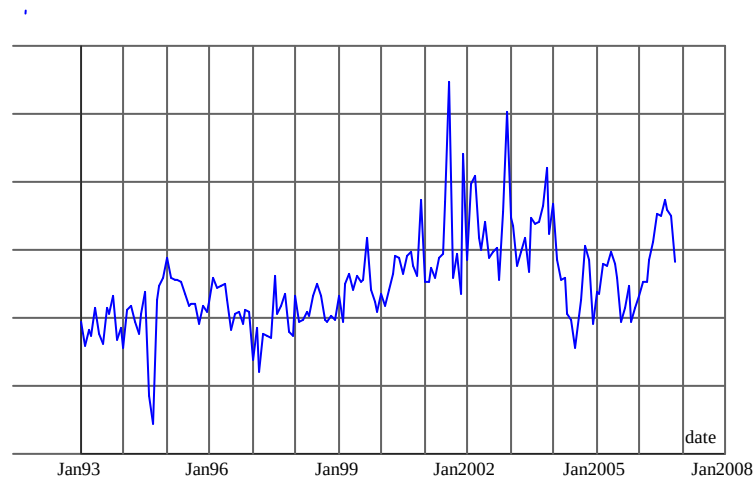
Fig(3.4) shows that a part from four peaks, the isolated irratic variation are resonably small. The peaks occured at July 1994, August 2001, December 2001 and December 2002.

Fig (3. 5): Final Seasonal Factor (X-12-ARIMA Method)



Fig(3.5) presents a picture for the seasonal factor.The seasonal factor is obvious, with increases in the value of the time series around the months of Jul, Aug and Sep.

Fig(3.6) gives a graphical picture of the seasonally adjusted series of the number of passangers using the X-12-ARIMA method

Fig (3. 6): Final Seasonal Adjusted Series (X-12-ARIMA Method)

To make sure that the seasonally adjusted series did not contain any more seasonal component, we applied, what is known as seasonal residuals test. This test is similar to stable seasonality test used on the original series. As shown in Table(3. 4), the test showed that there is no evidence of seasonality in the residuals of the entire series at 1% level. This implies that the adjustment is successful.

Table (3. 4): Test for the Presence of Residual Seasonality

F= 0.41	No evidence of residual seasonality in the entire series at the 1 per-cent level.
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In Table(3.5), the $M8$ test result, showed that there were some fluctuations in seasonally adjusted series, the value of the statistic for this test were equal to (1.347), which is greater than one. As the value of $M10$ test is equal to (1.764), indicating a change in the structure of the series, that must be taken into consideration.

The $M9$ test value equal to (0.746)- less than one - which shows that the linear movement in seasonal component of the series was less than 10%. This indicates to progress of seasonal pattern of the series. The value of $M11$ test was (1.626), which shows that the linear movement for the years is more than 10% (See Linde, 2005 for definition and how to calculated).

Table (3. 5): Monitoring and Quality Assessment Statistics

The size of the fluctuations in the seasonal component throughout the whole series.	M8 = 1.347
The average linear movement in the seasonal component throughout the whole series	M9 = 0.746
Same as 8, calculated for recent years only	M10 = 1.764
Same as 9, calculated for recent years only.	M11 = 1.626

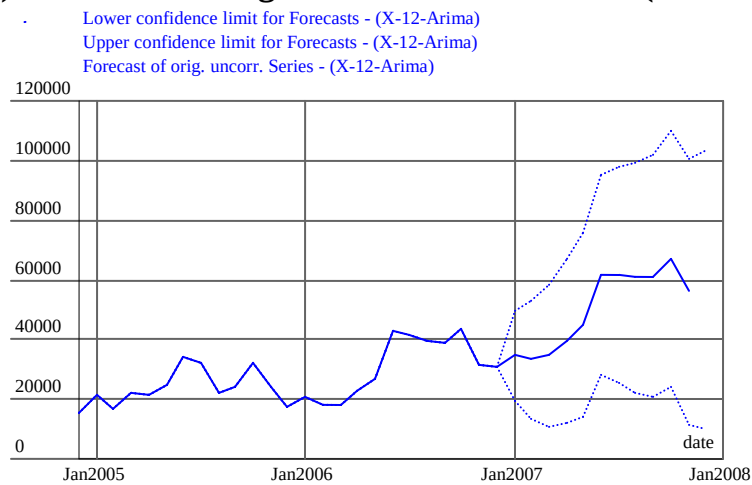
The above estimated model was used to forecast the values for the period Jan 2007 – Dec 2007. Table(3.6) and Fig(3.7) show these values and confidence intervals with 5% level of significance and the curve.

The forecast values and confidence intervals with coverage probability (0.95) .

Table (3. 6): Forecasts Values from Jan 2007 to Dec 2007 (X-12-ARIMA Method)

Date	Lower	Forecast	Upper
Jan. 2007	19508	34704	49899
Feb. 2007	13188	33240	53292
Mar. 2007	10423	34528	58633
Apr. 2007	12130	39679	67227
May. 2007	14356	44966	75575
Jun. 2007	28447	61838	95229
Jul. 2007	25673	61631	97589
Aug. 2007	22380	60733	99087
Sep. 2007	20701	61309	101916
Oct. 2007	24316	67059	109802
Nov. 2007	11274	56051	100828
Dec. 2007	9949	56671	103394

Fig (3. 7): Forecast of Original Uncorrelated Series (X-12-ARIMA Method)



Fig(3.8) and Fig(3.9), shows the forecast of trend and forecast of final seasonal adjusted series for the period (Jan 2007 – Dec 2007).

Fig (3. 8): Forecast of Final Trend (X-12-ARIMA Method)

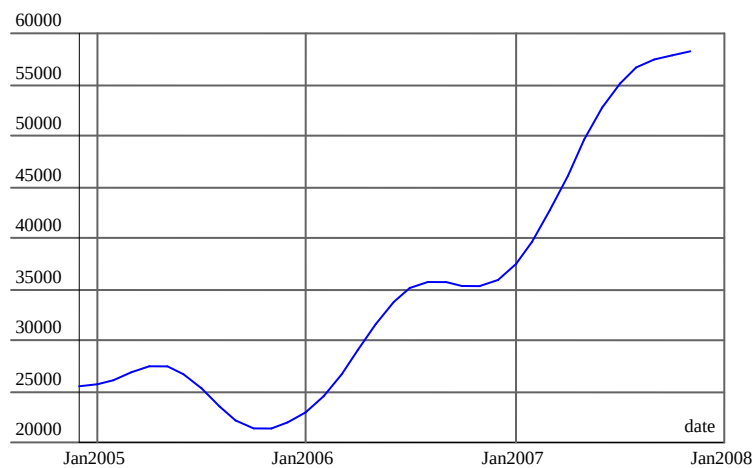
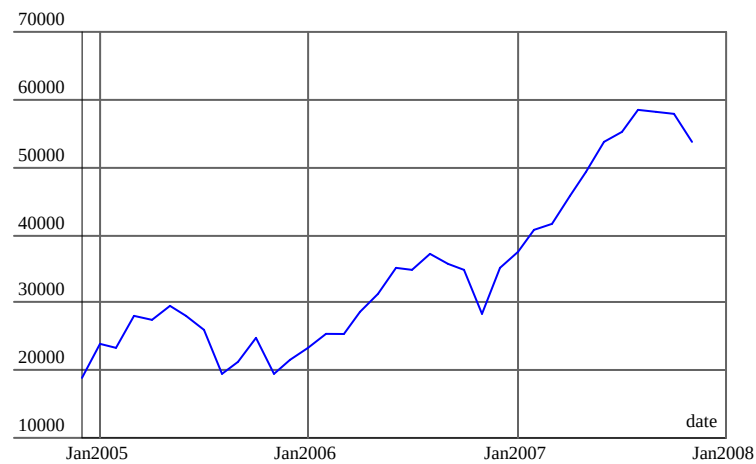


Fig (3. 9): Forecast of Final Seasonally Adjusted Series (X-12-ARIMA Method)



3.2 Comparison with the Class. Dec. Method:-

Recent studies suggested that simple time series forecasting method may sometimes, lead to more precise results than those obtained using advanced methods. To see whether this holds for seasonal adjustment of time series, in particular, we apply the simple classical decomposition method to the Sudan Airways passengers data.

Fig(3.10) to Fig(3.13), give graphically the trend/cycle, irregular, seasonal component and seasonally adjusted series.

Fig (3. 10): Trend/Cycle Components (Class. Dec. Method)

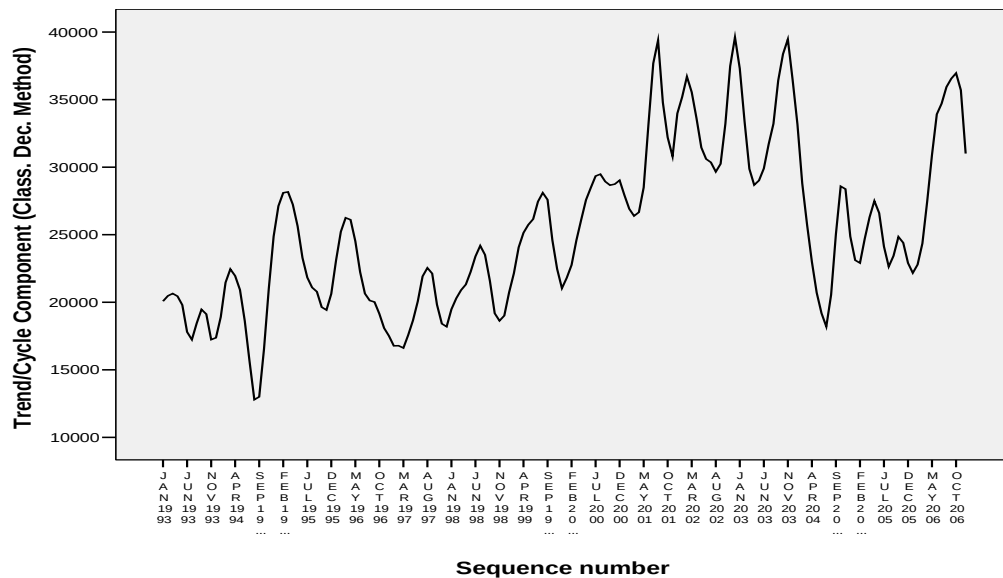
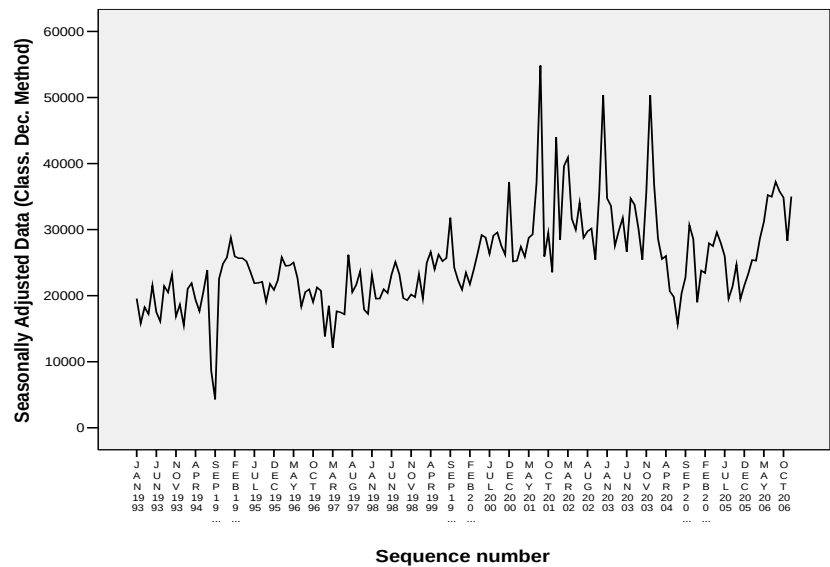
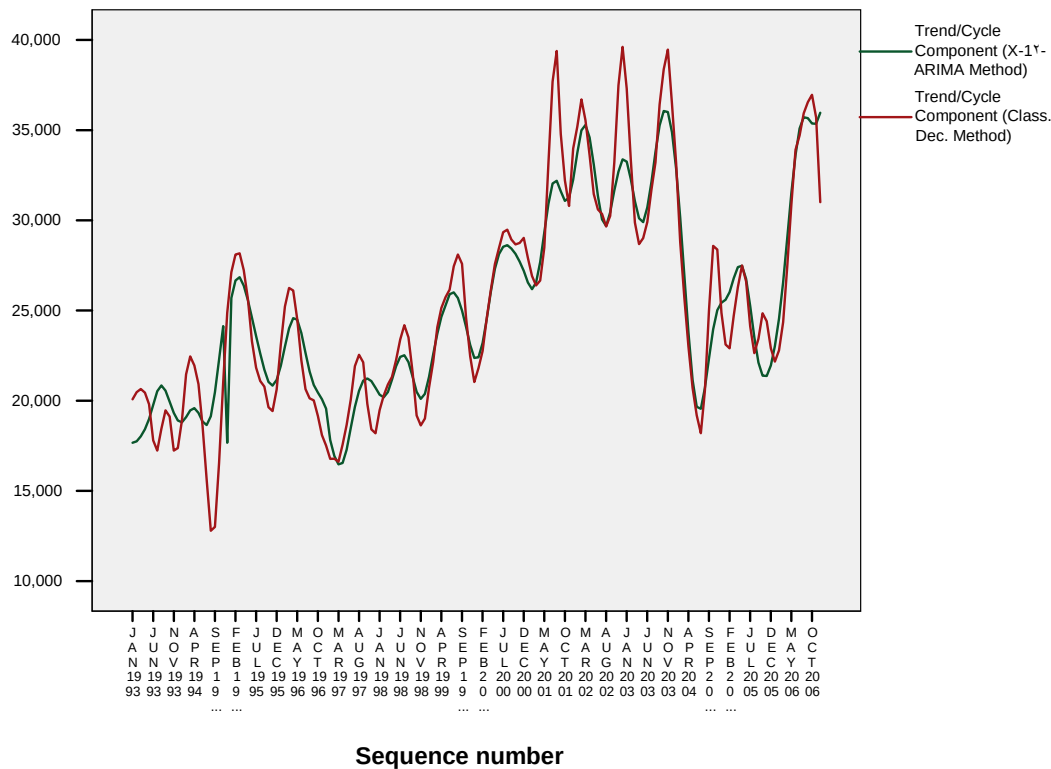


Fig (3. 13): Seasonal Adjusted Series (Class. Dec. Method)



Fig(3.14) to Fig(3.17), compares the curves of the two methods: X-12-ARIMA method and classical decomposition method of the various components, as well as, the seasonally adjusted series.

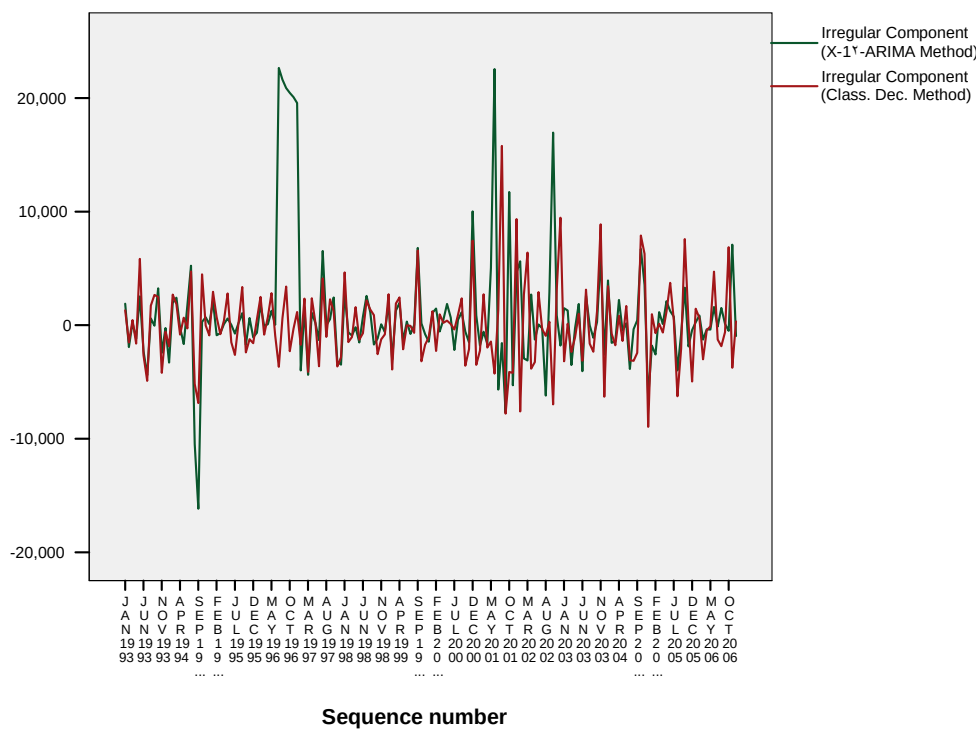
Fig (3. 14): Trend/Cycle Components of two Methods



We see from Fig(3.14) that there is a difference between the trend/cycle curves on using the two methods. The trend/cycle curve of the X-12-ARIMA method is smoother than that of the classical decomposition method, leading to a clearer picture of the long term behaviour of the time series.

Fig(3.15) shows curves of irregular component from the two methods. We see that the classical decomposition irregular curve is smoother than that of the X-12-ARIMA method curve. This result is consistent with that arrived in the previous paragraph, as it means that the X-12-ARIMA method managed to isolate more irregular variations than the classical decomposition method.

Fig (3. 15): Irregular Components of two Method



Fig(3.16) shows the curves of seasonal factor using the two methods. It is obvious that the factors from the classical decomposition method tend to be fixed for the same seasons (static), while the factors from X-1 Ψ -ARIMA method are different varying from year to year. This means that the X-1 Ψ -ARIMA method allows more flexible estimates of the seasonal factor.

Table (3.7) shows the observed and forecasted numbers of passengers in 2007 using X-12-ARIMA and classical decomposition method. We notice a great change and increase in the observed numbers of passengers in that year compared to previous years. This increase may be attributed to privatization of the company which lead to improvement in performance, and hence attraction of more passengers. Also new routs had been created in Africa and Europe leading to an increase in the number of flights and passengers.

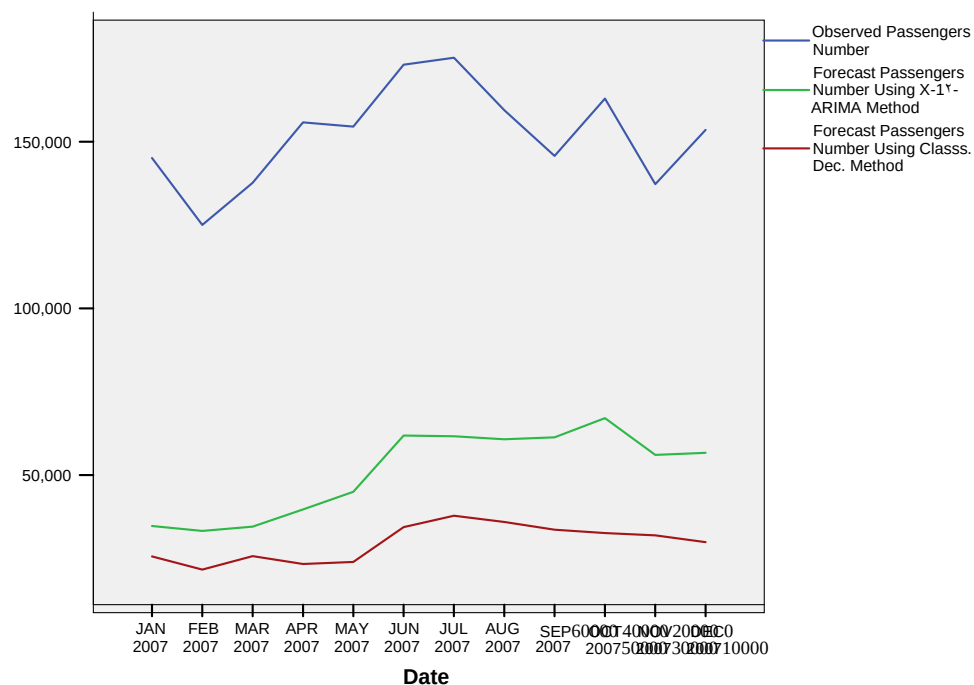
We see also that the forecasts through X-12-ARIMA is a relatively nearer to the observed values than those obtained through the classical decomposition method. This is confirmed by Fig (3.18) which shows that the forecast curve for X-12-ARIMA is also relatively nearer to the observed data curve than that curve for the classical decomposition method. On the other hand from Table(3.8) shows that the sum of squares error is 1.24099E+11 and 1.81398E+11 for the X-12-ARIMA and classical decomposition method respectively, which also pointed that the X-12-ARIMA forecast is better than the Class. Dec. Method forecast.

Table (3. 7): Observed and Forecast using the two methods of Passengers Number (Jan 2007-Dec 2007)

Date	Observed	X-12-ARIMA	Class. Dec.
Jan. 2007	145129	34704	25581
Feb. 2007	125059	33240	21627
Mar. 2007	137690	34528	25674
Apr. 2007	155806	39679	23289
May. 2007	154542	44966	23914
Jun. 2007	173115	61838	34375
Jul. 2007	175208	61631	37780
Aug. 2007	159473	60733	35910
Sep. 2007	145757	61309	33587
Oct. 2007	162945	67059	32588
Nov. 2007	137278	56051	31900
Dec. 2007	153547	56671	29863

Source: Civil Aviation Authority and Excel Program Output.

Fig (3. 18): Observed and Forecast using the two methods of Passengers Number (Jan 2007-Dec



Software (DEMETRA)

EUROSTAT is currently supporting the development of the seasonal adjustment software DEMETRA. This software embodies both X12 (U.S. Bureau of the Census, 2002) and TRAMO-SEATS (Gomez and Maravall, 1996); downloadable at www.bde.es and provides users with a friendly interface to these two programs. It is developed first for EUROSTAT internal needs, but it will be made available to the public on request.

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The Frequency Analysis Of A Linear Time-Invariant Relation Between Two Multivariate Stochastic Time Series

Ghazal, M. A.¹ and Metwally, A. M.²

Abstract

In this paper we focus on the analysis of a functional linear time-invariant relation between two multivariate stochastic time series. We concentrate on the analysis of stationary stochastic time series in the frequency domain. Our aim is to derive estimators for the parameters of this linear time-invariant model from observations of the two multivariate stochastic time series, especially estimator here is the coherence which represents a tool to check our model's goodness of fit. We taper the data before computing its Fourier transform and constructing estimates of the parameters of this model. In addition, we show under assumptions that these estimators are asymptotically normal.

Keywords. Linear time-invariant model; Multivariate time series; Stochastic; Stationary; Data taper; expanded Finite Fourier transform; Periodogram; Asymptotic normality; Wishart distribution.

1. Introduction

In this paper we concentrate on the analysis of a specific relation between two stochastic time series proposed in the so-called linear time-invariant model. This model was treated by Brillinger [1]. It will turn out that the results of Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] are of great importance for our estimation analysis of the characteristics described by the model . In the model an (output) stochastic time series is in a linear time-invariant way partly

¹Department of Mathematics, Faculty of Science, New Damietta, University of Damietta, A. R. Egypt.

²Department of Mathematics, Faculty of Science, New Damietta, University of Damietta, A. R. Egypt.

Cairo University-Institute of Statistical Studies and Research

determined by another (input) stochastic time series via a kind of filter operation. Suppose that

$$\begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix}, \quad t = 0, \pm 1, \pm 2, \dots, \quad (1.1)$$

is an $(r+s)$ vector-valued stationary stochastic time series with $\mathbf{X}(t)$ r vector valued and $\mathbf{Y}(t)$ s vector-valued. Suppose $\boldsymbol{\varepsilon}(t)$ is an s vector-valued stationary stochastic error series with mean $\mathbf{0}$ that is statistically independent of $\mathbf{X}(t)$ at all lags. The index t denotes the time of recording of the measurements. We assume that the tapered values $h(\frac{\cdot}{T})\mathbf{X}(t)$, $h(\frac{\cdot}{T})\mathbf{Y}(t)$, $t = 0, 1, 2, \dots, T-1$ are available, where T denotes the length of the time period of observations. The function $h^{(T)}(t) = h(\frac{\cdot}{T})$ is called a data window or taper with h satisfying the following standard assumption:

ASSUMPTION 1.1. The function $h(t)$, $t = 0, \pm 1, \pm 2, \dots$, is an even positive function of bounded variation with bounded support: $h(t) = 0$ for $|t| > 1$.

We further set

$$H_k(\lambda) = \int_0^1 [h(x)]^k \exp\{-i\lambda x\} dx, \quad -\infty < \lambda < \infty$$

and assume $H_2(0) \neq 0$. The properties and examples of data taper are mentioned in Brillinger [1]; Dahalhaus [4,5]; Ghazal and Trush [6] and other authors. Suppose that the series satisfies the following relation,

$$\mathbf{Y}(t) = \sum_{u=-\infty}^{\infty} \mathbf{a}(t-u)\mathbf{X}(u) + \boldsymbol{\varepsilon}(t), \quad (1.2)$$

Where the $s \times r$ matrix-valued filter (impulse-responses function) $\{\mathbf{a}(u)\}$ satisfies

$$\sum_{u=-\infty}^{\infty} |u| \|\mathbf{a}(u)\| < \infty. \quad (1.3)$$

Let

$$\begin{bmatrix} \mathbf{f}_{XX}(\lambda) & \mathbf{f}_{XY}(\lambda) \\ \mathbf{f}_{YX}(\lambda) & \mathbf{f}_{YY}(\lambda) \end{bmatrix}, \quad (1.4)$$

denotes the spectral density matrix of the series (1.1), the real-valued parameter λ denotes the frequency, and

$$\mathbf{A}(\lambda) = \sum_{u=-\infty}^{\infty} \mathbf{a}(u) \exp\{-i\lambda u\}, \quad -\infty < \lambda < \infty \quad (1.5)$$

denotes the transfer function of $\{\mathbf{a}(u)\}$. We now analyze the model (1.2) using our results from Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3]. We

start with an assumption which is analogous to Assumption 2.6.1 in Brillinger [1].

ASSUMPTION 1.2 The $(r + s)$ variate stochastic time series (1.1) has strictly stationary and all moments exist and absolutely summable cumulant functions of all orders.

We also have to state a mixing-condition analogously to Assumption 2.6.2 (1) in Brillinger [1] concerning the reduced versions of the cumulant functions of the $(r + s)$ variate stochastic time series (1.1).

ASSUMPTION 1.3 The $(r + s)$ variate stochastic time series

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix}$$

satisfies Assumption 1.2 and for all $k \geq 2$ and $a_1, a_2, \dots, a_k = 1, 2, \dots, r + s$ the reduced cumulant function of $Z_{a_1}, \dots, Z_{a_k}$ satisfies

$$\sum_{u_1=-\infty}^{\infty} \dots \sum_{u_{k-1}=-\infty}^{\infty} |u_j| |c_{a_1 \dots a_k}(u_1, \dots, u_{k-1})| < \infty, \quad \forall j = 1, 2, \dots, k-1,$$

where

$$\begin{aligned} c_{a_1 \dots a_k}(u_1, \dots, u_{k-1}) &= \text{cum}\{Z_{a_1}(u_1 + t), \dots, Z_{a_{k-1}}(u_{k-1} + t), Z_{a_k}(t)\} \\ &= c_{Z_{a_1} \dots Z_{a_k}}(u_1, \dots, u_{k-1}), \quad t; u_1, \dots, u_{k-1} = 0, \pm 1, \pm 2, \dots \end{aligned}$$

These assumptions enable us to apply the whole theory developed in Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] to our model. Let Assumption 1.3 be satisfied and suppose that for fixed λ the matrix $\mathbf{f}_{XX}(\lambda)$ is not singular. For the model (1.2) we then have

$$\mathbf{f}_{YX}(\lambda) = \mathbf{A}(\lambda) \mathbf{f}_{XX}(\lambda), \quad (1.6)$$

$$\begin{aligned} \mathbf{f}_{YY}(\lambda) &= \mathbf{A}(\lambda) \mathbf{f}_{XX}(\lambda) \overline{\mathbf{A}(\lambda)^T} + \mathbf{f}_{\varepsilon\varepsilon}(\lambda) \\ &= \mathbf{f}_{YX}(\lambda) \mathbf{f}_{XX}(\lambda)^{-1} \mathbf{f}_{XY}(\lambda) + \mathbf{f}_{\varepsilon\varepsilon}(\lambda), \end{aligned} \quad (1.7)$$

Where $\mathbf{f}_{\varepsilon\varepsilon}(\lambda)$ is the spectral density matrix of $\varepsilon(t)$ and $\overline{\mathbf{A}^T}$ denotes the complex conjugate transpose of a complex matrix $\mathbf{A}$. In Brillinger [1], the joint analysis of several vector-valued series is discussed. The theory parallels results from multivariate analysis including the estimation of the spectral analogs of multiple and partial correlation. The problem of regressing a stochastic time series on one or more other stochastic series is dealt with non-tapered data case in this Chapter. Also, in this Chapter Brillinger constructed

consistent and asymptotically normal estimates, based on data $\{\mathbf{X}(t), \mathbf{Y}(t)\} (t = 0, 1, 2, \dots, T-1)$ for $\mathbf{A}(\lambda)$, $\{\mathbf{a}(u)\}$, and $\mathbf{f}_{\varepsilon\varepsilon}(\lambda)$. While in this paper we taper the data (i.e., $\{h(t/T)\mathbf{X}(t), h(t/T)\mathbf{Y}(t)\} (t = 0, 1, 2, \dots, T-1)$) and show under assumptions (1.1-1.3) that the estimators of $\mathbf{A}(\lambda)$, $\{\mathbf{a}(u)\}$, and $\mathbf{f}_{\varepsilon\varepsilon}(\lambda)$ are asymptotically normal, see Theorems 2.1, 2.2 and 2.3 respectively. In addition, we prove the asymptotic normality of the multiple coherence in the univariate case, see Theorem 2.4. Alternate form of asymptotic distribution for the estimators of $\mathbf{A}(\lambda)$, $\mathbf{f}_{\varepsilon\varepsilon}(\lambda)$ and multiple coherence in the univariate case may be more relevant than that of Theorems (2.1, 2.3 and 2.4), see Proposition 2.5.

In section 2 we present estimators of the transfer function (a function which completely determines the relation between the two stochastic time series except for a constant), the spectral density matrix of error series, the impulse-responses function and the coherence (which represents a tool to check our model's goodness of fit). These estimates are based on the estimator of the second-order spectra. The major results give the asymptotic normality of these estimators. They are implications of the asymptotic behaviour of the estimator of the second order spectra.

2. Estimation of the transfer function, the spectral density matrix of error series, the impulse-responses function and the multiple coherence and their asymptotic normality

In this section we will construct estimates for the transfer function, the spectral density matrix of error series, the impulse-responses function and the multiple coherence and examine their asymptotic behaviour.

Now let us see how we can estimate the transfer function: If we suppose for fixed λ that the matrix $\mathbf{f}_{xx}(\lambda)$ is not singular, then we have according to (1.6), that

$$\mathbf{A}(\lambda) = \mathbf{f}_{yx}(\lambda)(\mathbf{f}_{xx}(\lambda))^{-1}. \quad (2.1)$$

The obvious estimate for the transfer function therefore is

$$\mathbf{A}^{(T)}(\lambda) = \mathbf{f}_{yx}^{(T)}(\lambda)(\mathbf{f}_{xx}^{(T)}(\lambda))^{-1}, \quad (2.2)$$

where

$$\begin{bmatrix} \mathbf{f}_{XX}^{(T)}(\lambda) & \mathbf{f}_{XY}^{(T)}(\lambda) \\ \mathbf{f}_{YX}^{(T)}(\lambda) & \mathbf{f}_{YY}^{(T)}(\lambda) \end{bmatrix}, \quad (2.3)$$

is an estimate of (1.4). Let $W(\alpha)$, $-\infty < \alpha < \infty$ be a weight function which is bounded, even, non-negative, equal to 0 for $|\alpha| > \pi$, and satisfies

$$\int_{-\pi}^{\pi} W(\alpha) d\alpha = 1.$$

Let B_T be a sequence of nonnegative bandwidths such that the weight function becomes concentrated as $T \rightarrow \infty$. Sometimes we shorten calculations by using the notation

$$W^{(T)}(\alpha) = B_T^{-1} W(B_T^{-1} \alpha). \quad (2.4)$$

The discrete expanded finite Fourier transform of (1.1) is

$$\begin{bmatrix} \mathbf{d}_X^{(T)}(\lambda) \\ \mathbf{d}_Y^{(T)}(\lambda) \end{bmatrix} = \frac{1}{\sqrt{2\pi \sum_{t=0}^{T-1} [h^{(T)}(t)]^2}} \sum_{t=0}^{T-1} h^{(T)}(t) \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix} \exp\{-i\lambda t\}. \quad (2.5)$$

The periodograms defined from these are

$$\begin{bmatrix} \mathbf{I}_{XX}^{(T)}(\lambda) & \mathbf{I}_{XY}^{(T)}(\lambda) \\ \mathbf{I}_{YX}^{(T)}(\lambda) & \mathbf{I}_{YY}^{(T)}(\lambda) \end{bmatrix} = \begin{bmatrix} \mathbf{d}_X^{(T)}(\lambda) \overline{\mathbf{d}_X^{(T)}(\lambda)^T} & \mathbf{d}_X^{(T)}(\lambda) \overline{\mathbf{d}_Y^{(T)}(\lambda)^T} \\ \mathbf{d}_Y^{(T)}(\lambda) \overline{\mathbf{d}_X^{(T)}(\lambda)^T} & \mathbf{d}_Y^{(T)}(\lambda) \overline{\mathbf{d}_Y^{(T)}(\lambda)^T} \end{bmatrix}. \quad (2.6)$$

In the case of (1.1) have mean $\mathbf{0}$, we now take as our estimate of (1.4),

$$\begin{bmatrix} \mathbf{f}_{XX}^{(T)}(\lambda) & \mathbf{f}_{XY}^{(T)}(\lambda) \\ \mathbf{f}_{YX}^{(T)}(\lambda) & \mathbf{f}_{YY}^{(T)}(\lambda) \end{bmatrix} = \begin{bmatrix} \int_{-\infty}^{\infty} W^{(T)}(\lambda - \alpha) \mathbf{I}_{XX}^{(T)}(\alpha) d\alpha & \int_{-\infty}^{\infty} W^{(T)}(\lambda - \alpha) \mathbf{I}_{XY}^{(T)}(\alpha) d\alpha \\ \int_{-\infty}^{\infty} W^{(T)}(\lambda - \alpha) \mathbf{I}_{YX}^{(T)}(\alpha) d\alpha & \int_{-\infty}^{\infty} W^{(T)}(\lambda - \alpha) \mathbf{I}_{YY}^{(T)}(\alpha) d\alpha \end{bmatrix}. \quad (2.7)$$

We also assume that the error series $\boldsymbol{\varepsilon}(t)$, $t = 0, \pm 1, \pm 2, \dots$, satisfies an appropriate mixing condition to guarantee that its span of dependence is weak (see, e.g., Brillinger [1], assumption 2.6.1). In the next theorem $N_r^C(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes the r variate complex Normal-distribution with r vector mean $\boldsymbol{\mu}$ and $(r \times r)$ dimensional covariance matrix $\boldsymbol{\Sigma}$ (see Brillinger [1], Section 4.2 for the definition) and the notation $\mathbf{U} \otimes \mathbf{V}$ denotes the Kronecker matrix-product of $\mathbf{U}$ with $\mathbf{V}$ and the notation $\text{vec} \mathbf{R}$ denotes the column vector obtained from a matrix $\mathbf{R}$ by placing its columns under one another successively.

We may prove

THEOREM 2.1 Suppose that

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix}, \quad t = 0, \pm 1, \pm 2, \dots,$$

be a strictly stationary $(r+s)$ vector-valued time series with mean $\mathbf{0}$ and satisfies Assumption 1.3. Assume for $\lambda \neq 0$ that $\mathbf{f}_{xx}(\lambda)$ is not singular. Let $B_T \rightarrow 0$, $B_T T \rightarrow \infty$ as $T \rightarrow \infty$. Then, as $T \rightarrow \infty$,

$$\sqrt{B_T T} \text{vec}(\mathbf{A}^{(T)}(\lambda) - \mathbf{A}(\lambda)) \xrightarrow{D} N_{rs}^C(\mathbf{0}, 2\pi H_4(0)(H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \times ((\mathbf{f}_{xx}(\lambda))^{-1} \otimes \mathbf{f}_{xx}(\lambda))) \quad (2.8)$$

PROOF: We have

$$\begin{aligned} \sqrt{B_T T}(\mathbf{A}^{(T)}(\lambda) - \mathbf{A}(\lambda)) &= \sqrt{B_T T}(\mathbf{f}_{yx}(\lambda)((\mathbf{f}_{xx}^{(T)}(\lambda))^{-1} - (\mathbf{f}_{xx}(\lambda))^{-1}) \\ &\quad + (\mathbf{f}_{yx}^{(T)}(\lambda) - \mathbf{f}_{yx}(\lambda))(\mathbf{f}_{xx}^{(T)}(\lambda))^{-1}) \\ &= \sqrt{B_T T}(\mathbf{f}_{yx}(\lambda)(\mathbf{f}_{xx}(\lambda))^{-1}(\mathbf{f}_{xx}(\lambda) - \mathbf{f}_{xx}^{(T)}(\lambda))(\mathbf{f}_{xx}^{(T)}(\lambda))^{-1} + (\mathbf{f}_{yx}^{(T)}(\lambda) - \mathbf{f}_{yx}(\lambda))(\mathbf{f}_{xx}^{(T)}(\lambda))^{-1}) \\ &= \mathbf{A}(\lambda)\sqrt{B_T T}(\mathbf{f}_{xx}(\lambda) - \mathbf{f}_{xx}^{(T)}(\lambda))(\mathbf{f}_{xx}(\lambda))^{-1} + \sqrt{B_T T}(\mathbf{f}_{yx}^{(T)}(\lambda) - \mathbf{f}_{yx}(\lambda))(\mathbf{f}_{xx}(\lambda))^{-1} \\ &\quad + \mathbf{A}(\lambda)\sqrt{B_T T}(\mathbf{f}_{xx}(\lambda) - \mathbf{f}_{xx}^{(T)}(\lambda))((\mathbf{f}_{xx}^{(T)}(\lambda))^{-1} - (\mathbf{f}_{xx}(\lambda))^{-1}) \\ &\quad + \sqrt{B_T T}(\mathbf{f}_{yx}^{(T)}(\lambda) - \mathbf{f}_{yx}(\lambda))((\mathbf{f}_{xx}^{(T)}(\lambda))^{-1} - (\mathbf{f}_{xx}(\lambda))^{-1}) \end{aligned} \quad (2.9)$$

We firstly show that the matrix components of the last two summands in (2.9) converge to zero in probability: The matrices $\sqrt{B_T T}(\mathbf{f}_{xx}(\lambda) - \mathbf{f}_{xx}^{(T)}(\lambda))$ and $\sqrt{B_T T}(\mathbf{f}_{yx}(\lambda) - \mathbf{f}_{yx}^{(T)}(\lambda))$ have asymptotically normally distributed components according to Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7]. For the factor $((\mathbf{f}_{xx}^{(T)}(\lambda))^{-1} - (\mathbf{f}_{xx}(\lambda))^{-1})$ the following inequality holds (see Stewart [8]):

$$\|(\mathbf{f}_{xx}^{(T)}(\lambda))^{-1} - (\mathbf{f}_{xx}(\lambda))^{-1}\| \leq \frac{\|(\mathbf{f}_{xx}(\lambda))^{-1}\|^2 \cdot \|\mathbf{f}_{xx}^{(T)}(\lambda) - \mathbf{f}_{xx}(\lambda)\|}{1 - \|(\mathbf{f}_{xx}(\lambda))^{-1}\| \cdot \|\mathbf{f}_{xx}^{(T)}(\lambda) - \mathbf{f}_{xx}(\lambda)\|}$$

for the Euclidean norm for the Euclidean norm $\|\cdot\|$. We know from Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7] that

$$\sqrt{B_T T} \text{vec}(\mathbf{f}_{xx}^{(T)}(\lambda) - \mathbf{f}_{xx}(\lambda)) \xrightarrow{D} N_r^C(\mathbf{0}, 2\pi H_4(0)(H_2(0))^{-2} \int (W(\alpha))^2 d\alpha (\mathbf{f}_{xx}(\lambda) \mathbf{f}_{xx}(-\lambda)))$$

The convergence in distribution is sufficient for $\sqrt{B_T T}(\mathbf{f}_{xx}^{(T)}(\lambda) - \mathbf{f}_{xx}(\lambda))$ to be tight, see Billingsley [9] for example. Tightness of $\sqrt{B_T T}(\mathbf{f}_{xx}^{(T)}(\lambda) - \mathbf{f}_{xx}(\lambda))$ means that $\forall \beta > 0$ there exists an $J_\beta > 0$ such that

$$\begin{aligned} \Pr(\|\sqrt{B_T T}(\mathbf{f}_{xx}^{(T)}(\lambda) - \mathbf{f}_{xx}(\lambda))\| \leq J_\beta) &\geq 1 - \beta, \quad \forall T \Leftrightarrow \\ \Pr(\|\mathbf{f}_{xx}^{(T)}(\lambda) - \mathbf{f}_{xx}(\lambda)\| \leq J_\beta (B_T T)^{-1/2}) &\geq 1 - \beta, \quad \forall T. \end{aligned}$$

This yields $\|\mathbf{f}_{xx}^{(t)}(\lambda) - \mathbf{f}_{xx}(\lambda)\|$ is of order $O_p((B_T T)^{-1/2})$ and hence componentwise converges to zero in probability the term on the right hand side of the inequality and hence $\|(\mathbf{f}_{xx}^{(t)}(\lambda))^{-1} - (\mathbf{f}_{xx}(\lambda))^{-1}\|$ converges to zero in probability. Therefore each component of $(\mathbf{f}_{xx}^{(t)}(\lambda))^{-1} - (\mathbf{f}_{xx}(\lambda))^{-1}$ converges to zero in probability. Since each matrix component of the last two summands in (2.9) hence is a finite sum of normally distributed variates multiplied with in probability to zero converging variates, we can conclude with the Lemma of Slutsky that all matrix components of the two summands converge to zero in probability. The limiting distribution of $\sqrt{B_T T}(\mathbf{A}^{(t)}(\lambda) - \mathbf{A}(\lambda))$ therefore is completely described by the first two summands in (2.9). This shows the asymptotic normality of $\sqrt{B_T T}(\mathbf{A}^{(t)}(\lambda) - \mathbf{A}(\lambda))$ according to Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7]. since the sum of normally distributed variates again is normally distributed. The asymptotic mean clearly is the zero vector because of the unbiasedness of $\mathbf{f}_{yx}^{(t)}(\lambda)$ and $\mathbf{f}_{xx}^{(t)}(\lambda)$. To finish the proof we have to assign the asymptotic variance-covariance matrix of $\sqrt{B_T T} \text{vec}(\mathbf{A}^{(t)}(\lambda) - \mathbf{A}(\lambda))$. For this we use Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7], properties of Kronecker matrix-product see Neudecker [10] and the representation (2.9).

$$\begin{aligned} \lim_{T \rightarrow \infty} \text{var}(\sqrt{B_T T} \text{vec}(\mathbf{A}^{(t)}(\lambda) - \mathbf{A}(\lambda))) &= \\ \lim_{T \rightarrow \infty} B_T T \text{var}(-\text{vec}(\mathbf{A}(\lambda) \mathbf{f}_{xx}^{(t)}(\lambda) (\mathbf{f}_{xx}(\lambda))^{-1}) + \text{vec}(\mathbf{f}_{yx}^{(t)}(\lambda) (\mathbf{f}_{xx}(\lambda))^{-1})) &= \\ = \lim_{T \rightarrow \infty} B_T T \text{var}(-(\mathbf{Q}_r \otimes \mathbf{A}(\lambda))((\mathbf{f}_{xx}(\lambda))^{-1} \otimes \mathbf{Q}_r) \text{vec}(\mathbf{f}_{xx}^{(t)}(\lambda)) &+ \\ + ((\mathbf{f}_{xx}(\lambda))^{-1} \otimes \mathbf{Q}_s) \text{vec}(\mathbf{f}_{yx}^{(t)}(\lambda))) &= \\ = \lim_{T \rightarrow \infty} (((\mathbf{f}_{xx}(\lambda))^{-1} \otimes \mathbf{A}(\lambda)) B_T T \text{var}(\text{vec}(\mathbf{f}_{xx}^{(t)}(\lambda))) ((\mathbf{f}_{xx}(\lambda))^{-1} \otimes \overline{\mathbf{A}(\lambda)^r}) & \end{aligned}$$

$$\begin{aligned}
& + \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) B_T T \text{var} \left(\text{vec} \left(\mathbf{f}_{YX}^{(T)}(\lambda) \right) \right) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) \\
& - \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{A}(\lambda) \right) B_T T \text{cov} \left(\text{vec} \left(\mathbf{f}_{XX}^{(T)}(\lambda) \right), \text{vec} \left(\mathbf{f}_{YX}^{(T)}(\lambda) \right) \right) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) \\
& - \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) B_T T \text{cov} \left(\text{vec} \left(\mathbf{f}_{YX}^{(T)}(\lambda) \right), \text{vec} \left(\mathbf{f}_{XX}^{(T)}(\lambda) \right) \right) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \overline{\mathbf{A}(\lambda)^T} \right) \\
& = 2\pi H_4(0) (H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{A}(\lambda) \right) \left(\mathbf{f}_{XX}(\lambda) \otimes \mathbf{f}_{XX}(\lambda) \right) \\
& \times \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \overline{\mathbf{A}(\lambda)^T} \right) \\
& + \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) \left(\mathbf{f}_{XX}(\lambda) \otimes \mathbf{f}_{YY}(\lambda) \right) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) \\
& - \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{A}(\lambda) \right) \left(\mathbf{f}_{XX}(\lambda) \otimes \mathbf{f}_{XY}(\lambda) \right) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) \\
& - \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) \left(\mathbf{f}_{XX}(\lambda) \otimes \mathbf{f}_{YX}(\lambda) \right) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \overline{\mathbf{A}(\lambda)^T} \right) \\
& = 2\pi H_4(0) (H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \left((\mathbf{Q}_r \otimes \mathbf{f}_{YX}(\lambda)) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes (\mathbf{f}_{XX}(\lambda))^{-1} \mathbf{f}_{XY}(\lambda) \right) \right. \\
& + (\mathbf{Q}_r \otimes \mathbf{f}_{YY}(\lambda)) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) - (\mathbf{Q}_r \otimes \mathbf{f}_{YX}(\lambda)) \left((\mathbf{f}_{XX}(\lambda))^{-1} \mathbf{f}_{XY}(\lambda) \right) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{Q}_s \right) \\
& \left. - (\mathbf{Q}_r \otimes \mathbf{f}_{YX}(\lambda)) \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes (\mathbf{f}_{XX}(\lambda))^{-1} \mathbf{f}_{XY}(\lambda) \right) \right) \\
& = 2\pi H_4(0) (H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes (\mathbf{f}_{YY}(\lambda) - \mathbf{f}_{YX}(\lambda) (\mathbf{f}_{XX}(\lambda))^{-1} \mathbf{f}_{XY}(\lambda)) \right) \\
& = 2\pi H_4(0) (H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \left((\mathbf{f}_{XX}(\lambda))^{-1} \otimes \mathbf{f}_{\varepsilon\varepsilon}(\lambda) \right),
\end{aligned}$$

where $\mathbf{Q}_j$ denote the $(j \times j)$ unit-matrix.

In the linear time-invariant model it will often be desirable to have an estimate of the impulse-response function $\{\mathbf{a}(u)\}$ (time domain), rather than of the transfer function $\mathbf{A}(\lambda)$ (frequency domain) only. It follows from the definition of $\mathbf{A}(\lambda)$ in (1.5) that

$$\mathbf{a}(u) = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathbf{A}(\alpha) \exp\{iu\alpha\} d\alpha. \quad (2.10)$$

For estimation of $\{\mathbf{a}(u)\}$ we follow the suggestion of Brillinger [1], and use

$$\mathbf{a}^{(T)}(u) = P_T^{-1} \sum_{p=0}^{P_T-1} \mathbf{A}^{(T)} \left(\frac{2\pi p}{P_T} \right) \exp \left\{ i \frac{2\pi p u}{P_T} \right\}, \quad u = 0, \pm 1, \pm 2, \dots, \quad (2.11)$$

Where P_T be a sequence of integers tending to ∞ with T . We may prove asymptotic normality for $\mathbf{a}^{(T)}(u)$.

THEOREM 2.2 Suppose that

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix}, \quad t = 0, \pm 1, \pm 2, \dots,$$

be a strictly stationary $(r+s)$ vector-valued time series with mean $\mathbf{0}$ and satisfies Assumption 1.3. Assume for $\lambda \neq 0$ that $\mathbf{f}_{xx}(\lambda)$ is not singular. Then if

$$B_T \rightarrow 0, B_T \leq P_T^{-1}, P_T \rightarrow \infty, P_T B_T T \rightarrow \infty \text{ as } T \rightarrow \infty,$$

$$\begin{aligned} \sqrt{P_T B_T T} \text{vec}(\mathbf{a}^{(r)}(u) - \mathbf{a}(u)) &\xrightarrow{D} N_{rs}^C(\mathbf{0}, 2\pi H_4(0)(H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \\ &\times \frac{1}{P_T} \sum_{p=0}^{P_T-1} \left(\left(\mathbf{f}_{xx} \left(\frac{2\pi p}{P_T} \right) \right)^{-1} \otimes \mathbf{f}_{xx} \left(\frac{2\pi p}{P_T} \right) \right) \end{aligned} \quad (2.12)$$

Note in particular that the asymptotic variance-covariance matrix is independent of the argument u .

PROOF: For all $u = 0, \pm 1, \pm 2, \dots$, we have

$$\begin{aligned} \text{vec}(\mathbf{a}^{(r)}(u)) &= P_T^{-1} \sum_{p=0}^{P_T-1} \exp\left\{i2\pi pu/P_T\right\} \text{vec}\left(\mathbf{A}\left(\frac{2\pi p}{P_T}\right)\right) \\ &+ P_T^{-1} \sum_{p=0}^{P_T-1} \exp\left\{i2\pi pu/P_T\right\} \left(\text{vec}\left(\mathbf{A}\left(\frac{2\pi p}{P_T}\right)\left(\mathbf{f}_{xx}\left(\frac{2\pi p}{P_T}\right) - \mathbf{f}_{xx}^{(r)}\left(\frac{2\pi p}{P_T}\right)\right)\left(\mathbf{f}_{xx}\left(\frac{2\pi p}{P_T}\right)\right)^{-1}\right) \right. \\ &+ \left. \text{vec}\left(\left(\mathbf{f}_{yx}^{(r)}\left(\frac{2\pi p}{P_T}\right) - \mathbf{f}_{yx}\left(\frac{2\pi p}{P_T}\right)\right)\left(\mathbf{f}_{xx}\left(\frac{2\pi p}{P_T}\right)\right)^{-1}\right) \right) \\ &+ P_T^{-1} \sum_{p=0}^{P_T-1} \exp\left\{i2\pi pu/P_T\right\} \left(\left(\mathbf{Q}_r \otimes \mathbf{A}\left(\frac{2\pi p}{P_T}\right) \right) \left(\left(\mathbf{f}_{xx}^{(r)}\left(\frac{2\pi p}{P_T}\right) \right)^{-1} - \left(\mathbf{f}_{xx}\left(\frac{2\pi p}{P_T}\right) \right)^{-1} \right) \otimes \mathbf{Q}_s \right) \\ &\times \text{vec}\left(\mathbf{f}_{xx}\left(\frac{2\pi p}{P_T}\right) - \mathbf{f}_{xx}^{(r)}\left(\frac{2\pi p}{P_T}\right)\right) + \left(\left(\mathbf{f}_{xx}^{(r)}\left(\frac{2\pi p}{P_T}\right) \right)^{-1} - \left(\mathbf{f}_{xx}\left(\frac{2\pi p}{P_T}\right) \right)^{-1} \right) \otimes \mathbf{Q}_s \\ &\times \text{vec}\left(\mathbf{f}_{yx}^{(r)}\left(\frac{2\pi p}{P_T}\right) - \mathbf{f}_{yx}\left(\frac{2\pi p}{P_T}\right)\right) \\ &= P_T^{-1} \sum_{p=0}^{P_T-1} \exp\left\{i2\pi pu/P_T\right\} \text{vec}\left(\mathbf{A}\left(\frac{2\pi p}{P_T}\right)\right) + P_T^{-1} \Psi_T(u) + P_T^{-1} \sum_{p=0}^{P_T-1} \exp\left\{i2\pi pu/P_T\right\} \Phi_T\left(\frac{2\pi p}{P_T}\right) \\ &= T_1 + T_2 + T_3. \end{aligned}$$

First we prove asymptotic normality of term T_2 . The term $\Psi_T(u)$ determines the

asymptotic behavior of $\text{vec}(\mathbf{a}^{(r)}(u))$. It directly follows from Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7] that $\Psi_T(u)$ is asymptotically normal of rate $\sqrt{B_T T}$ with zero mean vector. Applying Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7], Neudecker [10] and the fact that $\mathbf{f}_{XX}(\lambda)^r = \mathbf{f}_{XX}(-\lambda)$ for all λ , the asymptotic variance-covariance matrix of $\Psi_T(u)$ can be computed as follows. Let ascov denote "asymptotic covariance".

$$\begin{aligned}
& \sum_{p_1=0}^{P_T-1} \sum_{p_2=0}^{P_T-1} \exp\left\{i2\pi(p_1 - p_2)u/P_T\right\} B_T T \text{ascov} \left(\left(\left(\mathbf{f}_{XX}\left(2\pi p_1/P_T\right) \right)^{-1} \otimes \mathbf{Q}_s \right) \text{vec}\left(\mathbf{f}_{YX}^{(r)}\left(2\pi p_1/P_T\right)\right) \right. \\
& - \left(\left(\mathbf{f}_{XX}\left(2\pi p_1/P_T\right) \right)^{-1} \otimes \mathbf{A}\left(2\pi p_1/P_T\right) \right) \text{vec}\left(\mathbf{f}_{XX}^{(r)}\left(2\pi p_1/P_T\right)\right), \left(\left(\mathbf{f}_{XX}\left(2\pi p_2/P_T\right) \right)^{-1} \otimes \mathbf{Q}_s \right) \\
& \times \text{vec}\left(\mathbf{f}_{YX}^{(r)}\left(2\pi p_2/P_T\right)\right) \\
& - \left(\left(\mathbf{f}_{XX}\left(2\pi p_2/P_T\right) \right)^{-1} \otimes \mathbf{A}\left(2\pi p_2/P_T\right) \right) \text{vec}\left(\mathbf{f}_{XX}^{(r)}\left(2\pi p_2/P_T\right)\right) \Big) \\
& = \sum_{p_1=0}^{P_T-1} \sum_{p_2=0}^{P_T-1} \exp\left\{i2\pi(p_1 - p_2)u/P_T\right\} B_T T \left(\left(\left(\mathbf{f}_{XX}\left(2\pi p_1/P_T\right) \right)^{-1} \otimes \mathbf{Q}_s \right) \right. \\
& \times \text{ascov} \left(\text{vec}\left(\mathbf{f}_{YX}^{(r)}\left(2\pi p_1/P_T\right)\right), \text{vec}\left(\mathbf{f}_{YX}^{(r)}\left(2\pi p_2/P_T\right)\right) \right) \left(\left(\mathbf{f}_{XX}\left(2\pi p_2/P_T\right) \right)^{-1} \otimes \mathbf{Q}_s \right) \\
& - \left(\left(\mathbf{f}_{XX}\left(2\pi p_1/P_T\right) \right)^{-1} \otimes \mathbf{Q}_s \right) \text{ascov} \left(\text{vec}\left(\mathbf{f}_{YX}^{(r)}\left(2\pi p_1/P_T\right)\right), \text{vec}\left(\mathbf{f}_{XX}^{(r)}\left(2\pi p_2/P_T\right)\right) \right) \\
& \times \left(\left(\mathbf{f}_{XX}\left(2\pi p_2/P_T\right) \right)^{-1} \otimes \overline{\mathbf{A}\left(2\pi p_2/P_T\right)^r} \right) - \left(\left(\mathbf{f}_{XX}\left(2\pi p_1/P_T\right) \right)^{-1} \otimes \mathbf{A}\left(2\pi p_1/P_T\right) \right) \\
& \times \text{ascov} \left(\text{vec}\left(\mathbf{f}_{XX}^{(r)}\left(2\pi p_1/P_T\right)\right), \text{vec}\left(\mathbf{f}_{YX}^{(r)}\left(2\pi p_2/P_T\right)\right) \right) \left(\left(\mathbf{f}_{XX}\left(2\pi p_2/P_T\right) \right)^{-1} \otimes \mathbf{Q}_s \right) \Big)
\end{aligned}$$

$$\begin{aligned}
& + \left(\left(\mathbf{f}_{XX} \left(2\pi p_1 / P_T \right) \right)^{-1} \otimes \mathbf{A} \left(2\pi p_1 / P_T \right) \right) \\
& \times \text{ascov} \left(\text{vec} \left(\mathbf{f}_{XX}^{(T)} \left(2\pi p_1 / P_T \right) \right), \text{vec} \left(\mathbf{f}_{XX}^{(T)} \left(2\pi p_2 / P_T \right) \right) \right) \left(\left(\mathbf{f}_{XX} \left(2\pi p_2 / P_T \right) \right)^{-1} \otimes \overline{\mathbf{A} \left(2\pi p_2 / P_T \right)^T} \right) \\
& = 2\pi H_4(0) (H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \frac{1}{P_T} \sum_{p=0}^{P_T-1} \left(\left(\mathbf{f}_{XX} \left(2\pi p / P_T \right) \right)^{-1} \otimes \mathbf{f}_{\varepsilon} \left(2\pi p / P_T \right) \right).
\end{aligned}$$

Now let us see how we can estimate the spectral density matrix of $\varepsilon(t)$: If we suppose for fixed λ that the matrix $\mathbf{f}_{XX}(\lambda)$ is not singular, then we have according to (1.7) that

$$\mathbf{f}_{\varepsilon}(\lambda) = \mathbf{f}_{YY}(\lambda) - \mathbf{f}_{YX}(\lambda) (\mathbf{f}_{XX}(\lambda))^{-1} \mathbf{f}_{XY}(\lambda). \quad (2.13)$$

The obvious estimate for the spectral density matrix therefore is

$$\mathbf{g}_{\varepsilon}^{(T)}(\lambda) = \mathbf{f}_{YY}^{(T)}(\lambda) - \mathbf{f}_{YX}^{(T)}(\lambda) (\mathbf{f}_{XX}^{(T)}(\lambda))^{-1} \mathbf{f}_{XY}^{(T)}(\lambda). \quad (2.14)$$

THEOREM 2.3 Suppose that

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix}, \quad t = 0, \pm 1, \pm 2, \dots,$$

be a strictly stationary $(r+s)$ vector-valued time series with mean $\mathbf{0}$ and satisfies Assumption 1.3. Assume for $\lambda \neq 0$ that $\mathbf{f}_{XX}(\lambda)$ is not singular. Then if

$$B_T \rightarrow 0, \quad B_T T \rightarrow \infty \text{ as } T \rightarrow \infty,$$

$$\sqrt{B_T T} \text{vec}(\mathbf{g}_{\varepsilon}^{(T)}(\lambda) - \mathbf{f}_{\varepsilon}(\lambda)) \xrightarrow{D} N_{rs}^C(\mathbf{0}, 2\pi H_4(0) (H_2(0))^{-2} \times \int (W(\alpha))^2 d\alpha (\mathbf{f}_{\varepsilon}(\lambda) \otimes \mathbf{f}_{\varepsilon}(\lambda))) \quad (2.15)$$

PROOF: Let

$$\hat{\mathbf{f}}_{YX}^{(T)}(\lambda) = \mathbf{f}_{YX}^{(T)}(\lambda) - \mathbf{f}_{YX}(\lambda)$$

with a similar notation for $\hat{\mathbf{f}}_{YY}^{(T)}(\lambda), \hat{\mathbf{f}}_{XX}^{(T)}(\lambda)$. We have the Taylor expansion

$$\begin{aligned}
\mathbf{g}_{\varepsilon}^{(T)}(\lambda) &= \mathbf{f}_{\varepsilon}(\lambda) + \hat{\mathbf{f}}_{YY}^{(T)}(\lambda) - \hat{\mathbf{f}}_{YX}^{(T)}(\lambda) (\mathbf{f}_{XX}(\lambda))^{-1} \mathbf{f}_{XY}(\lambda) - \mathbf{f}_{YX}(\lambda) (\mathbf{f}_{XX}(\lambda))^{-1} \hat{\mathbf{f}}_{XY}^{(T)}(\lambda) \\
&+ \mathbf{f}_{YX}(\lambda) (\mathbf{f}_{XX}(\lambda))^{-1} \hat{\mathbf{f}}_{XX}^{(T)}(\lambda) (\mathbf{f}_{XX}(\lambda))^{-1} \mathbf{f}_{XY}(\lambda) + \dots
\end{aligned}$$

Now the indicated first- and second-order moments follow from Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7]. The asymptotic normality follows as in Theorem 2.1.

Now let $r = s = 1$. The next step will be the estimation of the multiple coherence in this univariate case. For fixed λ we suppose $f_{xx}(\lambda) \neq 0$ and $f_{yy}(\lambda) \neq 0$. We have

$$|R_{yx}(\lambda)|^2 = \frac{f_{yx}(\lambda)f_{xy}(\lambda)}{f_{yy}(\lambda)f_{xx}(\lambda)}. \quad (2.16)$$

The obvious estimate for the multiple coherence of $Y(t)$ with $X(t)$ at frequency λ therefore is

$$|R_{yx}^{(T)}(\lambda)|^2 = \frac{f_{yx}^{(T)}(\lambda)f_{xy}^{(T)}(\lambda)}{f_{yy}^{(T)}(\lambda)f_{xx}^{(T)}(\lambda)}.$$

The next theorem describes the asymptotic behaviour of $|R_{yx}^{(T)}(\lambda)|^2$.

THEOREM 2.4 Suppose that

$$Z(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}, \quad t = 0, \pm 1, \pm 2, \dots,$$

be a strictly stationary real-valued time series with mean 0 and satisfies Assumption 1.3. Assume for $\lambda \neq 0$ that $f_{xx}(\lambda), f_{yy}(\lambda)$ are not singular. Then, as $T \rightarrow \infty$,

$$\begin{aligned} \sqrt{B_T T} \left(|R_{yx}^{(T)}(\lambda)|^2 - |R_{yx}(\lambda)|^2 \right) &\xrightarrow{D} N^C \left(0, 4\pi H_4(0) (H_2(0))^{-2} \right. \\ &\quad \left. \times \int (W(\alpha))^2 d\alpha |R_{yx}(\lambda)|^2 (1 - |R_{yx}(\lambda)|^2)^2 \right). \end{aligned} \quad (2.17)$$

PROOF: We start with a Taylor expansion of

$$\begin{aligned} |R_{yx}^{(T)}(\lambda)|^2 &= |R_{yx}(\lambda)|^2 \left(1 + (f_{yx}^{(T)}(\lambda) - f_{yx}(\lambda)) f_{yx}(\lambda)^{-1} + (f_{xy}^{(T)}(\lambda) - f_{xy}(\lambda)) f_{xy}(\lambda)^{-1} \right. \\ &\quad \left. + (f_{yy}(\lambda) - f_{yy}^{(T)}(\lambda)) f_{yy}(\lambda)^{-1} + (f_{xx}(\lambda) - f_{xx}^{(T)}(\lambda)) f_{xx}(\lambda)^{-1} \right) + K. \end{aligned} \quad (2.18)$$

Applying Corollary 3 of Mann and Wald [11] yields $K = o_p\left((B_T T)^{-1/2}\right)$ Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7] hence gives the asymptotic normality of $\sqrt{B_T T}\left(\left|R_{YX}^{(T)}(\lambda)\right|^2 - |R_{YX}(\lambda)|^2\right)$ with asymptotic mean zero. The asymptotic variance has to be calculated using the Taylor expansion (2.18):

$$\begin{aligned} \lim_{T \rightarrow \infty} \text{var}\left(\sqrt{B_T T}\left(\left|R_{YX}^{(T)}(\lambda)\right|^2 - |R_{YX}(\lambda)|^2\right)\right) &= \lim_{T \rightarrow \infty} B_T T \left(\left|R_{YX}(\lambda)\right|^2\right)^2 \text{var}\left(f_{YX}^{(T)}(\lambda)(f_{YX}(\lambda))^{-1}\right. \\ &+ f_{XY}^{(T)}(\lambda)(f_{XY}(\lambda))^{-1} + f_{YY}^{(T)}(\lambda)(f_{YY}(\lambda))^{-1} + f_{XX}^{(T)}(\lambda)(f_{XX}(\lambda))^{-1}) \\ &= \lim_{T \rightarrow \infty} B_T T \left(\left|R_{YX}(\lambda)\right|^2\right)^2 \left((f_{YX}(\lambda)f_{XY}(\lambda))^{-1} \text{var}\left(f_{YX}^{(T)}(\lambda)\right) + (f_{YX}(\lambda)f_{XY}(\lambda))^{-1} \text{var}\left(f_{XY}^{(T)}(\lambda)\right)\right. \\ &+ (f_{YY}(\lambda))^{-2} \text{var}\left(f_{YY}^{(T)}(\lambda)\right) + (f_{XX}(\lambda))^{-2} \text{var}\left(f_{XX}^{(T)}(\lambda)\right) \\ &+ (f_{YX}(\lambda))^{-1} \text{cov}\left(f_{YX}^{(T)}(\lambda), f_{XY}^{(T)}(\lambda)(f_{XY}(\lambda))^{-1} - f_{YY}^{(T)}(\lambda)(f_{YY}(\lambda))^{-1} - f_{XX}^{(T)}(\lambda)(f_{XX}(\lambda))^{-1}\right) \\ &+ (f_{XY}(\lambda))^{-1} \text{cov}\left(f_{XY}^{(T)}(\lambda), f_{YX}^{(T)}(\lambda)(f_{YX}(\lambda))^{-1} - f_{YY}^{(T)}(\lambda)(f_{YY}(\lambda))^{-1} - f_{XX}^{(T)}(\lambda)(f_{XX}(\lambda))^{-1}\right) \\ &- (f_{YY}(\lambda))^{-1} \text{cov}\left(f_{YY}^{(T)}(\lambda), f_{YX}^{(T)}(\lambda)(f_{YX}(\lambda))^{-1} + f_{XY}^{(T)}(\lambda)(f_{XY}(\lambda))^{-1} - f_{XX}^{(T)}(\lambda)(f_{XX}(\lambda))^{-1}\right) \\ &\left.- (f_{XX}(\lambda))^{-1} \text{cov}\left(f_{XX}^{(T)}(\lambda), f_{YX}^{(T)}(\lambda)(f_{YX}(\lambda))^{-1} + f_{XY}^{(T)}(\lambda)(f_{XY}(\lambda))^{-1} - f_{YY}^{(T)}(\lambda)(f_{YY}(\lambda))^{-1}\right)\right). \end{aligned}$$

From Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3] and Ghazal and Farag [7] we now know the asymptotic covariances of the second-order spectra. Because of that we get

$$\begin{aligned} \lim_{T \rightarrow \infty} \text{var}\left(\sqrt{B_T T}\left(\left|R_{YX}^{(T)}(\lambda)\right|^2 - |R_{YX}(\lambda)|^2\right)\right) &= 2\pi H_4(0)(H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \left(\left|R_{YX}(\lambda)\right|^2\right)^2 \\ &\times \left(\left(2\left(\left|R_{YX}(\lambda)\right|^2\right)^{-1} + 2\right) + (-1) + (-1) - \left(2 - |R_{YX}(\lambda)|^2\right) - \left(2 - |R_{YX}(\lambda)|^2\right)\right) \\ &= 2\pi H_4(0)(H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \left|R_{YX}(\lambda)\right|^2 \left(2 - 4|R_{YX}(\lambda)|^2 + 2\left(\left|R_{YX}(\lambda)\right|^2\right)^2\right) \\ &= 4\pi H_4(0)(H_2(0))^{-2} \int (W(\alpha))^2 d\alpha \left|R_{YX}(\lambda)\right|^2 \left(1 - |R_{YX}(\lambda)|^2\right)^2. \end{aligned}$$

The multiple coherence in the general case and the partial coherency can be estimated in the same manner using the asymptotic results of the estimation of the second-order spectra, see Brillinger [1]. Asymptotic normality of the

estimator of the partial coherency there is shown while for the multiple coherence only the asymptotic structure of the first and second moments of the estimator is stated.

Similar results hold for the case $\lambda = 0$. Real-valued statistics and distributions will be involved in this case.

On occasion an alternate form of asymptotic distribution may be more relevant than that of Theorem 2.1. Because of the known erratic asymptotic behavior of periodogram ordinates (Brillinger [1], sec. 5.2), we combine estimates from neighboring frequency bands by forming moving averages of length $(2m+1)$ if $\lambda \neq 0$ and $(2m)$ if $\lambda = 0$ on (2.6); this smoothing operation yields the quantities

$$\begin{bmatrix} \mathbf{f}_{XX}^{(T)}(\lambda) & \mathbf{f}_{XY}^{(T)}(\lambda) \\ \mathbf{f}_{YX}^{(T)}(\lambda) & \mathbf{f}_{YY}^{(T)}(\lambda) \end{bmatrix} = \begin{bmatrix} (2m+1)^{-1} \sum_{s=-m}^m \mathbf{I}_{XX}^{(T)} \left(2\pi(s(T)+s)/T \right) \\ (2m+1)^{-1} \sum_{s=-m}^m \mathbf{I}_{YX}^{(T)} \left(2\pi(s(T)+s)/T \right) \\ (2m+1)^{-1} \sum_{s=-m}^m \mathbf{I}_{XY}^{(T)} \left(2\pi(s(T)+s)/T \right) \\ (2m+1)^{-1} \sum_{s=-m}^m \mathbf{I}_{YY}^{(T)} \left(2\pi(s(T)+s)/T \right) \end{bmatrix}, \quad \text{if } \lambda \neq 0 \quad (2.19)$$

$$= \begin{bmatrix} (2m)^{-1} \left\{ \sum_{s=-m}^{-1} \mathbf{I}_{XX}^{(T)} \left(2\pi s/T \right) + \sum_{s=1}^m \mathbf{I}_{XX}^{(T)} \left(2\pi s/T \right) \right\} \\ (2m)^{-1} \left\{ \sum_{s=-m}^{-1} \mathbf{I}_{YX}^{(T)} \left(2\pi s/T \right) + \sum_{s=1}^m \mathbf{I}_{YX}^{(T)} \left(2\pi s/T \right) \right\} \\ (2m)^{-1} \left\{ \sum_{s=-m}^{-1} \mathbf{I}_{XY}^{(T)} \left(2\pi s/T \right) + \sum_{s=1}^m \mathbf{I}_{XY}^{(T)} \left(2\pi s/T \right) \right\} \\ (2m)^{-1} \left\{ \sum_{s=-m}^{-1} \mathbf{I}_{YY}^{(T)} \left(2\pi s/T \right) + \sum_{s=1}^m \mathbf{I}_{YY}^{(T)} \left(2\pi s/T \right) \right\} \end{bmatrix}, \quad \text{if } \lambda = 0. \quad (2.20)$$

If the $(r \times r)$ matrix $\mathbf{f}_{XX}^{(T)}(\lambda)$ is not singular, estimate $\mathbf{A}(\lambda)$ in (1.5) by

$$\mathbf{A}^{(T)}(\lambda) = \mathbf{f}_{YX}^{(T)}(\lambda) \left(\mathbf{f}_{XX}^{(T)}(\lambda) \right)^{-1},$$

and $\mathbf{f}_{\varepsilon\varepsilon}(\lambda)$ by

$$\begin{aligned}\mathbf{g}_{\varepsilon\varepsilon}^{(T)}(\lambda) &= \frac{2m+1}{2m+1-r} \left[\mathbf{f}_{YY}^{(T)}(\lambda) - \mathbf{f}_{YX}^{(T)}(\lambda) [\mathbf{f}_{XX}^{(T)}(\lambda)]^{-1} \mathbf{f}_{XY}^{(T)}(\lambda) \right] \quad \text{if } \lambda \neq 0, \\ &= \frac{2m}{2m-r} \left[\mathbf{f}_{YY}^{(T)}(\lambda) - \mathbf{f}_{YX}^{(T)}(\lambda) [\mathbf{f}_{XX}^{(T)}(\lambda)]^{-1} \mathbf{f}_{XY}^{(T)}(\lambda) \right] \quad \text{if } \lambda = 0.\end{aligned}$$

In the next theorem $W_r^C(G, \mu)$ denotes the r variate complex Wishart distribution with G degrees of freedom and mean μ (see Brillinger [1], Section 4.2 for the definition).

Then for $m, s(T)$ are integers with $2\pi s(T)/T \rightarrow \lambda$ as $T \rightarrow \infty$, we may prove

PROPOSITION 2.5 Suppose that

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix}, \quad t = 0, \pm 1, \pm 2, \dots,$$

be a strictly stationary $(r+s)$ vector-valued time series with mean $\mathbf{0}$ and satisfies Assumption 1.3. Then

$$\begin{bmatrix} \mathbf{f}_{XX}^{(T)}(\lambda) & \mathbf{f}_{XY}^{(T)}(\lambda) \\ \mathbf{f}_{YX}^{(T)}(\lambda) & \mathbf{f}_{YY}^{(T)}(\lambda) \end{bmatrix}$$

is asymptotically,

$$(2m+1)^{-1} W_{r+s}^C \left(2m+1, \begin{bmatrix} \mathbf{f}_{XX}(\lambda) & \mathbf{f}_{XY}(\lambda) \\ \mathbf{f}_{YX}(\lambda) & \mathbf{f}_{YY}(\lambda) \end{bmatrix} \right), \quad \text{if } \lambda \neq 0 \quad (2.21)$$

and asymptotically,

$$(2m)^{-1} W_{r+s} \left(2m, \begin{bmatrix} \mathbf{f}_{XX}(0) & \mathbf{f}_{XY}(0) \\ \mathbf{f}_{YX}(0) & \mathbf{f}_{YY}(0) \end{bmatrix} \right), \quad \text{if } \lambda = 0. \quad (2.22)$$

In consequence the limiting distribution of $\mathbf{A}^{(T)}(\lambda)$ and $\mathbf{g}_{\varepsilon\varepsilon}^{(T)}(\lambda)$ will be that of

$$\mathbf{W}_{YX} (\mathbf{W}_{XX})^{-1},$$

$$\frac{2m+1}{2m+1-r} (\mathbf{W}_{YY} - \mathbf{W}_{YX} (\mathbf{W}_{XX})^{-1} \mathbf{W}_{XY}), \quad \text{if } \lambda \neq 0$$

$$\frac{2m}{2m-r} (\mathbf{W}_{YY} - \mathbf{W}_{YX} (\mathbf{W}_{XX})^{-1} \mathbf{W}_{XY}), \quad \text{if } \lambda = 0$$

respectively where

$$\begin{bmatrix} \mathbf{W}_{XX} & \mathbf{W}_{XY} \\ \mathbf{W}_{YX} & \mathbf{W}_{YY} \end{bmatrix}$$

has the distribution (2.21) if $\lambda \neq 0$ and (2.22) if $\lambda = 0$. Also in the case $r = s = 1$,

$$\left| R_{YX}^{(r)}(\lambda) \right|^2 \text{ tends to } \frac{W_{YX} W_{XY}}{W_{YY} W_{XX}}.$$

For example, the asymptotic distribution of $\mathbf{g}_{\varepsilon\varepsilon}^{(r)}(\lambda)$, will be

$$(2m+1-r)^{-1} W_s^C(2m+1-r, \mathbf{f}_{\varepsilon\varepsilon}(\lambda)), \quad \text{if } \lambda \neq 0.$$

PROOF: Let

$$\mathbf{L}_X^{(r)}(\lambda) = \sum_{t=0}^{T-1} h^{(r)}(t) \mathbf{X}(t) \exp \{-i\lambda t\} = \text{Re } \mathbf{L}_X^{(r)}(\lambda) + i \text{Im } \mathbf{L}_X^{(r)}(\lambda),$$

from Brillinger [1], Ghazal and Farag [2]; Ghazal, Hennawy and Farag [3],

$\mathbf{L}_X^{(r)}(\lambda)$, $\lambda \neq 0$ is asymptotically independent $N_r^c(\mathbf{0}, 2\pi TH_2(0) \mathbf{f}_{XX}(\lambda))$ variate and

$\mathbf{L}_X^{(r)}(\lambda)$, $\lambda = 0$ is asymptotically independent $N_r(\mathbf{0}, 2\pi TH_2(0) \mathbf{f}_{XX}(\lambda))$ variate.

Theorem P5.1 of Brillinger [1] now indicates that

$$\mathbf{I}_{XX}^{(r)}(\lambda) = (2\pi TH_2(0))^{-1} \mathbf{L}_X^{(r)}(\lambda) \overline{\mathbf{L}_X^{(r)}(\lambda)^T},$$

are asymptotically independent

$$W_r^c(\mathbf{I}, \mathbf{f}_{XX}(\lambda)), \quad \text{if } \lambda \neq 0$$

$$W_r(\mathbf{I}, \mathbf{f}_{XX}(\lambda)), \quad \text{if } \lambda = 0,$$

where $\mathbf{I}$ denotes to identity matrix. Then $\mathbf{f}_{XX}^{(r)}(\lambda)$ is asymptotically

$$(2m+1)^{-1} W_r^c(2m+1, \mathbf{f}_{XX}(\lambda)), \quad \text{if } \lambda \neq 0$$

$$(2m)^{-1} W_r(2m, \mathbf{f}_{XX}(\lambda)), \quad \text{if } \lambda = 0.$$

Similar results for $\mathbf{I}_{XY}^{(r)}(\lambda)$, $\mathbf{I}_{YX}^{(r)}(\lambda)$, $\mathbf{I}_{YY}^{(r)}(\lambda)$ and $\mathbf{f}_{XY}^{(r)}(\lambda)$, $\mathbf{f}_{YX}^{(r)}(\lambda)$, $\mathbf{f}_{YY}^{(r)}(\lambda)$. The second part follows from Theorem P5.1 of Brillinger [1] (see also theorem of Slutsky (Mann and Wald [11])).

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