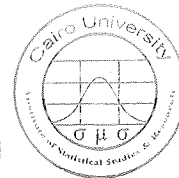




Cairo University

Institute of Statistical Studies and Research



**The 48<sup>th</sup> Annual Conference on Statistics , Computer  
Sciences and Operation Research**

**Mathematical Statistics**

**23-26 Dec ,2013**



# Index

## MATHEMTICAL STATISTICS

|   |                                                                                                                                                                                                  |         |
|---|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------|
| 1 | Modified Goodness of Fit Tests for the Weibull Distribution Based on Moving Extreme Ranked Set Sampliong<br><b>Amal S.Hassan , A.M.Abd-Elfattah , Heba F.Nagy</b>                                | 1-14    |
| 2 | Estimation of the Parameters of Bur Type III Distribution in Presence of Outliers<br><b>A.M. Abd-Elfattah , E.A. Elsherpieny , S.E.Hemeda</b>                                                    | 15-24   |
| 3 | On the Lomax-Poisson Distribution<br><b>A.M.Abd-Elfattah , Amal S.Hassan, Asmaa M.Hussein</b>                                                                                                    | 25-39   |
| 4 | Maximum Likelihood Estimation of the Parameters for Exponentiated Weibull Poisson Distribution Based on Generalized Order Statistics<br><b>A.M. Abd-Elfattah , Amal S.Hassan , Said, G.Nassr</b> | 40-56   |
| 5 | Multiple Step Stress Accelerated Life for Weibull Poisson Distribution With Type I Censoring<br><b>Amal S.Hassan , Assar,S.M , Shelbaia,A</b>                                                    | 57-71   |
| 6 | On the Chaotic Behaviour of Nonlinear Time Series Models<br><b>A. Elhassanein , A. SH. Al-henawy</b>                                                                                             | 72-83   |
| 7 | Estimation of Stress-Strength Parameter for Rayleigh Distribution Based on Progressive Type-II Censoring<br><b>A.M. Abd-Elfattah , M.H. Abu-Moussa , M.M.El-Fahham</b>                           | 84-97   |
| 8 | On Markov Chain-Gambler's Ruin Problem With Ties Allowed<br><b>M.A.El-Shehawey , M.E. El-Tantawey , Gh.A.Al-Shreef</b>                                                                           | 98-112  |
| 9 | Parameters Estimation of the Bivariate Weibull Distribution by EM Algorithm Based on Censored Samples<br><b>Sahar,A.M. Ibrahim , Radwan,E.Bedar</b>                                              | 113-122 |

1. The first part of the paper is a review of the literature on the topic of the paper.

2. The second part of the paper is a description of the methodology used in the study.

3. The third part of the paper is a discussion of the results of the study.

4. The fourth part of the paper is a conclusion.

5. The fifth part of the paper is a list of references.



## Modified Goodness of Fit Tests for the Weibull Distribution Based on Moving Extreme Ranked Set Sampling

Amal S. Hassan, A.M. Abd-Elfattah, and Heba F. Nagy<sup>1</sup>

### Abstract

Moving extreme ranked set sampling is a very useful modified version of the usual ranked set sampling that allows for increasing the set size without introducing too much ranking error. This article deals with modified empirical distribution function goodness of fit tests for Weibull distribution based on moving extreme ranked set sampling. Tables of critical values for the modified Kolmogorov-Smirnov, Cramer-von-Mises, Anderson-Darling, Watson and Kuiper goodness of fit tests for Weibull distribution with unknown parameters are created. Functional relationship between the critical values of these test statistics are examined for each set size, number of cycles and significance level. Powers of test statistics for a number of alternative distributions are given through a simulation. Furthermore, power efficiencies of these test statistics based on moving extreme ranked set sampling relative to simple random sampling are created for the same sample size. The resulting of power efficiencies showed that the modified tests under moving extreme ranked set sampling are more efficient than their corresponding in simple random sampling. In addition the Watson test statistics has the highest efficiency for all alternative hypotheses.

**Keywords:** Moving extreme Ranked Set Sampling; Simple Random Sample; Anderson-Darling test statistic; Cramer-von Mises test statistic; Critical values; Power test; Kolmogorov-Smirnov statistic, Watson statistic.

### 1. Introduction

Ranked set sampling (RSS) is a development that enables one to provide more structure to the collected sample items. RSS was introduced by McIntyre (1952) for estimating pasture yields. This technique is useful for cases when the variable of interest can be more easily ranked than quantified. RSS has many applications in different fields such as agricultural, medical and biological areas.

There are two factors that affect the efficiency of RSS, the set size and the ranking errors. The larger the set size, the more the efficiency of RSS. Thus the larger the set size, the more troublesome in visual ranking and ranking errors for this reason several authors modified RSS to reduce the error in ranking and to make visual ranking tractable by an experimenter. Samawi *et al.* (1996) investigated extreme ranked set sample (ERSS), i.e., they quantify the smallest and the largest order statistics instead of detailed ranking. Al-Odat and Al-Saleh (2001) introduced moving extreme ranked set sampling (MERSS).

---

<sup>1</sup> Department of Mathematical Statistics in Institute of Statistical Studies and Research (ISSR).Cairo University, Egypt

This method uses only extremes with varied set size to reduce error in ranking. MERSS does not need a complete ranking but RSS needs the ranking of all elements of each set. Samawi and Al-Saleh (2013) introduced two types of MERSS which are based on the maximum and minimum order observations and denoted by  $MERSS_{max}$  and  $MERSS_{min}$  respectively. They provided estimation of the odds ratio between two independent groups using two types of MERSS. Theoretical properties of the suggested estimator are derived and compared with its counterpart estimator using simple random sampling (SRS). They found that the estimator based on MERSS is always valid and has some advantages over that based on SRS. The MERSS procedure as described by Samawi and Al-Saleh (2013) can be summarized as follows:

**Step 1:** Select  $m$  random samples of size 1, 2, 3...  $m$ , respectively.

**Step 2:** Identify the maximum of each set, visually or by any cost free method without actual measurement of the variable of interest.

**Step 3:** Measure accurately the selected judgment identified maxima.

**Step 4:** Repeat the above steps  $r$  times in order to obtain a sample of large size  $n = rm$ . This sample will be denoted by  $MERSS_{max}$ .

**Step 5:** Repeat the above steps through identifying the minimum of each set instead of the maximum to have a moving extreme ranked set sample as  $MERSS_{min}$ .

Goodness-of-fit (GOF) tests are designed to measure how well the observed sample data fits some proposed model. One class of GOF tests can be used consists of tests based on the distance between the empirical and hypothesized distribution functions. Five of the known tests in this class are Kolmogrov-Smirnov, Cramer-von-Mises, Anderson-Darling, Watson and Kuiper tests. These tests are valid when there are no unknown parameters in the hypothesized distribution. These tests become extremely conservative if they are used in case where unknown parameters must be estimated from the sample data.

Through the last two decades, the goodness of fit tests based on data collected via RSS technique and its modifications have not taken the attention of authors. Stockes and Sager (1988) showed that the empirical distribution function of RSS is an unbiased estimator for the population distribution function and has greater precision even if the ranking is imperfect. Then, they proposed a Kolmogrov-Smirnov GOF test based on the empirical distribution function (EDF). Al-Subh *et al.* (2009) gave a comparison study for the power of a set of empirical distribution function goodness of fit tests for the logistic distribution under SRS and RSS. Ibrahim *et al.* (2009) proposed ERSS method to improve the power of empirical distribution function GOF tests for logistic distribution through a simulation study. Shahabuddin *et al.* (2009) investigated the performance of several GOF tests under SRS and RSS. Hassan (2012) established tables of critical values for the exponentiated Pareto distribution under ERSS. She investigated the power of the modified test statistics under ERSS and SRS for a number of alternative distributions and showed that the modified tests under ERSS are more efficient than their corresponding in SRS.

In the literature, there were no studies that had been performed about the GOF tests based on MERSS. Therefore, the main objective in this article is to create Tables of critical values for the Weibull distribution under  $MERSS_{min}$  and  $MERSS_{max}$ . The power comparisons and the efficiency of a set of modified EDF tests are investigated for a number of alternative distributions based on  $MERSS_{min}$ ,  $MERSS_{max}$  and SRS.

This article is organized as follows. In section 2, maximum likelihood estimator of the unknown parameters from Weibull distribution based on  $MERSS_{min}$  and  $MERSS_{max}$  is obtained. Section 3 deals with the set of modified EDF goodness of fit tests under SRS and two types of MERSS. Section 4 generates the percentage points for the modified test statistics through simulation technique. In the same section response functions that give the percentage points for the modified test statistics are obtained to avoid using a large number of tables of critical values. Section 5 deals with the power efficiency for the modified test statistics under  $MERSS_{min}$  and  $MERSS_{max}$  relative to SRS. Finally concluding remarks are presented in Section 6.

## 2. Maximum Likelihood Estimation

In this Section the maximum likelihood estimators (MLEs) of the unknown parameters for the Weibull distribution will be obtained based on  $MERSS_{max}$  and  $MERSS_{min}$ .

The Weibull distribution is an important distribution for modeling and lifetime data analysis in biological, medical and engineering sciences. It can therefore model a great variety of data and life characteristics. It is used extensively in reliability applications to model failure times. The cumulative distribution function (CDF) and probability density function (PDF) of the Weibull distribution are given, respectively, by

$$F(x; \lambda, \alpha) = 1 - e^{-\left(\frac{x}{\lambda}\right)^\alpha} \quad ; x > 0, \alpha > 0, \lambda > 0, \quad (2.1)$$

and,

$$f(x; \lambda, \alpha) = \frac{\alpha}{\lambda^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^\alpha} \quad ; x > 0, \alpha > 0, \lambda > 0, \quad (2.2)$$

### 2.1 MLE Based on $MERSS_{max}$

Let  $\{X_{(m:m)_j}, X_{(m-1:m-1)_j}, \dots, X_{(1:1)_j}\}$ , for  $j = 1, \dots, r$ , be a  $MERSS_{max}$  of size  $n = rm$ , where  $m$  is the set size with  $r$  number of cycles. If judgement ranking error are accurate then  $x_{(i:i)}$ , for  $i = 1, \dots, m$ , has the same distribution as the largest order statistic of a SRS of size  $i$  from PDF (2.1), therefore

$$f(x_{(i:i)_j}) = \frac{i}{\lambda^\alpha} x_{(i:i)_j}^{\alpha-1} e^{-\left(\frac{x_{(i:i)_j}}{\lambda}\right)^\alpha} [1 - e^{-\left(\frac{x_{(i:i)_j}}{\lambda}\right)^\alpha}]^{i-1},$$

The likelihood function of the sample  $\{X_{(m:m)_j}, X_{(m-1:m-1)_j}, \dots, X_{(1:1)_j}\}$  is given by

$$l(\alpha, \lambda) = \prod_{j=1}^r \prod_{i=1}^m \frac{i}{\lambda^\alpha} x_{(i:j)f}^{\alpha-1} e^{-\left(\frac{x_{(i:j)f}}{\lambda}\right)^\alpha} [1 - e^{-\left(\frac{x_{(i:j)f}}{\lambda}\right)^\alpha}]^{i-1}, \quad (2.3)$$

The log likelihood function  $\ell \equiv \ln l(\alpha, \lambda)$  is given by

$$\ell = \sum_{j=1}^r \sum_{i=1}^m \left\{ \ln i + \ln \alpha - \alpha \ln \lambda + (\alpha - 1) \ln x_{(i:j)f} - \left(\frac{x_{(i:j)f}}{\lambda}\right)^\alpha + (i - 1) \ln [1 - e^{-\left(\frac{x_{(i:j)f}}{\lambda}\right)^\alpha}] \right\}. \quad (2.4)$$

The first partial derivatives of the logarithm of the likelihood function with respect to  $\alpha$  and  $\lambda$ , and the following normal equations, are obtained as follows

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{j=1}^r \sum_{i=1}^m \ln \left( \frac{x_{(i:j)f}}{\lambda} \right) - \sum_{j=1}^r \sum_{i=1}^m \left( \frac{x_{(i:j)f}}{\lambda} \right)^\alpha \ln \left( \frac{x_{(i:j)f}}{\lambda} \right) \\ &+ \sum_{j=1}^r \sum_{i=1}^m \frac{(i-1)}{\left( e^{\left(\frac{x_{(i:j)f}}{\lambda}\right)^\alpha} - 1 \right)} \left( \frac{x_{(i:j)f}}{\lambda} \right)^\alpha \ln \left( \frac{x_{(i:j)f}}{\lambda} \right) = 0, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \frac{\partial \ell}{\partial \lambda} &= -\frac{n}{\lambda} + \sum_{j=1}^r \sum_{i=1}^m \left\{ \alpha \left( \frac{x_{(i:j)f}}{\lambda} \right)^{\alpha-1} \right\} \\ &- \sum_{j=1}^r \sum_{i=1}^m \left\{ \frac{(i-1)}{\left( e^{\left(\frac{x_{(i:j)f}}{\lambda}\right)^\alpha} - 1 \right)} \alpha \left( \frac{x_{(i:j)f}}{\lambda} \right)^{\alpha-1} \right\} = 0. \end{aligned} \quad (2.6)$$

Obviously, it is difficult to obtain a closed form solution to the non linear equations (2.5) and (2.6). Therefore, an iterative procedure is applied to solve these equations numerically using MathCAD (14) program.

## 2.2 MLE Based on MERSS<sub>min</sub>

Let  $\{Y_{(1:m)f}, Y_{(1:m-1)f}, Y_{(1:m-2)f}, \dots, Y_{(1:1)f}\}$  be MERSS<sub>min</sub> of size  $n = rm$ . If judgement ranking error are accurate then, for  $i = 1, \dots, m$ ,  $y_{1i}$  has the same distribution as the 1<sup>st</sup> order statistics of a SRS of size  $i$  from PDF (2.1), therefore

$$f(y_{(1:i)f}) = \frac{i}{\lambda^\alpha} y_{(1:i)f}^{\alpha-1} e^{-\left(\frac{y_{(1:i)f}}{\lambda}\right)^\alpha}$$

The likelihood function of the sample  $\{Y_{(1:m)f}, Y_{(1:m-1)f}, Y_{(1:m-2)f}, \dots, Y_{(1:1)f}\}$  is given by

$$l(\alpha, \lambda) = \prod_{j=1}^r \prod_{i=1}^m \frac{i}{\lambda^\alpha} y_{(1:i)f}^{\alpha-1} e^{-\left(\frac{y_{(1:i)f}}{\lambda}\right)^\alpha}$$

The log likelihood function  $\ell \equiv \ln l(\alpha, \lambda)$  is given by

$$\ell = \sum_{j=1}^r \sum_{i=1}^m \left\{ \ln i + \ln \alpha - \alpha \ln \lambda + (\alpha - 1) \ln y_{(i:j)} - i \left( \frac{y_{(i:j)}}{\lambda} \right)^\alpha \right\}.$$

Furthermore, the normal equations from maximum likelihood for MERSS<sub>min</sub> are as follows

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{j=1}^r \sum_{i=1}^m \ln \left( \frac{y_{(i:j)}}{\lambda} \right) - \sum_{j=1}^r \sum_{i=1}^m i \left( \frac{y_{(i:j)}}{\lambda} \right)^\alpha \ln \left( \frac{y_{(i:j)}}{\lambda} \right) = 0, \quad (2.7)$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{j=1}^r \sum_{i=1}^m \left( i \alpha \left( \frac{y_{(i:j)}}{\lambda^{\alpha+1}} \right) \right) - \frac{n \alpha}{\lambda} = 0. \quad (2.8)$$

An iterative procedure is applied to solve equations (2.7) and (2.8), numerically using MathCAD (14) program.

### 3. Modified EDF Goodness of Fit Tests

In this Section modified EDF goodness of fit tests based on SRS, MERSS<sub>min</sub> and MERSS<sub>max</sub> will be discussed

#### 3.1 Tests Based on SRS

A goodness of fit test based on the EDF, where the parameters are estimated is called modified goodness of fit test. The objective is to test the statistical hypothesis

$$H_0 : F(x) = F_0(x) \quad \forall x, \quad \text{vs} \quad H_1 : F(x) \neq F_0(x) \quad \text{for some } x, \quad (3.1)$$

where  $F_0(x)$  is a hypothesized distribution function based on a random sample  $X_1, X_2, \dots, X_n$  from the distribution function for Weibull distribution with two unknown parameters defined in (2.2).

The following set of the modified EDF goodness of fit tests defined as follows

a) The Kolmogorov smirnov (KS) test statistic  $D$  is

$$D = \max \left\{ \max_{1 \leq i \leq n} \left[ \frac{i}{n} - F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}) \right], \max_{1 \leq i \leq n} \left[ F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}) - \frac{i-1}{n} \right] \right\} \quad (3.2)$$

b) The Kuiper test statistic  $V$  is a modification of (KS) and takes the following form

$$V = \left\{ \max_{1 \leq i \leq n} \left[ \frac{i}{n} - F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}) \right] + \max_{1 \leq i \leq n} \left[ F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}) - \frac{i-1}{n} \right] \right\} \quad (3.3)$$

- c) The Cramer-von Mises (CvM) statistic  $W^2$  is presented by the following formula

$$W^2 = \left(\frac{1}{12n}\right) + \sum_{i=1}^n \left[F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}) - \left(\frac{2i-1}{2n}\right)^2\right] \quad (3.4)$$

- d) The Watson statistic  $U^2$  is a modification of (CvM) statistic and takes the following form

$$U^2 = W^2 - n \left( \frac{\sum_{i=1}^n F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda})}{n} - \frac{1}{2} \right)^2 \quad (3.5)$$

- e) The Anderson-Darling (AD) statistic  $A^2$  takes the following form

$$A^2 = -n - \frac{1}{n} \left\{ \sum_{i=1}^n (2i-1) [\ln(F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda})) + \ln(1 - F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}))] \right\} \quad (3.6)$$

Let us denote the test statistics (3.2)-(3.6) by T, under SRS.

### 3.2 Tests Based on MERSS<sub>max</sub>

To test the hypothesis based on MERSS<sub>max</sub>, Let  $\{X_{(m:m)_j}, X_{(m-1:m-1)_j}, \dots, X_{(1:1)_j}\}$  be a random sample selected via the largest order statistic. According to Al-Subh *et al.* (2009), testing the hypotheses

$H_0: F(x) = F_0(x) \forall x$ , vs  $H_1: F(x) \neq F_0(x)$  for some  $x$ , is equivalent to testing the hypotheses,

$$H_0^*: G(x) = G_0(x) \forall x, \text{ vs } H_1^*: G(x) \neq G_0(x) \text{ for some } x, \quad (3.7)$$

where,  $G(x)$  and  $G_0(x)$  are the CDF's of the MERSS<sub>max</sub> of random samples chosen from  $F(x)$  and  $F_0(x)$  respectively,  $i = 1, \dots, n$ .

### 3.2 Tests Based on MERSS<sub>min</sub>

To test the hypothesis based on MERSS<sub>min</sub>, Let  $\{Y_{(1:m)_j}, Y_{(1:m-1)_j}, Y_{(1:m-2)_j}, \dots, Y_{(1:1)_j}\}$  be a random sample selected via the first order statistic. According to Al-Subh *et al.* (2009), testing the hypotheses

$H_0: F(y) = F_0(y) \forall y$ , vs  $H_1: F(y) \neq F_0(y)$  for some  $y$ , is equivalent to testing the hypotheses,

$$H_0^{**}: K(y) = K_0(y) \forall y, \text{ vs } H_1^{**}: K(y) \neq K_0(y) \text{ for some } y, \quad (3.8)$$

where,  $K(y)$  and  $K_0(y)$  are the CDF's of the MERSS<sub>min</sub> of random samples chosen from  $F(y)$  and  $F_0(y)$  respectively,  $i = 1, \dots, n$ .

Thus the goodness of fit tests for the hypothesis (3.7) and (3.8), denoted by  $T_1^*$  for MERSS<sub>max</sub> and  $T_2^*$  for MERSS<sub>min</sub>, can be performed using the test statistic

$T$  as defined in the beginning Section, but by using the data  $\{X_{(m:m)_j}, X_{(m-1:m-1)_j}, \dots, X_{(1:1)_j}\}$  and  $\{Y_{(l:m)_j}, Y_{(l:m-1)_j}, Y_{(l:m-2)_j}, \dots, Y_{(1:1)_j}\}$  respectively.

#### 4. Percentage Points of the Modified Test Statistics

The aim in this Section is to obtain percentage points of  $T_i^*$ ,  $i = 1, 2$ , for the Weibull distribution based on  $MERSS_{\min}$  and  $MERSS_{\max}$ . Monte Carlo simulation is used to create critical values for the proposed test statistics mentioned in Section 3 for the Weibull distribution with unknown parameters using the following steps:

**Step 1:** Generate a sample from the Weibull distribution with  $\alpha = 0.5$  and  $\lambda = 1$ , for set sizes  $m = 2, 3, 4, 5$ , and cycles  $r = 3, 7, 9$  using the  $i^{th}$  order statistic, then  $\{X_{(m:m)_j}, X_{(m-1:m-1)_j}, \dots, X_{(1:1)_j}\}$  be a random sample generated based on  $MERSS_{\max}$  and  $\{Y_{(l:m)_j}, Y_{(l:m-1)_j}, Y_{(l:m-2)_j}, \dots, Y_{(1:1)_j}\}$  be a random sample generated based on  $MERSS_{\min}$ .

**Step 2:** This random sample is used to estimate the unknown parameters  $\alpha, \lambda$  using maximum likelihood method of estimation by solving the non linear equations (2.5) and (2.6) based on  $MERSS_{\max}$  and equations (2.7) and (2.8) based on  $MERSS_{\min}$ .

**Step 3:** The resulting MLEs of the unknown parameters are used to determine the hypothesized CDF of the Weibull distribution.

**Step 4:** Obtain the EDF for  $MERSS_{\max}$ , as

$$\hat{F}_{MERSS_{\max}}(x) = \frac{1}{n} \sum_{j=1}^r \sum_{i=1}^m I(X_{(i:j)_j} \leq x), \quad I(X_{(i:j)_j}) = \begin{cases} 1 & X_{(i:j)_j} \leq x \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

Similarly, obtain the EDF for  $MERSS_{\min}$ , as

$$\hat{F}_{MERSS_{\min}}(y) = \frac{1}{n} \sum_{j=1}^r \sum_{i=1}^m I(Y_{(i:l)_j} \leq y), \quad I(Y_{(i:l)_j}) = \begin{cases} 1 & Y_{(i:l)_j} \leq y \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

**Step 5:** Use steps 3 and 4 to calculate the modified KS, V, CvM, AD and U test statistics.

**Step 6:** This procedure is repeated 5000 times, thus generating 5000 independent values of the appropriate test statistics. These 5000 values are then arranged in an ascending order and the values of these test statistics are calculated at various significance levels, i.e.,  $\gamma = 0.01, 0.025, 0.05, 0.10, 0.15, 0.20$  and  $0.25$ . The obtained values are the critical values for that particular test under each sample size used. (Tables of critical values under needed).

To avoid using a large number of tables of critical values, response functions are estimated which give the predicted critical values for each

combination of the set size, number of cycles and significance levels. The response function for the critical values are obtained using the following equation

$$\text{critical value}_{\text{test statistic}} = b_0 + b_1(\text{set size}) + b_2(\text{number of cycles}) + b_3(\text{significance level } \gamma) \quad (4.3)$$

The estimated coefficients of the response functions, t-statistics and their  $R^2$  values are given in the following tables under  $\text{MERSS}_{\min}$  and  $\text{MERSS}_{\max}$  respectively using (4.3).

Table (1): Response function for different test statistics under  $\text{MERSS}_{\min}$

| Test statistic | Estimated coefficients |               |               |               | $R^2$ |
|----------------|------------------------|---------------|---------------|---------------|-------|
|                | $b_0$                  | $b_1$         | $b_2$         | $b_3$         |       |
| D              | 0.412(64.9)            | -0.025(-19.6) | -0.161(-25.2) | -0.284(-16.8) | 0.923 |
| V              | 0.690(68.3)            | -0.039(-19.2) | -0.025(-24.9) | -0.501(-18.7) | 0.925 |
| $W^2$          | 0.129(20.6)            | 0.007(5.3)    | 0.002(3.02)   | -0.403(-24.1) | 0.851 |
| $A^2$          | 0.745(21.8)            | 0.041(6.01)   | 0.013(3.8)    | -2.203(-24.2) | 0.855 |
| $U^2$          | 0.123(20.3)            | 0.006(5.2)    | 0.002(2.9)    | -0.381(-23.6) | 0.846 |

The number in parentheses is the value of the t-statistic used to test the hypothesis that the true value of the coefficient is different from zero.

Table (2): Response function for different test statistics under  $\text{MERSS}_{\max}$

| Test statistic | Estimated coefficients |               |               |               | $R^2$ |
|----------------|------------------------|---------------|---------------|---------------|-------|
|                | $b_0$                  | $b_1$         | $b_2$         | $b_3$         |       |
| D              | 0.439(59.6)            | -0.020(-13.4) | -0.015(-19.9) | -0.369(-18.9) | 0.896 |
| V              | 0.716(57.9)            | -0.026(-10.4) | -0.021(-16.6) | -0.688(-20.9) | 0.884 |
| $W^2$          | 0.080(5.7)             | 0.038(13.6)   | 0.015(10.8)   | -0.753(-20.4) | 0.869 |
| $A^2$          | 0.026(0.16)            | 0.295(8.9)    | 0.210(12.6)   | -6.094(-13.9) | 0.800 |
| $U^2$          | 0.062(5.4)             | 0.035(15.2)   | 0.014(11.7)   | -0.653(-21.2) | 0.883 |

The number in parentheses is the value of the t-statistic used to test the hypothesis that the true value of the coefficient is different from zero.

## 5. Power efficiency

In this Section a power study is carried out to investigate the power of the modified test statistics to the null hypothesis under SRS,  $\text{MERSS}_{\min}$  and  $\text{MERSS}_{\max}$ . The power of a test is useful in assessing the goodness of a test or in comparing competing tests. Power comparisons are made among KS, Kuiper, CvM, AD and Watson test statistics for the Weibull distribution with unknown parameters. In this study the null hypothesis  $H_0$  is that the random sample comes from the Weibull distribution and the alternative hypothesis  $H_1$  is that the sample follows some other distributions.



The power is determined by generating 5000 random sample for each of the four alternatives:

1. Normal distribution, denoted by  $N(3, 1)$ .
2. Lognormal distribution, denoted by  $LN(3, 1)$ .
3. Uniform distribution, denoted by  $U(0, 1)$ .
4. Exponential distribution, denoted by  $Exp(2)$ .

For each test, all test statistics are calculated and compared to its respective critical values and counted the number of rejections of the null hypothesis. The power results for the tests at the significance level  $\gamma=0.01$  and  $0.05$ .

The procedure for calculating the power of  $T_i^*$ ,  $i=1,2$ , under the alternative distributions are as follows

**Step (1):** Let  $\{X_{(m:m)_f}, X_{(m-1:m-1)_f}, \dots, X_{(1:1)_f}\}$  and  $\{Y_{(1:m)_f}, Y_{(1:m-1)_f}, Y_{(1:m-2)_f}, \dots, Y_{(1:1)_f}\}$  be a random sample from the alternative distributions for largest and smallest order statistics respectively.

**Step (2):** Obtain the EDF as defined in (4.1) and (4.2).

**Step (3):** Calculate the value of  $T_i^*$ ,  $i=1,2$ , as defined in tests from (3.2)-(3.6) but using data  $\{X_{(m:m)_f}, X_{(m-1:m-1)_f}, \dots, X_{(1:1)_f}\}$  and  $\{Y_{(1:m)_f}, Y_{(1:m-1)_f}, Y_{(1:m-2)_f}, \dots, Y_{(1:1)_f}\}$ . If the critical values of the test statistics for the alternative distribution exceed the corresponding critical values, then the null hypothesis  $H_0$  will be rejected at the significance level (0.05 and 0.01).

**Step (4):** Repeat the above steps from (1-3) 5000 times to generate 5000 independent sets of the test statistics.

**Step (5):** The power of each test is obtained by counting the number of rejections of the null hypothesis divided by 5000. Figures 1 and 2 represent the power of each test statistics for each of four alternative distributions based on  $MERSS_{min}$  and  $MERSS_{max}$

**Step (6):** By the similar way the power of each test statistics is obtained under SRS but using the data  $X_1, X_2, \dots, X_n$ .

**Step (7):** The efficiency of test statistics,  $T_i^*$ ,  $i=1,2$ , under  $MERSS_{max}$  and  $MERSS_{min}$  relative to test statistics  $T$ , under SRS is calculated, where the relative efficiency is defined as the ratio of the powers ,

$$eff(T^*, T) = \frac{\text{power of } T_i^*}{\text{power of } T}, \quad i=1,2$$

The efficiency values of tests at the significance level  $\gamma=0.01$  and  $0.05$  are presented in Table (3) for  $MERSS_{min}$  and in Table (4) for  $MERSS_{max}$ .

From the simulation results given in Tables (1-4), the following remarks may be observed.

1. For different significance levels and the same sample sizes, the change of critical values for all test statistics for  $MERSS_{max}$  are greater than that the corresponding for  $MERSS_{min}$ . As the set size increases, the critical values for test statistics decrease monotonically, for Kolmogrov-Smirnov and Kuiper test statistics.
2. Based on  $MERSS_{max}$  and  $MERSS_{min}$  the efficiency of the all modified tests is greater than one which indicated that the power for test statistics under  $MERSS_{max}$  and  $MERSS_{min}$  is larger than the corresponding under SRS.
3. As the number of cycles increases, the efficiencies for each test statistics increase.
4. Anderson-Darling test statistics has the highest power and Watson test has the smallest power for all alternative hypotheses based on  $MERSS_{max}$  and  $MERSS_{min}$ .
5. The power for  $MERSS_{max}$  and  $MERSS_{min}$  are the same as the number of cycles increase, which leads to their equality in their efficiencies. ( See Figures (1) and (2))
6. Powers efficiency of modified EDF tests are broadly in the following order of Ascending power  $U^2 \rightarrow V \rightarrow D \rightarrow W^2 \rightarrow A^2$

## 6. Conclusion

For different significance levels and sample sizes, the modified EDF tests of fit for Weibull distribution under  $MERSS_{min}$ ,  $MERSS_{max}$  and SRS are investigated. A power study is made using four alternative families of distributions based on SRS,  $MERSS_{min}$  and  $MERSS_{max}$ . This study shows that the efficiency of GOF tests can be much improved if the sample is collected via the  $MERSS_{min}$  or  $MERSS_{max}$ .

Furthermore, the modified EDF tests under  $MERSS_{min}$  and  $MERSS_{max}$  are more efficient than their corresponding in SRS. The efficiency for all tests varies for different values of set sizes and number of cycles. The critical values for the Anderson-Darling test statistic are greater than the other GOF tests in case of  $MERSS_{max}$  and  $MERSS_{min}$ .

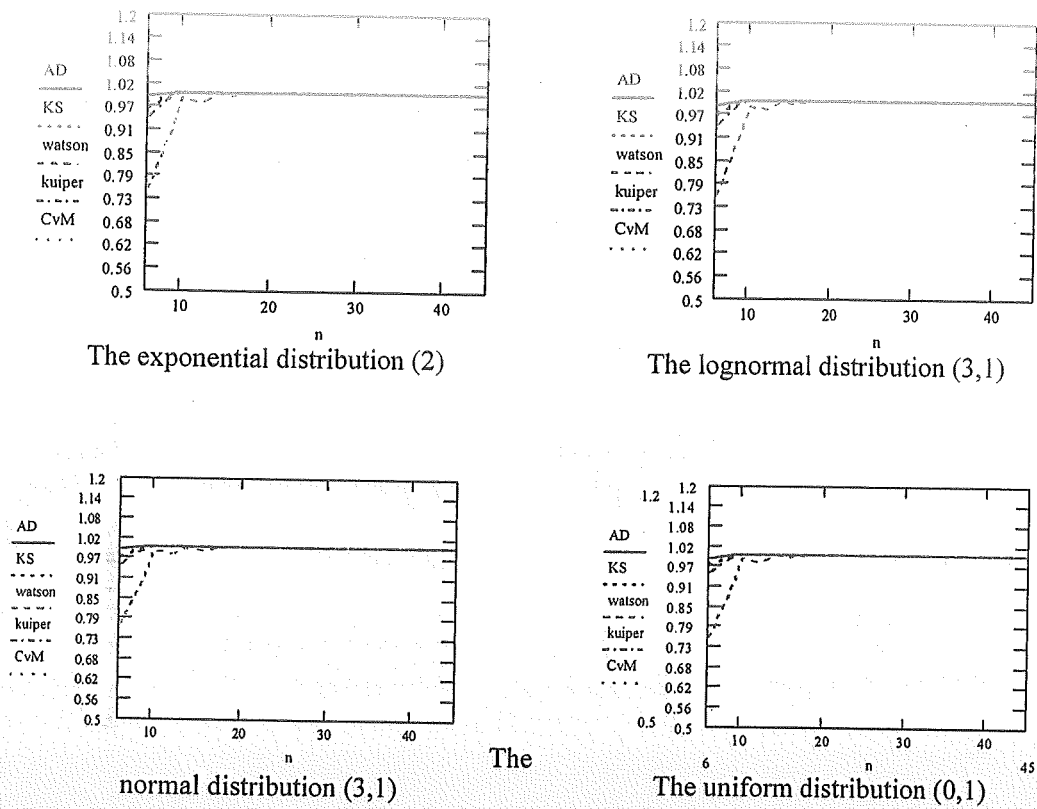
In general, the Watson statistic, Kolmogrov-Smirnov, and Kuiper tests appear to be the best EDF test statistics. The Anderson-Darling test statistic tends to be least powerful among the five EDF considered here. On the other hand, the Watson test statistic is the most powerful goodness-of-fit test among the competitors. The efficiency of the modified EDF tests increases as the sample size increases in most cases for both sampling schemes;  $MERSS_{min}$  and  $MERSS_{max}$ .

Table (3): Efficiency of test statistics for Weibull distribution with estimated parameters based on  $MERSS_{min}$ .

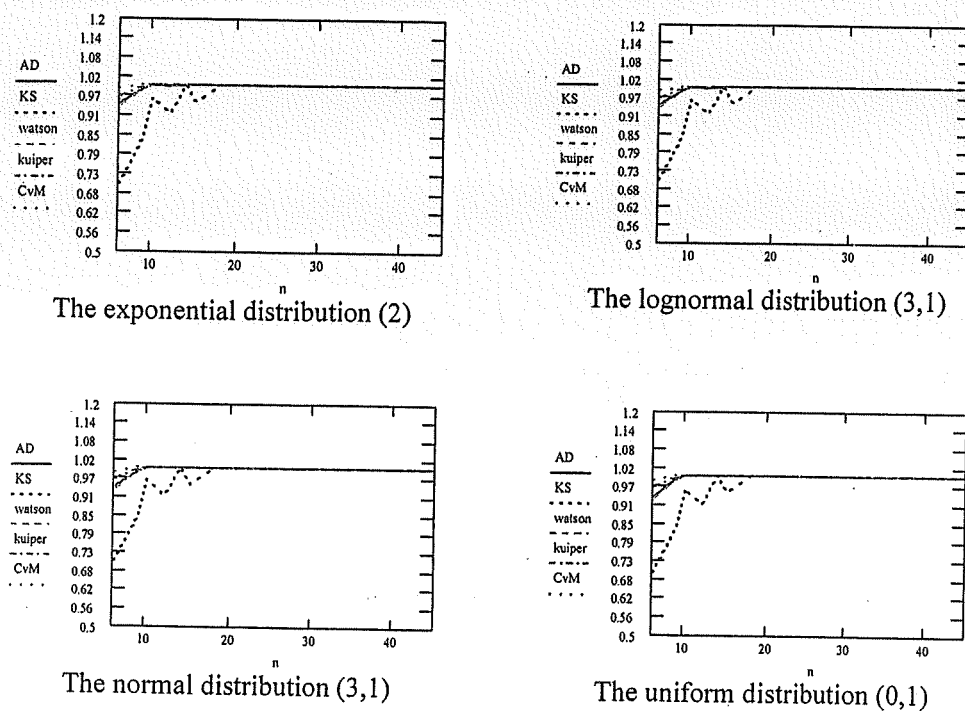
| Number of Cycles (r) | Set Size (m) | Test Statistics | Significance level $\gamma$ |       |           |       |          |       |         |       |
|----------------------|--------------|-----------------|-----------------------------|-------|-----------|-------|----------|-------|---------|-------|
|                      |              |                 | 0.01                        | 0.05  | 0.01      | 0.05  | 0.01     | 0.05  | 0.01    | 0.05  |
|                      |              |                 | Alternatives                |       |           |       |          |       |         |       |
|                      |              |                 | exp (2)                     |       | LN (3, 1) |       | N (3, 1) |       | U (0,1) |       |
| 3                    | 2            | D               | 3.125                       | 2.096 | 2.994     | 2.071 | 3.128    | 2.075 | 3.163   | 2.109 |
|                      |              | V               | 3.432                       | 2.424 | 3.295     | 2.339 | 3.462    | 2.391 | 3.485   | 2.442 |
|                      |              | $W^2$           | 2.499                       | 1.904 | 2.444     | 1.848 | 2.479    | 1.840 | 2.514   | 1.908 |
|                      |              | $A^2$           | 2.427                       | 1.832 | 2.442     | 1.807 | 2.381    | 1.784 | 2.432   | 1.830 |
|                      |              | $U^2$           | 5.692                       | 3.322 | 5.505     | 3.184 | 5.664    | 3.333 | 5.701   | 3.339 |
|                      | 3            | D               | 3.525                       | 2.316 | 3.529     | 2.269 | 3.468    | 2.259 | 3.543   | 2.277 |
|                      |              | V               | 3.621                       | 2.513 | 3.612     | 2.448 | 3.601    | 2.468 | 3.644   | 2.473 |
|                      |              | $W^2$           | 2.706                       | 1.957 | 2.701     | 1.930 | 2.664    | 1.910 | 2.743   | 1.928 |
|                      |              | $A^2$           | 2.657                       | 1.866 | 2.622     | 1.848 | 2.622    | 1.826 | 2.643   | 1.838 |
|                      |              | $U^2$           | 8.010                       | 4.348 | 7.396     | 4.182 | 7.546    | 4.178 | 7.923   | 4.300 |
|                      | 4            | D               | 3.734                       | 2.268 | 3.864     | 2.291 | 3.857    | 2.288 | 3.809   | 2.291 |
|                      |              | V               | 3.777                       | 2.431 | 3.895     | 2.452 | 3.857    | 2.500 | 3.824   | 2.437 |
|                      |              | $W^2$           | 2.775                       | 2.000 | 2.879     | 2.022 | 2.846    | 2.016 | 2.806   | 2.002 |
|                      |              | $A^2$           | 2.571                       | 1.876 | 2.625     | 1.894 | 2.584    | 1.883 | 2.632   | 1.901 |
|                      |              | $U^2$           | 10.609                      | 4.793 | 10.644    | 4.866 | 9.809    | 4.771 | 9.840   | 4.660 |
|                      | 5            | D               | 3.968                       | 2.463 | 3.876     | 2.410 | 3.906    | 2.370 | 3.968   | 2.463 |
|                      |              | V               | 3.937                       | 2.577 | 3.846     | 2.525 | 3.857    | 2.494 | 3.937   | 2.577 |
|                      |              | $W^2$           | 3.021                       | 2.041 | 2.924     | 2.033 | 2.956    | 2.045 | 3.021   | 2.041 |
|                      |              | $A^2$           | 2.747                       | 1.946 | 2.688     | 1.927 | 2.717    | 1.912 | 2.747   | 1.946 |
|                      |              | $U^2$           | 11.805                      | 5.161 | 11.435    | 4.892 | 11.926   | 5.151 | 11.866  | 5.172 |
| 7                    | 2            | D               | 3.831                       | 2.227 | 3.774     | 2.262 | 3.876    | 2.237 | 3.831   | 2.309 |
|                      |              | V               | 3.922                       | 2.320 | 3.831     | 2.370 | 3.906    | 2.353 | 3.937   | 2.433 |
|                      |              | $W^2$           | 2.695                       | 1.916 | 2.710     | 1.957 | 2.725    | 1.916 | 2.747   | 1.976 |
|                      |              | $A^2$           | 2.653                       | 1.808 | 2.674     | 1.842 | 2.667    | 1.869 | 2.695   | 1.862 |
|                      |              | $U^2$           | 9.881                       | 4.464 | 9.980     | 4.464 | 9.784    | 4.566 | 10.396  | 4.566 |
|                      | 3            | D               | 3.891                       | 2.433 | 3.906     | 2.404 | 3.774    | 2.364 | 3.891   | 2.433 |
|                      |              | V               | 3.968                       | 2.445 | 4.000     | 2.415 | 3.861    | 2.387 | 3.968   | 2.445 |
|                      |              | $W^2$           | 3.058                       | 2.037 | 2.985     | 2.020 | 2.941    | 1.996 | 3.058   | 2.037 |
|                      |              | $A^2$           | 2.959                       | 1.916 | 2.890     | 1.898 | 2.874    | 1.845 | 2.959   | 1.916 |
|                      |              | $U^2$           | 10.989                      | 5.025 | 11.494    | 4.950 | 11.236   | 5.000 | 10.989  | 5.025 |
|                      | 4            | D               | 4.237                       | 2.525 | 4.098     | 2.481 | 4.049    | 2.469 | 4.237   | 2.525 |
|                      |              | V               | 4.310                       | 2.564 | 4.167     | 2.506 | 4.098    | 2.488 | 4.310   | 2.564 |
|                      |              | $W^2$           | 2.985                       | 2.110 | 2.933     | 2.088 | 2.907    | 2.045 | 2.985   | 2.110 |
|                      |              | $A^2$           | 2.770                       | 1.972 | 2.717     | 1.938 | 2.674    | 1.908 | 2.770   | 1.972 |
|                      |              | $U^2$           | 12.987                      | 5.435 | 12.500    | 5.348 | 12.346   | 5.076 | 12.987  | 5.435 |
|                      | 5            | D               | 3.984                       | 2.519 | 4.115     | 2.532 | 4.219    | 2.513 | 3.984   | 2.519 |
|                      |              | V               | 4.049                       | 2.564 | 4.167     | 2.604 | 4.274    | 2.584 | 4.049   | 2.564 |
|                      |              | $W^2$           | 3.268                       | 2.045 | 3.279     | 2.041 | 3.322    | 2.058 | 3.268   | 2.045 |
|                      |              | $A^2$           | 2.994                       | 1.946 | 3.058     | 1.957 | 3.106    | 1.953 | 2.994   | 1.946 |
|                      |              | $U^2$           | 13.158                      | 4.525 | 14.085    | 4.975 | 14.286   | 4.926 | 13.158  | 4.525 |
| 9                    | 2            | D               | 3.745                       | 2.347 | 3.937     | 2.421 | 3.906    | 2.375 | 3.745   | 2.347 |
|                      |              | V               | 3.759                       | 2.404 | 3.984     | 2.519 | 3.891    | 2.457 | 3.759   | 2.404 |
|                      |              | $W^2$           | 2.849                       | 2.020 | 2.950     | 2.079 | 2.907    | 2.037 | 2.849   | 2.020 |
|                      |              | $A^2$           | 2.770                       | 1.916 | 2.841     | 1.946 | 2.717    | 1.912 | 2.770   | 1.916 |
|                      |              | $U^2$           | 11.494                      | 4.950 | 11.905    | 5.051 | 11.111   | 4.785 | 11.494  | 4.950 |
|                      | 3            | D               | 4.167                       | 2.532 | 4.098     | 2.457 | 4.202    | 2.500 | 4.167   | 2.532 |
|                      |              | V               | 4.219                       | 2.591 | 4.115     | 2.506 | 4.237    | 2.564 | 4.219   | 2.591 |
|                      |              | $W^2$           | 3.021                       | 2.096 | 2.985     | 2.088 | 3.077    | 2.096 | 3.021   | 2.096 |
|                      |              | $A^2$           | 2.976                       | 1.934 | 2.976     | 1.927 | 3.030    | 1.938 | 2.976   | 1.934 |
|                      |              | $U^2$           | 13.333                      | 5.051 | 12.821    | 5.464 | 12.346   | 5.128 | 13.333  | 5.051 |
|                      | 4            | D               | 4.255                       | 2.439 | 4.274     | 2.410 | 4.484    | 2.451 | 4.255   | 2.439 |
|                      |              | V               | 4.292                       | 2.451 | 4.274     | 2.421 | 4.505    | 2.457 | 4.292   | 2.451 |
|                      |              | $W^2$           | 3.003                       | 2.037 | 2.994     | 2.033 | 3.115    | 2.058 | 3.003   | 2.037 |
|                      |              | $A^2$           | 2.857                       | 1.934 | 2.882     | 1.938 | 3.003    | 1.957 | 2.857   | 1.934 |
|                      |              | $U^2$           | 11.364                      | 4.878 | 11.628    | 4.831 | 12.346   | 5.102 | 11.364  | 4.878 |
|                      | 5            | D               | 3.922                       | 2.273 | 4.049     | 2.381 | 3.953    | 2.326 | 3.922   | 2.273 |
|                      |              | V               | 3.953                       | 2.309 | 4.115     | 2.427 | 3.984    | 2.370 | 3.953   | 2.309 |
|                      |              | $W^2$           | 3.067                       | 1.965 | 3.215     | 2.070 | 3.175    | 2.024 | 3.067   | 1.965 |
|                      |              | $A^2$           | 2.825                       | 1.802 | 3.040     | 1.905 | 2.915    | 1.862 | 2.825   | 1.802 |
|                      |              | $U^2$           | 13.889                      | 4.651 | 14.085    | 4.878 | 12.821   | 4.630 | 13.889  | 4.651 |

Table (4): Efficiency of test statistics for Weibull distribution with estimated parameters based MERSS<sub>max</sub>.

| Number of Cycles (r) | Set Size (m) | Test Statistics | Significance level $\gamma$ |       |           |       |          |       |         |       |
|----------------------|--------------|-----------------|-----------------------------|-------|-----------|-------|----------|-------|---------|-------|
|                      |              |                 | 0.01                        | 0.05  | 0.01      | 0.05  | 0.01     | 0.05  | 0.01    | 0.05  |
|                      |              |                 | Alternatives                |       |           |       |          |       |         |       |
|                      |              |                 | exp (2)                     |       | LN (3, 1) |       | N (3, 1) |       | U (0,1) |       |
| 3                    | 2            | D               | 3.034                       | 2.059 | 2.909     | 2.028 | 2.987    | 2.013 | 2.983   | 2.037 |
|                      |              | V               | 3.367                       | 2.419 | 3.216     | 2.332 | 3.316    | 2.356 | 3.314   | 2.395 |
|                      |              | W <sup>2</sup>  | 2.436                       | 1.886 | 2.379     | 1.835 | 2.399    | 1.819 | 2.420   | 1.885 |
|                      |              | A <sup>2</sup>  | 2.402                       | 1.826 | 2.424     | 1.803 | 2.342    | 1.772 | 2.398   | 1.821 |
|                      |              | U <sup>2</sup>  | 4.860                       | 3.101 | 4.703     | 2.929 | 4.745    | 3.048 | 4.879   | 3.066 |
|                      | 3            | D               | 3.421                       | 2.307 | 3.407     | 2.253 | 3.359    | 2.250 | 3.432   | 2.268 |
|                      |              | V               | 3.518                       | 2.503 | 3.491     | 2.433 | 3.495    | 2.460 | 3.533   | 2.463 |
|                      |              | W <sup>2</sup>  | 2.670                       | 1.957 | 2.658     | 1.923 | 2.632    | 1.906 | 2.704   | 1.923 |
|                      |              | A <sup>2</sup>  | 2.644                       | 1.864 | 2.612     | 1.847 | 2.609    | 1.826 | 2.630   | 1.836 |
|                      |              | U <sup>2</sup>  | 6.324                       | 4.029 | 5.730     | 3.818 | 6.028    | 3.863 | 6.154   | 3.944 |
|                      | 4            | D               | 3.715                       | 2.261 | 3.845     | 2.284 | 3.834    | 2.284 | 3.790   | 2.289 |
|                      |              | V               | 3.754                       | 2.426 | 3.871     | 2.447 | 3.830    | 2.495 | 3.801   | 2.434 |
|                      |              | W <sup>2</sup>  | 2.769                       | 1.996 | 2.873     | 2.020 | 2.840    | 2.014 | 2.801   | 2.002 |
|                      |              | A <sup>2</sup>  | 2.571                       | 1.876 | 2.625     | 1.894 | 2.584    | 1.883 | 2.632   | 1.901 |
|                      |              | U <sup>2</sup>  | 9.230                       | 4.502 | 9.149     | 4.537 | 8.436    | 4.439 | 8.457   | 4.344 |
|                      | 5            | D               | 3.960                       | 2.461 | 3.864     | 2.407 | 3.898    | 2.367 | 3.960   | 2.461 |
|                      |              | V               | 3.929                       | 2.575 | 3.835     | 2.523 | 3.853    | 2.491 | 3.929   | 2.575 |
|                      |              | W <sup>2</sup>  | 3.018                       | 2.041 | 2.924     | 2.033 | 2.956    | 2.045 | 3.018   | 2.041 |
|                      |              | A <sup>2</sup>  | 2.747                       | 1.946 | 2.688     | 1.927 | 2.717    | 1.912 | 2.747   | 1.946 |
|                      |              | U <sup>2</sup>  | 10.402                      | 4.953 | 9.965     | 4.655 | 10.506   | 4.927 | 10.256  | 4.938 |
| 7                    | 2            | D               | 3.831                       | 2.227 | 3.774     | 2.262 | 3.876    | 2.237 | 3.831   | 2.309 |
|                      |              | V               | 3.922                       | 2.320 | 3.831     | 2.370 | 3.906    | 2.353 | 3.937   | 2.433 |
|                      |              | W <sup>2</sup>  | 2.695                       | 1.916 | 2.710     | 1.957 | 2.725    | 1.916 | 2.747   | 1.976 |
|                      |              | A <sup>2</sup>  | 2.653                       | 1.808 | 2.674     | 1.842 | 2.667    | 1.869 | 2.695   | 1.862 |
|                      |              | U <sup>2</sup>  | 9.782                       | 4.460 | 9.910     | 4.460 | 9.696    | 4.557 | 10.323  | 4.557 |
|                      | 3            | D               | 3.891                       | 2.433 | 3.906     | 2.404 | 3.774    | 2.364 | 3.891   | 2.433 |
|                      |              | V               | 3.968                       | 2.445 | 4.000     | 2.415 | 3.861    | 2.387 | 3.968   | 2.445 |
|                      |              | W <sup>2</sup>  | 3.058                       | 2.037 | 2.985     | 2.020 | 2.941    | 1.996 | 3.058   | 2.037 |
|                      |              | A <sup>2</sup>  | 2.959                       | 1.916 | 2.890     | 1.898 | 2.874    | 1.845 | 2.959   | 1.916 |
|                      |              | U <sup>2</sup>  | 10.989                      | 5.025 | 11.483    | 4.950 | 11.236   | 5.000 | 10.989  | 5.025 |
|                      | 4            | D               | 4.237                       | 2.525 | 4.098     | 2.481 | 4.049    | 2.469 | 4.237   | 2.525 |
|                      |              | V               | 4.310                       | 2.564 | 4.167     | 2.506 | 4.098    | 2.488 | 4.310   | 2.564 |
|                      |              | W <sup>2</sup>  | 2.985                       | 2.110 | 2.933     | 2.088 | 2.907    | 2.045 | 2.985   | 2.110 |
|                      |              | A <sup>2</sup>  | 2.770                       | 1.972 | 2.717     | 1.938 | 2.674    | 1.908 | 2.770   | 1.972 |
|                      |              | U <sup>2</sup>  | 12.987                      | 5.435 | 12.500    | 5.348 | 12.346   | 5.076 | 12.987  | 5.435 |
|                      | 5            | D               | 3.984                       | 2.519 | 4.115     | 2.532 | 4.219    | 2.513 | 3.984   | 2.519 |
|                      |              | V               | 4.049                       | 2.564 | 4.167     | 2.604 | 4.274    | 2.584 | 4.049   | 2.564 |
|                      |              | W <sup>2</sup>  | 3.268                       | 2.045 | 3.279     | 2.041 | 3.322    | 2.058 | 3.268   | 2.045 |
|                      |              | A <sup>2</sup>  | 2.994                       | 1.946 | 3.058     | 1.957 | 3.106    | 1.953 | 2.994   | 1.946 |
|                      |              | U <sup>2</sup>  | 13.158                      | 4.525 | 14.085    | 4.975 | 14.286   | 4.926 | 13.158  | 4.525 |
| 9                    | 2            | D               | 3.745                       | 2.347 | 3.937     | 2.421 | 3.906    | 2.375 | 3.745   | 2.347 |
|                      |              | V               | 3.759                       | 2.404 | 3.984     | 2.519 | 3.891    | 2.457 | 3.759   | 2.404 |
|                      |              | W <sup>2</sup>  | 2.849                       | 2.020 | 2.950     | 2.079 | 2.907    | 2.037 | 2.849   | 2.020 |
|                      |              | A <sup>2</sup>  | 2.770                       | 1.916 | 2.841     | 1.946 | 2.717    | 1.912 | 2.770   | 1.916 |
|                      |              | U <sup>2</sup>  | 11.494                      | 4.950 | 11.905    | 5.051 | 11.111   | 4.785 | 11.494  | 4.950 |
|                      | 3            | D               | 4.167                       | 2.532 | 4.098     | 2.457 | 4.202    | 2.500 | 4.167   | 2.532 |
|                      |              | V               | 4.219                       | 2.591 | 4.115     | 2.506 | 4.237    | 2.564 | 4.219   | 2.591 |
|                      |              | W <sup>2</sup>  | 3.021                       | 2.096 | 2.985     | 2.088 | 3.077    | 2.096 | 3.021   | 2.096 |
|                      |              | A <sup>2</sup>  | 2.976                       | 1.934 | 2.976     | 1.927 | 3.030    | 1.938 | 2.976   | 1.934 |
|                      |              | U <sup>2</sup>  | 13.333                      | 5.051 | 12.821    | 5.464 | 12.346   | 5.128 | 13.333  | 5.051 |
|                      | 4            | D               | 4.255                       | 2.439 | 4.274     | 2.410 | 4.484    | 2.451 | 4.255   | 2.439 |
|                      |              | V               | 4.292                       | 2.451 | 4.274     | 2.421 | 4.505    | 2.457 | 4.292   | 2.451 |
|                      |              | W <sup>2</sup>  | 3.003                       | 2.037 | 2.994     | 2.033 | 3.115    | 2.058 | 3.003   | 2.037 |
|                      |              | A <sup>2</sup>  | 2.857                       | 1.934 | 2.882     | 1.938 | 3.003    | 1.957 | 2.857   | 1.934 |
|                      |              | U <sup>2</sup>  | 11.364                      | 4.878 | 11.628    | 4.831 | 12.346   | 5.102 | 11.364  | 4.878 |
|                      | 5            | D               | 3.922                       | 2.273 | 4.049     | 2.381 | 3.953    | 2.326 | 3.922   | 2.273 |
|                      |              | V               | 3.953                       | 2.309 | 4.115     | 2.427 | 3.984    | 2.370 | 3.953   | 2.309 |
|                      |              | W <sup>2</sup>  | 3.067                       | 1.965 | 3.215     | 2.070 | 3.175    | 2.024 | 3.067   | 1.965 |
|                      |              | A <sup>2</sup>  | 2.825                       | 1.802 | 3.040     | 1.905 | 2.915    | 1.862 | 2.825   | 1.802 |
|                      |              | U <sup>2</sup>  | 13.889                      | 4.651 | 14.085    | 4.878 | 12.821   | 4.630 | 13.889  | 4.651 |



**Figure (1):** Power comparison for Weibull distribution for different alternative distributions under  $MERSS_{min}$



**Figure (2):** Power comparison for Weibull distribution for different alternative distributions under  $MERSS_{max}$

### References

1. Al-Odat, M.T. and Al-Saieh, M. F. (2001). A variation of ranked set sampling. *Journal of Applied Statistical Science*, 10, 137-146.
2. Al-Subh, S.A., Alodat, M.T. Ibrahim, K. and Jemain, A. (2009). EDF goodness of fit tests of logistic distribution under selective order statistics, *Pakistan Journal of Statistics*, 25(3), 265-274.
3. Hassan, A. S. (2012). Modified goodness of fit tests for exponentiated Pareto distribution under selective ranked set sampling. *Australian Journal of Basic and Applied Sciences*, 6(1), 173-189.
4. Ibrahim, K., Alodat, A.A., Jemain. A.A. and Al-Subh, A. (2009). Modified EDF goodness of fit tests for logistic distribution under SRS and ERSS. *Applied Mathematical Sciences*, 44(3), 2175-2194.
5. McIntyre, G.A. (1952). A method of unbiased selective sampling using ranked sets. *Australian Journal of Agricultural Research*, 3,385-390.
6. Samawi, H.M., Ahmed M.S. and Abu-Dayyeh, W. (1996). Estimating the population mean using extreme ranked set sampling. *Biometrical Journal*, 38(5), 577-586.
7. Samawi, H. M., and Al-Saleh, M. F. (2013). Valid estimation of odds ratio using two types of moving extreme ranked set sampling. *Journal of the Korean Statistical Society*, 42(1), 17-24.
8. Shahabuddin, F.A., Ibrahim K. and Jemain, A.Z. (2009). On the comparison of several goodness of fit tests under simple random sampling and ranked set sampling. *World Academy of Science, Engineering and Technology*, 54, 77-80.
9. Stokes, S. L., and Sager, T. W. (1988). Characterization of a ranked-set sample with application to estimating distribution. *Journal of the American Statistical Association*, 83(402), 374-381.

## Estimation of the Parameters of Bur Type III Distribution in Presence of Outliers

A.M.Abd-Elfattah <sup>[1]</sup>, E.A. Elsherpieny <sup>[2]</sup> and S.E.Hemeda <sup>[3]</sup>

**Abstract:** In this paper, the unknown parameters for Burr type III distribution in the presence of k outliers are estimated. The problem discussed using the moment estimation method to estimate the shape and scale parameters. Some properties are obtained. A numerical study is used to study the properties of estimators.

**Keywords:** Outliers, Burr family distributions, Burr type III distribution, pdf, c.d.f, moment estimation, rth moment, mean variance, median, mode, Hazard function, Reliability function, numerical study and conclusion.

### 1. Introduction

In the last few years there has been an increasing interest in the problem of detecting possible outliers in some distributions of a sequence of random variables. When outliers occur in the data, the assumption of normally distributed errors is violated and reduces the Performance of the linear regression models. Outliers can occur for a variety of reasons including data entry errors, scoring error or sample which is a typical. Outliers can be described in terms of their location and impact. See Grubbs (1969). The proper treatment of outlying observation had long been a subject of study. Traditionally, the main emphasis has been to detect outliers by some test of significance. Outlier detection is a primary step in many data-mining applications. In presence of outliers, special attention should be taken to assure the robustness of the used estimators.

Outlier detection for data mining is often based on distance measures, clustering and spatial methods. Outlier mining is an important issue in knowledge discovery and data mining and has attracted increasing interests in recent years. Class outlier is promising research direction. Outliers can have deleterious effects on statistical analyses. First, they generally serve to increase error variance and reduce the power of statistical tests. Second, if non-randomly distributed they can decrease normality (and in multivariate analyses, violate assumptions of sphericity and multivariate normality), altering the odds of making both Type I and Type II errors. Third, they can seriously bias or influence estimates that may be of substantive interest, see Rasmussen and Vicente (1988).

Burr (1942) obtained twelve distribution functions (Burr family of distributions) includes twelve types of cumulative distribution functions, which yield a variety of density shapes. Dixit, Moore and Barnett (1996), studied a set of random variables distributed according to exponentiated gamma distribution in the presence of k outliers.

---

[1] Professor in Mathematical Statistics department, I.S.S.R., Cairo University.

[2] Professor in Mathematical Statistics department, I.S.S.R., Cairo University.

[3] Researcher and student in Master degree.



Hossain and Nath (1997) discussed the estimation of parameters in the presence of outliers for a Burr XII distribution.

Recently Dixit and Nasiri (2001) considered estimation of parameters of the exponential distribution in the presence of outliers generated from uniform distribution. Nooghabi and Nasiri (2009) estimated the parameters of gamma distribution in the presence of outliers generated from uniform distribution. Nooghabi (2010) studied the efficient estimation of pdf, CDF and rth moment for the exponentiated Pareto distribution in the presence of outliers. Deiri (2011) studied the estimation of  $P(Y < X)$  for exponential distribution in the presence of two outliers. Ghanizadeh (2011) studied the estimation of  $P(Y < X)$  in the Rayleigh distribution in the presence of  $k$  outliers. Makhdoom and Nasiri (2011) discussed the estimation parameters of weighted exponential distribution in the presence of outliers generated from exponential distribution. Abd-Elfattah and Alharbey (2012) discussed the Bayesian estimation for Burr type III based on trimmed samples.

The objective of this paper is to study the parameters estimation of the unknown parameters of the Burr type III in the presence of  $k$  outliers. The problem discussed using the moment method to estimate the shape and scale parameters of the Burr type III distribution in the presence of  $k$  outliers.

## 2. Burr type III distribution

The Burr family distributions have an important position in the field of life testing because of its uses to fit business failure data; also it includes the well-known exponential and Weibull distributions as special cases. Burr (1942) has suggested this family of distributions by solving the following differential equation:

$$\frac{dy}{dx} = y(1-y)g(x, y) \quad , \quad y = F(x) \quad (1)$$

Where  $g(x, y) > 0$ ,  $0 \leq y \leq 1$ ,  $x$  in the range over which the solution is being satisfied.

By using different forms of  $g(x, y)$  and Pearson systems, Burr obtained twelve distribution functions which listed in Burr family distributions which yield a variety of density shapes. Many standard theoretical distributions, including the Weibull, exponential logistic, generalized logistic, Gompertz, normal, extreme value, and uniform distributions are special cases or limiting cases of Burr family of distributions. The simple closed form of these distributions has been applied to studies in simulation. Burr (1942) developed the family of Burr distributions as an outgrowth into methods for fitting cumulative frequency functions rather than probability density functions to frequency data.

Some of the forms of Burr distributions are related by simple transformation like the Burr type III distribution can be obtained from Burr type II distribution by replacing  $X$  with  $\ln(X)$ , see Johnson et al. (1995) Similarly, Burr type XII distribution can be derived from Burr type III distribution by replacing  $X$  with  $1/X$ , see Burr and Cislak (1968). The Burr type I family is more commonly known as the uniform distribution. The



Burr III, IV, V, IX, and XII families have a variety of density shapes. Burr Types III and XII are the simplest functionally and therefore, the two distributions are the most desirable for statistical modeling. The Burr type III distribution is much richer than Burr type XII distribution. The probability density function (p.d.f) of Burr type III distribution is given by

$$f(x; \alpha, \lambda) = \alpha \lambda x^{-(\lambda+1)} (1+x^{-\lambda})^{-(\alpha+1)}, \quad x > 0, \lambda > 0, \alpha > 0 \quad (2)$$

And its cumulative distribution function (c.d.f) is given by

$$F(x; \lambda, \alpha) = (1+x^{-\lambda})^{-\alpha}, \quad x > 0 \quad (3)$$

Where  $\lambda$  and  $\alpha$  are the scale and shape parameters, according to Dixit and Nasiri (2001), and Nasiri and Pazira (2010).

Abd-Elfattah and Alharbey (2012) obtained the  $r^{\text{th}}$  moment of Burr( $\lambda, \theta$ ) type III distribution as

$$\begin{aligned} E(X^r) &= \mu'_r = \int_0^{\infty} x^r f(x; \lambda, \alpha) dx \\ &= \int_0^{\infty} \frac{x^r x^{-(\lambda+1)}}{(1+x^{-\lambda})^{\alpha+1}} dx = \alpha \beta(1-\frac{r}{\lambda}, \alpha + \frac{r}{\lambda}), \quad r < \lambda \end{aligned} \quad (4)$$

Where  $\beta(a, b)$  is the standard beta function and

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a, b > 0$$

The expectation of the distribution will be

$$E(x) = \mu'_1 = \frac{\Gamma(1-\frac{1}{\lambda})\Gamma(\alpha+\frac{1}{\lambda})}{\Gamma(\alpha)}, \quad \lambda > 1 \quad (5)$$

The variance

$$Var(x) = \frac{\Gamma(\alpha)\lambda_2 - \lambda_1^2}{\Gamma^2(\alpha)}, \quad \lambda > 2 \quad (6)$$

Where,

$$\lambda_i = \Gamma(1-i/\lambda) \Gamma(\alpha+i/\lambda), \quad i = 1, 2, \dots, r$$

The mode

$$x_m = \left( \frac{\lambda(\alpha+1)}{\lambda+1} - 1 \right)^{\frac{1}{\lambda}}, \quad \lambda \alpha \geq 1, \lambda > 0, \alpha > 0 \quad (7)$$

and the median

$$x_{med} = \frac{1}{\left[ 2^{1/\alpha} - 1 \right]^{1/\lambda}}, \quad \lambda > 0, \alpha > 0 \quad (8)$$

### 3. Estimation of parameters for Burr type III distribution in the presence of k outliers

Nasiri (2001), and Nasiri and Pazira (2010) assumed that  $X_1, X_2, \dots, X_n$  is a set of random variables such that k outliers of them are distributed with (p.d.f),  $g(x; \theta, \beta)$  where

$$g(x; \theta, \beta) = \theta \beta x^{-(\beta+1)} (1+x^{-\beta})^{-(\theta+1)}, \quad x > 0, \theta > 0, \beta > 0 \quad (9)$$

Where  $\theta$  and  $\beta$  are the scale and the shape parameters of Burr type III distribution and the remaining (n-k) random variables are distributed with (p.d.f) where,  $f(x; \alpha, \lambda, \theta, \beta)$  given in(2).

The joint distribution of  $(X_1, X_2, \dots, X_n)$  with presence of k outliers can be expressed as

$$f(x_1, x_2, \dots, x_n; \lambda, \alpha, \theta, \beta) = \frac{(n-k)!k!}{n!} \prod_{i=1}^n f(x_i, \lambda, \alpha) \cdot \sum_d \prod_{i=1}^k \frac{g(x_i, \theta, \beta)}{f(x_i; \lambda, \alpha)} \quad (10)$$

$$\begin{aligned} f(x_1, x_2, \dots, x_n; \lambda, \alpha, \theta, \beta) &= \frac{(n-k)!k!}{n!} \prod_{i=1}^n \lambda \alpha x_i^{-(\lambda+1)} (1+x_i^{-\lambda})^{-(\alpha+1)} \\ &\quad \cdot \sum_d \prod_{i=1}^k \frac{\beta \theta x_{A_i}^{-(\beta+1)} (1+x_{A_i}^{-\beta})^{-(\theta+1)}}{\lambda \alpha x_{A_i}^{-(\lambda+1)} (1+x_{A_i}^{-\lambda})^{-(\alpha+1)}} \\ &= \frac{(n-k)!k!}{n!} \lambda^{n-k} \alpha^{n-k} \beta^k \theta^k \cdot \prod_{i=1}^n [x_i^{-(\lambda+1)} (1+x_i^{-\lambda})^{-(\alpha+1)}] \\ &\quad \cdot \sum_d \left[ \prod_{i=1}^k x_{A_i}^{\lambda-\beta} (1+x_{A_i}^{-\lambda})^{\theta-\alpha} \cdot \frac{(1+x_{A_i}^{-\beta})^{-(\theta+1)}}{(1+x_{A_i}^{-\lambda})^{-(\alpha+1)}} \right] \\ f(x_1, x_2, \dots, x_n; \lambda, \alpha, \theta, \beta) &= \frac{(n-k)!k!}{n!} \lambda^{n-k} \alpha^{n-k} \beta^k \theta^k \cdot \prod_{i=1}^n [x_i^{-(\lambda+1)} (1+x_i^{-\lambda})^{-(\alpha+1)}] \\ &\quad \cdot \sum_d \left[ \prod_{i=1}^k x_{A_i}^{\lambda-\beta} (1+x_{A_i}^{-\lambda})^{\theta-\alpha} \cdot \frac{(1+x_{A_i}^{-\beta})^{-(\theta+1)}}{(1+x_{A_i}^{-\lambda})^{-(\alpha+1)}} \right] \quad (11) \end{aligned}$$

Where,

$$\sum_d = \sum_{A_1}^{n-k+1} \sum_{A_2=A_1+1}^{n-k+2} \dots \sum_{A_k=A_{k-1}+1}^n$$

See Dixit (1996), Dixit and Nasiri (2001), and Nasiri and Pazira (2010).

From (11) the marginal distribution of X is

$$f(x; \lambda, \alpha, \theta, \beta) = \frac{n-k}{n} \lambda \alpha x^{-(\lambda+1)} (1+x^{-\lambda})^{-(\alpha+1)} + \frac{k}{n} \theta \beta x^{-(\beta+1)} (1+x^{-\beta})^{-(\theta+1)}, \quad x > 0 \quad (12)$$

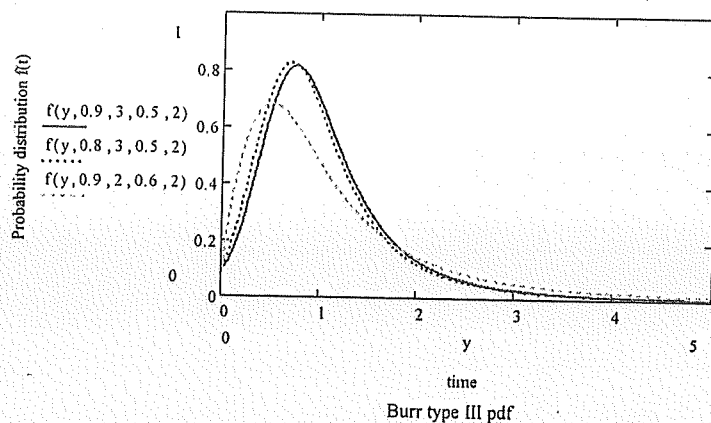
The hazard function is

$$H(x; \alpha, \lambda, \theta, \beta) = \frac{f(x; \alpha, \lambda, \theta, \beta)}{1-F(x; \alpha, \lambda, \theta, \beta)} \quad (13)$$

The Reliability function is

$$R(x; \alpha, \lambda, \theta, \beta) = 1 - F(x; \alpha, \lambda, \theta, \beta) \quad (14)$$

**Fig 1: The probability density function distribution**



**Fig 2: The Cumulative density function distribution**

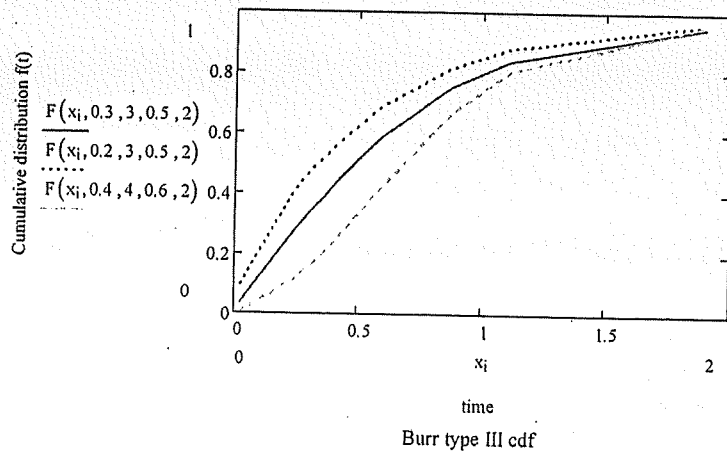


Fig 3: The Hazard function

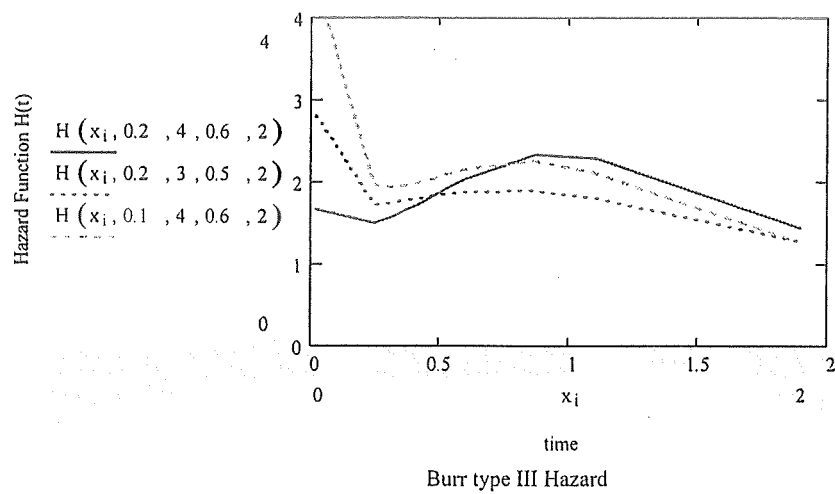
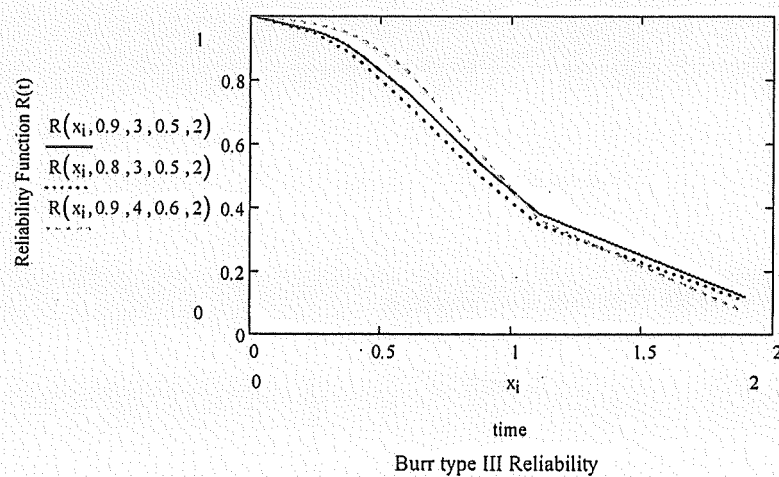


Fig4: the Reliability function



### 3.1 Method of Moment

The  $r^{\text{th}}$  moment of Burr  $(\lambda, \theta)$  type III distribution in the presence of  $k$  outliers can be obtained from the equations (4),(5)and (6) as

$$E(x^r) = \mu'_r = \alpha\beta(1 - \frac{r}{\lambda}, \alpha + \frac{r}{\lambda}) = \frac{\Gamma(1 - \frac{r}{\lambda}).\Gamma(\alpha + \frac{r}{\lambda})}{\Gamma(\alpha)}, r < \lambda \quad (15)$$

When  $r=1$  in (13), we obtain

$$\mu = E(x) = \frac{n-k}{n} \cdot \frac{\Gamma(1-\frac{1}{\lambda})\Gamma(\alpha+\frac{1}{\lambda})}{\Gamma(\alpha)} + \frac{k}{n} \cdot \frac{\Gamma(1-\frac{1}{\beta})\Gamma(\theta+\frac{1}{\beta})}{\Gamma(\theta)}, \quad \lambda > 1, \beta > 1 \quad (16)$$

The variance will be

$$Var(x) = \frac{n-k}{n} \cdot \frac{\Gamma(\alpha)\lambda_2 - \lambda_1^2}{\Gamma^2(\alpha)} + \frac{n}{k} \cdot \frac{\Gamma(\theta)\lambda_2 - \lambda_1^2}{\Gamma^2(\theta)}, \quad \lambda > 2, \theta > 2 \quad (17)$$

Where,

$$\lambda_1 = \Gamma(1-\frac{1}{\lambda})\Gamma(\alpha+\frac{1}{\lambda}), \quad \lambda > 1 \text{ and } \lambda_2 = \Gamma(1-\frac{2}{\lambda})\Gamma(\alpha+\frac{2}{\lambda}), \quad \lambda > 2$$

Therefore,

$$Var(x) = \frac{n-k}{n} \cdot \frac{\Gamma(\alpha)\Gamma(1-\frac{2}{\lambda})\Gamma(\alpha+\frac{2}{\lambda}) - [\Gamma(1-\frac{2}{\lambda})\Gamma(\alpha+\frac{2}{\lambda})]^2}{\Gamma^2(\alpha)} + \frac{n}{k} \cdot \frac{\Gamma(\theta)\Gamma(1-\frac{2}{\beta})\Gamma(\theta+\frac{2}{\beta}) - [\Gamma(1-\frac{2}{\beta})\Gamma(\theta+\frac{2}{\beta})]^2}{\Gamma^2(\theta)}, \quad \lambda > 2, \theta > 2 \quad (18)$$

The mode will be

$$\text{mod}(x) = \frac{n-k}{n} \left( \frac{\lambda(\alpha+1)}{\lambda+1} - 1 \right)^{\frac{1}{\lambda}} + \frac{n}{k} \left( \frac{\beta(\theta+1)}{\beta+1} - 1 \right)^{\frac{1}{\beta}}, \quad \beta\theta \geq 1, \theta > 0, \beta > 0 \quad (19)$$

and the median will be

$$\text{med}(x) = \frac{n-k}{n} \cdot \left( 2^{\frac{1}{\alpha}} - 1 \right)^{-\frac{1}{\lambda}} + \frac{n}{k} \cdot \left( 2^{\frac{1}{\theta}} - 1 \right)^{-\frac{1}{\beta}}, \quad \theta > 0, \beta > 0 \quad (20)$$

#### 4. Numerical study

In this paper, we have discussed the problem of estimating parameters of burr type III in the presence of k outliers. In order to have some idea about Bias and Mean Square Error (MSE) of method of moment. We generate random numbers from burr type III (with and without outliers) performing sampling experiments using MathCAD, then we obtain the estimators for  $\lambda, \alpha, \beta, \theta$ . the values of biases and MSEs of these estimates are presented in Table1,2 and k=1,2, for  $\alpha = 0.2, \lambda = 2.8, \theta = 0.3, \beta = 2.9$ . We report the average estimates and the MSEs based on 1000 replications with sample sizes (n =25, 50, 75 and 100) in Table 1 and 2.

Table 1:  $k = 1, \alpha = 0.2, \lambda = 2.8, \theta = 0.3, \beta = 2.9$ 

| n   |           | Mean    | Bias     | MSE     | SE      | Skewness | Kurtosis |
|-----|-----------|---------|----------|---------|---------|----------|----------|
| 25  | $\alpha$  | 0.08024 | 0.11976  | 0.01919 | 0.00278 | 1.888    | 5.174    |
|     | $\lambda$ | 1.3867  | 1.4133   | 2.01191 | 0.015   | 2.240876 | 6.463    |
|     | $\theta$  | 0.09411 | 0.20589  | 0.04800 | 0.003   | 2.15034  | 6.246    |
|     | $\beta$   | 1.6113  | 1.2887   | 1.79416 | 0.01461 | 2.276    | 6.569    |
| 50  | $\alpha$  | 0.1063  | 0.0937   | 0.01632 | 0.00174 | 1.078    | 2.361    |
|     | $\lambda$ | 1.42264 | 1.37736  | 1.93840 | 0.011   | 2.252478 | 6.499    |
|     | $\theta$  | 0.13686 | 0.16314  | 0.04474 | 0.00269 | 1.51592  | 3.654    |
|     | $\beta$   | 1.67114 | 1.22886  | 1.69514 | 0.0086  | 1.835    | 4.97     |
| 75  | $\alpha$  | 0.09239 | 0.10761  | 0.01371 | 0.00062 | 0.351    | 1.365    |
|     | $\lambda$ | 1.53694 | 1.26306  | 1.84672 | 0.00279 | 2.26295  | 6.529    |
|     | $\theta$  | 0.30357 | -0.00357 | 0.55431 | 0.00993 | 2.27488  | 6.564    |
|     | $\beta$   | 1.56065 | 1.33935  | 2.15072 | 0.00796 | 0.811    | 4.233    |
| 100 | $\alpha$  | 0.14797 | 0.05203  | 0.01948 | 0.0013  | 1.357    | 3.882    |
|     | $\lambda$ | 1.42195 | 1.37805  | 1.92004 | 0.00012 | 2.13737  | 6.12     |
|     | $\theta$  | 0.46549 | -0.16549 | 1.33230 | 0.01142 | 2.21657  | 6.378    |
|     | $\beta$   | 1.52318 | 1.37682  | 2.23275 | 0.00581 | 1.235    | 3.629    |

Table 2 :  $k = 2, \alpha = 0.2, \lambda = 2.8, \theta = 0.3, \beta = 2.9$ 

| n   |           | Mean    | Bias     | MSE     | SE      | Skewness  | Kurtosis |
|-----|-----------|---------|----------|---------|---------|-----------|----------|
| 25  | $\alpha$  | 0.05404 | 0.14596  | 0.02314 | 0.00171 | 1.969     | 5.719    |
|     | $\lambda$ | 1.3697  | 1.4303   | 2.04632 | 0.01521 | 0.206903  | 1.633    |
|     | $\theta$  | 0.09152 | 0.20848  | 0.04923 | 0.00304 | 2.06611   | 6.053    |
|     | $\beta$   | 1.50803 | 1.39197  | 1.96406 | 0.00651 | 2.042     | 5.912    |
| 50  | $\alpha$  | 0.05921 | 0.14079  | 0.02078 | 0.00062 | 2.202     | 6.371    |
|     | $\lambda$ | 1.36235 | 1.43765  | 2.06714 | 0.01231 | 0.398964  | 2.635    |
|     | $\theta$  | 0.08135 | 0.21865  | 0.04822 | 0.0002  | 2.27449   | 6.563    |
|     | $\beta$   | 1.4846  | 1.4154   | 2.00618 | 0.00106 | -0.758    | 3.912    |
| 75  | $\alpha$  | 0.20176 | -0.00176 | 0.03462 | 0.00248 | 1.267     | 3.566    |
|     | $\lambda$ | 1.35892 | 1.44108  | 2.07732 | 0.00312 | 0.97926   | 2.75     |
|     | $\theta$  | 0.22994 | 0.07006  | 0.05021 | 0.00284 | 0.34484   | 0.963    |
|     | $\beta$   | 1.69355 | 1.20645  | 1.52541 | 0.00352 | 0.127     | 1.087    |
| 100 | $\alpha$  | 0.14645 | 0.05355  | 0.01512 | 0.00111 | 0.986     | 2.795    |
|     | $\lambda$ | 1.33933 | 1.46067  | 2.13604 | 0.0004  | -1.993007 | 5.734    |
|     | $\theta$  | 0.19059 | 0.10941  | 0.04337 | 0.00177 | 0.7271    | 1.705    |
|     | $\beta$   | 1.70267 | 1.19733  | 1.49619 | 0.0025  | 0.33      | 0.952    |

## 5. Conclusion:

In this paper, we obtained the classical estimators for shape and scale parameters of Burr type III distribution in the presence of k outliers. The estimators for  $\lambda, \alpha, \beta, \theta$  of Burr type III distribution in the presence of k outliers and the values of biases and MSEs of these estimates are presented in Table1 and 2, when k=1,2 for

$\alpha = 0.2, \lambda = 2.8, \theta = 0.3, \beta = 2.9$ . We report the average estimates, the MSE, the Bias, the Skewness and kurtosis based on 10 replications with sample sizes ( $n = 25, 50, 75$  and  $100$ ) in table 1 and 2.

We report the average estimates and the MSE<sub>s</sub> based on 10 replications. We observed from Table 1 ( $k=1$ ) and Table 2 ( $k=2$ ). The moment estimators of  $\lambda, \alpha, \beta, \theta$  are unbiased for some cases especially when the sample size increases. In Table 1 ( $k=1$ ) the MSE for estimators less than in Table 2 ( $k=2$ ). Table 1 and 2 show that, the MSE<sub>s</sub> for estimators decrease with increasing the sample size  $n$ .

## References

- [1] Abd-Elfattah, A.M & Alharbey, A.H (2012). "Bayesian estimation for Burr type III distribution based on trimmed samples". ISRN Applied mathematics, Vol.3, 1–18.
- [2] Burr, I.W. (1942). "Cumulative frequency functions". The annals of Mathematical Statistics, Vol.13, 215-232.
- [3] Burr, I.W. & P.J. Cislak (1986). "On a general system of distribution: I. Its curve shape characteristics; II. The sample median". J .Amer. Statist. Assoc, Vol.36, 627-635.
- [4] Deiri, E. (2011). "Estimation of parameters of Gamma distribution in the presence of two outliers". Australian Journal of Basic & Applied Science, Vol.5, No.6, p 1496-1502.
- [5] Dixit, U.J. Moore.K.L. & Barnett. (1996). "On the estimation of power of the scale parameter of the Exponential distribution in the presence of outliers generated from Uniform distribution". Metron, Vol.54, 201-211.
- [6] Dixit, U.J. & Nasiri.P.E. (2001). " Estimation of the parameters of the Exponential distribution in the presence of outliers generated from Uniform distribution". Metron, 49(3-4):187-198.
- [7] Ghanizadeh, A. (2011). "Estimation of  $P(Y < X)$  in the Rayleigh distribution in the presence of  $k$  outliers", Journal of Applied Sciences, 11(17): 3192–3197.
- [8] Grubbs, F.E. (1969). "Procedures for detecting outlying observations in Samples." Technometrics, 11 (1): 1–21.
- [9] Hossain, A.M. & Nath, S.K. (1997). "Estimation of parameters in the presence of outliers for Burr XII distribution", Communications in Statistics, Vol.26(3):637–652.
- [10] Jonson et al. (1995). "Continuous univariate distribution", New York. Wiley, Vol.1, 2nd Ed.

- [11] Makhdoom, I. (2011). "Estimation of the parameters of Minimax distribution in the presence of outliers", *Int.J.Acad. Res*, (3)500-5007.
- [12] Nasiri, P., & Pazira, H. (2010). "Estimation of  $P(Y < X)$  for exponential distribution in the presence of k outliers". *Journal of Applied Science*, 21765, 21766.
- [13] Nooghabi, M. J., Nooghabi, H. J. (2010). "Efficient estimation of pdf and cdf for the Exponentiated Pareto distribution". *Journal of Mathematics*, Vol.14, (19).
- [14] Nooghabi, M. J., Nooghabi, H. J., & Nasiri, P. F. (2009). "Estimation of parameters of the Gamma distribution in the presence of outliers generated from Uniform distribution", *Pak. J. Statist*, Vol. 25 (1), 15-26.
- [15] Rasmussen, Vicente K.J. (1988). Coping with human errors through system design: implications for ecological interface design. *International Journal of Man-Machine Studies* (1989) 31, 517-534.83.



## On the Lomax-Poisson Distribution

Abd-Elfattah, A. M<sup>1</sup> ,

Amal S. Hassan<sup>2</sup>

and Asmaa M. Hussein<sup>3</sup>

### Abstract

In this paper, a new three-parameters lifetime distribution, called Lomax-Poisson, distribution with decreasing failure rate is introduced. This distribution is obtained by mixing Lomax distribution and zero truncated Poisson distribution. Closed-form expressions for the density, cumulative distribution, survival and failure rate are provided. The  $r$ th raw moment of the new distribution, and moment generating function are obtained. Estimations of the unknown parameters using different methods is obtained. Based on a Monte Carlo simulation study, comparisons are made through the different estimators by through their biases and mean squared errors.

**Keywords:** Lomax distribution; Zero truncated distribution; Lifetime distribution; Decreasing Failure rate.

### 1-Introduction

The study of the lifetime (length) of organisms, devices, structures, materials, etc., is very important in the biological and engineering sciences. A substantial part of such study is devoted to modeling the lifetime data by a failure distribution. Sometimes physical considerations of the failure mechanism may lead to a specific distribution but more often, the choice is made on the basis of how well the actual observations of times to failure appear to be fitted by the distribution.

In recent years, new classes of distributions have been proposed based on modifications of the exponential and Weibull distributions. Adamidis and Loukas (1998), introduced a distribution with decreasing failure rate. This distribution is known as exponential-geometric distribution and is obtained by compounding an exponential distribution with a geometric distribution. Adamidis et al. (2005) proposed a distribution with monotone decreasing, increasing, or constant. This distribution is known as extended exponential-geometric distribution. Kus (2007) introduced a two-parameter distribution known as exponential-Poisson, with decreasing failure rate, by compounding an exponential with Poisson distributions.

---

1, 2- Institute of Statistical Studies and Research, Cairo University

3- phd, student in Institute of Statistical Studies and Research, Cairo University

In the same fashion, Tahmasbi and Rezaei (2008) introduced an exponential-logarithmic distribution, which has decreasing failure rate. A two-parameter distribution family with decreasing failure rate arising by mixing power-series distribution has been introduced by Chahkandi and Ganjali (2009). The Weibull-Poisson distribution with decreasing failure rate which generalizes the exponential-Poisson was introduced by Hemmati et al. (2011). They studied various properties of this distribution and discussed the estimation of the parameters by considering the expectation maximization algorithm. Also, the asymptotic variance-covariance matrices of these estimates are obtained. Silva et al. (2010) introduced the so-called generalized exponential geometric distribution, having a decreasing, increasing, and upside-down bathtub failure rates depending on its parameters. Such failure rate curves can be observed, for example, in the course of a disease, whose mortality reaches a peak after some finite period, then declines gradually. Louzada et al. (2011) proposed a new lifetime distribution with increasing failure rate, called the complementary exponential geometric distribution. The exponential truncated Poisson with increasing failure rate introduced by Rezaei and Tahmasbi (2012).

The main aim of this article is to create a new decreasing failure rate distribution namely the Lomax-Poisson (LP) distribution. Various properties of the new mixed distribution are discussed. Different estimation procedures are discussed for different parameter values and sample sizes. Numerical results are illustrated.

The remainder of the article is devoted to a comprehensive study of this new distribution. In Section 2, the density and cumulative distribution functions of the LP distribution are derived. Additionally, forms for the median, reliability and decreasing failure rate functions are derived. Expressions for the moment generating function, and hence the mean and variance, are calculated in Section 3. In the same Section forms of the  $r$ th raw moments are derived. The estimation methods and procedures are covered in Section 4. Using simulation technique, the biases and mean square errors (MSEs) in the estimation methods are analyzed and the guideline of the best estimation method is laid down in Section 5. Finally, conclusion is presented in Section 6.

## 2-The Lomax-Poisson Distribution

Let  $W_1, W_2, \dots, W_Z$  be a random sample from Lomax distribution with scale parameter ( $\beta > 0$ ) and shape parameter ( $\alpha > 0$ ) has the following probability density (PDF) and cumulative distribution functions (CDF), respectively,

$$f(w; \alpha, \beta) = \frac{\alpha}{\beta} \left(1 + \frac{w}{\beta}\right)^{-(\alpha+1)}, \quad w, \alpha, \beta > 0, \quad (2.1)$$

and 
$$F(w; \alpha, \beta) = 1 - \left(1 + \frac{w}{\beta}\right)^{-\alpha}, \quad w, \alpha, \beta > 0, \quad (2.2)$$

Now suppose  $Z$  is a zero truncated Poisson variables with probability mass function

$$f(z; \lambda) = \frac{e^{-\lambda} \lambda^z}{(1-e^{-\lambda})} \Gamma^{-1}(z+1), \quad \lambda > 0, z \in \mathbb{Z} = \{1, 2, 3, \dots\} \quad (2.3)$$

where  $\Gamma(\cdot)$  is gamma function . Assuming that the random variables  $Z$  and  $W$  are independent, define  $X = \min(W_1, W_2, \dots, W_Z)$ , then,

$$f(x|z) = \frac{\alpha z}{\beta} \left(1 + \frac{x}{\beta}\right)^{-\alpha z - 1} \quad (2.4)$$

From (2.3) and (2.4) the marginal probability density function of  $X$  is given by

$$f(x; \alpha, \beta, \lambda) = \frac{\alpha \lambda \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)}}{\beta (e^\lambda - 1)} e^{\lambda \left(1 + \frac{x}{\beta}\right)^{-\alpha}}, x > 0, \alpha, \beta, \lambda > 0, \quad (2.5)$$

where  $\alpha$ ,  $\lambda$  are the shape parameters and  $\beta$  is the scale parameter of the Lomax Poisson distribution.

The Lomax Poisson has the following distribution function

$$F(x; \alpha, \beta, \lambda) = \frac{1}{(e^\lambda - 1)} \left[ e^\lambda - e^{\lambda \left(1 + \frac{x}{\beta}\right)^{-\alpha}} \right], x > 0, \alpha, \beta, \lambda > 0 \quad (2.6)$$

The probability density function and cumulative distribution function of the Lomax Poisson distribution are displayed in Figures (1) and (2) for selected values of the parameters  $\alpha, \beta$  and  $\lambda$ .

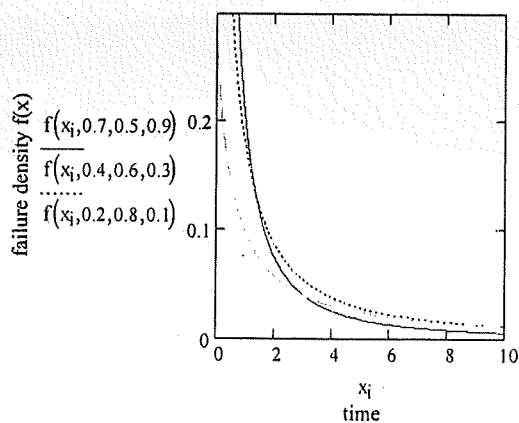


Figure (1): pdf of Lomax- Poisson distribution

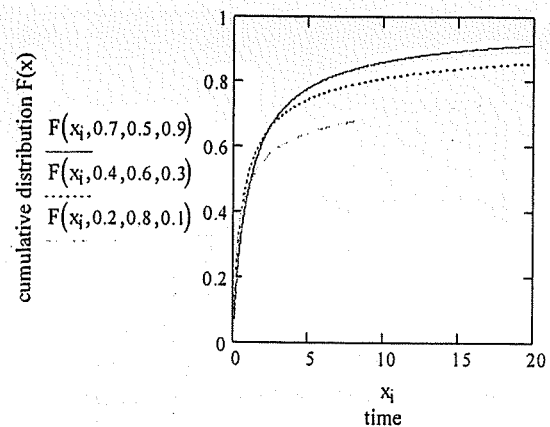


Figure (2): cdf of the Lomax- Poisson distribution

The median (second quartile) of the Lomax-Poisson distribution will be obtained using (2.6)

$$m = \beta \left[ \frac{\ln \left[ \frac{(1-e^\lambda)}{2} \right]}{\lambda} + 1 \right]^{\frac{-1}{\alpha}} - 1$$

The median of a Lomax-Poisson random variable occurs at  $\beta \left[ \frac{\ln \left[ \frac{(1-e^\lambda)}{2} \right]}{\lambda} + 1 \right]^{\frac{-1}{\alpha}} - 1$  and clearly is decreasing function.

Now let's look at the reliability and hazard rate functions of the Lomax-Poisson distribution

$$R(x; \alpha, \beta, \lambda) = \frac{e^{\lambda \left(1 + \frac{x}{\beta}\right)^{-\alpha}} - 1}{(e^\lambda - 1)}, \quad (2.7)$$

$$h(x; \alpha, \beta, \lambda) = \frac{\alpha \lambda \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} e^{\lambda \left(1 + \frac{x}{\beta}\right)^{-\alpha}}}{\beta \left[ 1 - e^{-\lambda \left(1 + \frac{x}{\beta}\right)^{-\alpha}} \right]}, \quad x > 0. \quad (2.8)$$

The reliability and hazard rate functions of the Lomax-Poisson distribution are displayed in Figures (3) and (4) for selected values of the parameters  $\alpha, \beta$  and  $\lambda$ .

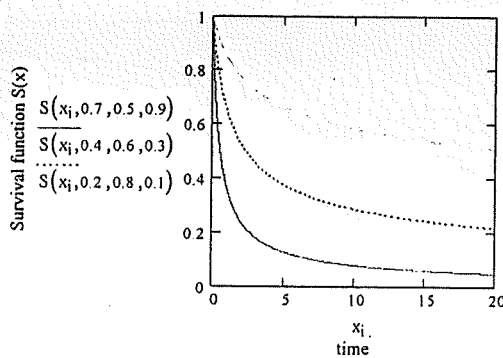


Figure (3): the reliability function of the Lomax-Poisson distribution

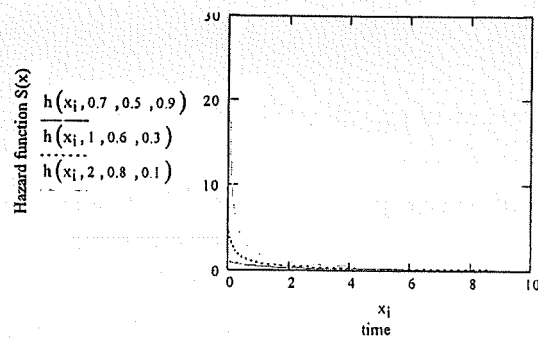


Figure (4): the hazard function of the Lomax-Poisson distribution

It is clear from Figure (4), that the hazard rate function of the Lomax Poisson distribution has decreasing failure rate.

### 3- Distributional Properties

In this Section some properties of Lomax-Poisson distribution will be obtained such as the  $r$ th moment and the moment generating function, in particular the mean, and variance will be obtained.

#### (3.1) The $r$ th Moment

The  $r$ th moment of the Lomax-Poisson distribution about the origin is given by

$$E(X^r) = \frac{\alpha \lambda}{\beta(e^\lambda - 1)} \int_0^\infty x^r \left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} e^{\lambda \left(1 + \frac{x}{\beta}\right)^{-\alpha}} dx$$

$$E(X^r) = \frac{(-\beta)^r}{(e^\lambda - 1)} \int_0^\lambda \left(1 - \left(\frac{z}{\lambda}\right)^{-\frac{1}{\alpha}}\right)^r e^z dz$$

By using binomial expression  $E(X^r)$  can be expressed as,

$$E(X^r) = \frac{(-\beta)^r}{(e^\lambda - 1)} \sum_{j=0}^r (-1)^j \binom{r}{j} \lambda^{\frac{j}{\alpha}} \sum_{k=0}^\infty \frac{\lambda^{k - \frac{j}{\alpha} + 1}}{(k - \frac{j}{\alpha} + 1)k!}, \quad r = 1, 2, 3, \dots \quad (3.1)$$

In particular, the mean of the Lomax Poisson distribution will be obtained by substitute  $r = 1$  in (3.1), then

$$E(X) = \frac{\beta}{(e^\lambda - 1)} \left[ \lambda^{\frac{1}{\alpha}} \sum_{k=0}^\infty \frac{\lambda^{k - \frac{1}{\alpha} + 1}}{(k - \frac{1}{\alpha} + 1)k!} - \sum_{k=0}^\infty \frac{\lambda^{k+1}}{(k+1)k!} \right] \quad (3.2)$$

For the more, the second moment of Lomax-Poisson distribution is given by substitute  $r = 2$  in (3.1), then

$$E(X^2) = \frac{\beta^2}{(e^\lambda - 1)} \left[ \lambda^{\frac{2}{\alpha}} \sum_{k=0}^\infty \frac{\lambda^{k - \frac{2}{\alpha} + 1}}{(k - \frac{2}{\alpha} + 1)k!} - 2\lambda^{\frac{1}{\alpha}} \sum_{k=0}^\infty \frac{\lambda^{k - \frac{1}{\alpha} + 1}}{(k - \frac{1}{\alpha} + 1)k!} + \sum_{k=0}^\infty \frac{\lambda^{k+1}}{(k+1)k!} \right] \quad (3.3)$$

Therefore, the variance of Lomax-Poisson takes the following form

$$V(X) = \frac{\beta^2}{(e^\lambda - 1)} \left[ \lambda^{\frac{2}{\alpha}} \sum_{k=0}^\infty \frac{\lambda^{k - \frac{2}{\alpha} + 1}}{(k - \frac{2}{\alpha} + 1)k!} - 2\lambda^{\frac{1}{\alpha}} \sum_{k=0}^\infty \frac{\lambda^{k - \frac{1}{\alpha} + 1}}{(k - \frac{1}{\alpha} + 1)k!} + \sum_{k=0}^\infty \frac{\lambda^{k+1}}{(k+1)k!} \right] - \left[ \frac{\beta}{(e^\lambda - 1)} \left[ \lambda^{\frac{1}{\alpha}} \sum_{k=0}^\infty \frac{\lambda^{k - \frac{1}{\alpha} + 1}}{(k - \frac{1}{\alpha} + 1)k!} - \sum_{k=0}^\infty \frac{\lambda^{k+1}}{(k+1)k!} \right] \right]^2$$

### (3.2) Moment Generating Function

The moment generating function of the Lomax-Poisson distribution is given by the following

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r$$

From (3.1), the moment generating function takes the following form

$$M_X(t) = \frac{1}{(e^{\lambda}-1)} \sum_{r=0}^{\infty} \frac{t^r (-\beta)^r}{r!} \sum_{j=0}^r (-1)^j \binom{r}{j} \lambda^{\frac{j}{\alpha}} \sum_{k=0}^{\infty} \frac{\lambda^{k-\frac{j}{\alpha}+1}}{(k-\frac{j}{\alpha}+1)k!} \quad (3.4)$$

## 4- Different Methods of Estimation

In this Section different methods of estimation of the unknown parameters for Lomax Poisson distribution will be studied. Four method of estimation will be considered namely, maximum likelihood, moments, least square and weighted least squares.

### (4.1) Maximum Likelihood Method (ML)

Maximum likelihood method of estimation is one of the most popular methods for estimating the unknown parameters of continuous distributions because of its attractive properties, such as consistency, asymptotic unbiased, asymptotic efficiency, and asymptotic normality.

Let  $x_1, x_2, \dots, x_n$  be observed random sample of size  $n$  from a Lomax-Poisson distribution with parameters  $\alpha, \beta$ , and  $\lambda$ , the likelihood function based on observed sample is given by

$$L(\alpha, \beta, \lambda; x) = \alpha^n \lambda^n \beta^{-n} - (e^{\lambda} - 1)^n \prod_{i=1}^n \left(1 + \frac{x_i}{\beta}\right)^{-(\alpha+1)} e^{\lambda \sum_{i=1}^n \left(1 + \frac{x_i}{\beta}\right)^{-\alpha}}$$

For simplicity write  $L(\alpha, \beta, \lambda; x)$  to be  $L$ , taking the natural logarithm of  $L$  then,

$$\ln L = n \ln \alpha + n \ln \lambda - n \ln \beta - n \ln(e^{\lambda} - 1) - (\alpha + 1) \sum_{i=1}^n \ln \left(1 + \frac{x_i}{\beta}\right) + \lambda \sum_{i=1}^n \left(1 + \frac{x_i}{\beta}\right)^{-\alpha} \quad (4.1)$$

The maximum likelihood estimates (MLEs) of  $\alpha, \beta, \lambda$ , say  $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ , are obtained by setting the first partial derivatives of (4.1) to zero with respect to  $\alpha, \beta$  and  $\lambda$ , respectively. These simultaneous equations are

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \ln \left(1 + \frac{x_i}{\beta}\right) - \lambda \sum_{i=1}^n \left(1 + \frac{x_i}{\beta}\right)^{-\alpha} \ln \left(1 + \frac{x_i}{\beta}\right) = 0 \quad (4.2)$$

$$\frac{\partial \ln L}{\partial \beta} = -\frac{n}{\beta} + (\hat{\alpha} + 1) \sum_{i=1}^n \frac{1}{\left(1 + \frac{x_i}{\hat{\beta}}\right)} \left(\frac{x_i}{\hat{\beta}^2}\right) + \hat{\alpha} \hat{\lambda} \sum_{i=1}^n \left(1 + \frac{x_i}{\hat{\beta}}\right)^{-(\hat{\alpha}+1)} \frac{x_i}{\hat{\beta}^2} = 0 \quad (4.3)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\hat{\lambda}} - \frac{n e^{\hat{\lambda}}}{(e^{\hat{\lambda}} - 1)} + \sum_{i=1}^n \left(1 + \frac{x_i}{\hat{\beta}}\right)^{-\hat{\alpha}} = 0 \quad (4.4)$$

It may be noted that this is an implicit equations in  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{\lambda}$ , so it can not be solved analytically. Numerical iteration method will be proposed to solve these nonlinear equations.

#### (4.2) Moments Method (MM)

Let  $X_1, X_2, \dots, X_n$  be an independent and identically distributed random variables, each has the Lomax- Poisson distribution with parameters  $\alpha, \beta$  and  $\lambda$ , then their first, second and third population moments are given respectively from (3.1) as

$$E(X) = \frac{\beta}{(e^{\lambda}-1)} \left[ \lambda^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \frac{\lambda^{k-\frac{1}{\alpha}+1}}{(k-\frac{1}{\alpha}+1)k!} - \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)k!} \right],$$

$$E(X^2) = \frac{\beta^2}{(e^{\lambda}-1)} \left[ \lambda^{\frac{2}{\alpha}} \sum_{k=0}^{\infty} \frac{\lambda^{k-\frac{2}{\alpha}+1}}{(k-\frac{2}{\alpha}+1)k!} - 2\lambda^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \frac{\lambda^{k-\frac{1}{\alpha}+1}}{(k-\frac{1}{\alpha}+1)k!} + \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)k!} \right],$$

and

$$E(X^3) = \frac{\beta^3}{(e^{\lambda}-1)} \left[ \lambda^{\frac{3}{\alpha}} \sum_{k=0}^{\infty} \frac{\lambda^{k-\frac{3}{\alpha}+1}}{(k-\frac{3}{\alpha}+1)k!} - 3\lambda^{\frac{2}{\alpha}} \sum_{k=0}^{\infty} \frac{\lambda^{k-\frac{2}{\alpha}+1}}{(k-\frac{2}{\alpha}+1)k!} + 3\lambda^{\frac{1}{\alpha}} \sum_{k=0}^{\infty} \frac{\lambda^{k-\frac{1}{\alpha}+1}}{(k-\frac{1}{\alpha}+1)k!} - \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)k!} \right].$$

Let  $x_1, x_2, \dots, x_n$  be a random sample each has LP distribution, then the  $r$ th sample moment is  $m_r = \frac{1}{n} \sum_{i=1}^n x_i^r$ . In particular, the first, the second and the third sample moments are given respectively

$$m_1 = \frac{1}{n} \sum_{i=1}^n x_i, \quad m_2 = \frac{1}{n} \sum_{i=1}^n x_i^2, \quad \text{and} \quad m_3 = \frac{1}{n} \sum_{i=1}^n x_i^3.$$

The moment estimators (MEs) of  $\alpha, \beta$  and  $\lambda$  denoted by  $\hat{\alpha}_1, \hat{\beta}_1$  and  $\hat{\lambda}_1$  are obtained by equating the sample moments with the corresponding population moment as

$$\frac{\hat{\beta}_1}{(e^{\hat{\lambda}_1}-1)} \left[ \hat{\lambda}_1^{\frac{1}{\hat{\alpha}_1}} \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k-\frac{1}{\hat{\alpha}_1}+1}}{(k-\frac{1}{\hat{\alpha}_1}+1)k!} - \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k+1}}{(k+1)k!} \right] = \frac{1}{n} \sum_{i=1}^n x_i \quad (4.5)$$

$$\frac{\hat{\beta}_1^2}{(e^{\hat{\lambda}_1}-1)} \left[ \hat{\lambda}_1^{\frac{2}{\hat{\alpha}_1}} \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k-\frac{2}{\hat{\alpha}_1}+1}}{(k-\frac{2}{\hat{\alpha}_1}+1)k!} - 2\hat{\lambda}_1^{\frac{1}{\hat{\alpha}_1}} \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k-\frac{1}{\hat{\alpha}_1}+1}}{(k-\frac{1}{\hat{\alpha}_1}+1)k!} + \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k+1}}{(k+1)k!} \right] = \frac{1}{n} \sum_{i=1}^n x_i^2 \quad (4.6)$$

$$\frac{\hat{\beta}_1^3}{(e^{\hat{\lambda}_1}-1)} \left[ \hat{\lambda}_1^{\frac{3}{\hat{\alpha}_1}} \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k-\frac{3}{\hat{\alpha}_1}+1}}{(k-\frac{3}{\hat{\alpha}_1}+1)k!} - 3\hat{\lambda}_1^{\frac{2}{\hat{\alpha}_1}} \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k-\frac{2}{\hat{\alpha}_1}+1}}{(k-\frac{2}{\hat{\alpha}_1}+1)k!} + 3\hat{\lambda}_1^{\frac{1}{\hat{\alpha}_1}} \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k-\frac{1}{\hat{\alpha}_1}+1}}{(k-\frac{1}{\hat{\alpha}_1}+1)k!} - \sum_{k=0}^{\infty} \frac{\hat{\lambda}_1^{k+1}}{(k+1)k!} \right] = \frac{1}{n} \sum_{i=1}^n x_i^3 \quad (4.7)$$

The moment estimators of the parameters  $\alpha, \beta$  and  $\lambda$  of the Lomax Poisson distribution can be obtained by solving equations (4.5), (4.6) and (4.7) numerically.

#### (4.3)Least Squares and Weighted least Squares Methods

For the estimation of probability distribution parameters, the least square (LS) method is extensively used in reliability engineering and mathematics problems. The method can be described as follows: Suppose  $x_1, x_2, \dots, x_n$  is a random sample of size  $n$  from distribution function (2.6) and  $x_{(1)} < x_{(2)} < \dots < x_{(n)}$  denotes the corresponding ordered sample. According to Johnson et al. (1995)

$$E\left(F(X_{(i)})\right) = \frac{i}{n+1}, i = 1, 2, \dots, n \quad \text{and} \quad V\left(F(Y_{(i)})\right) = \frac{i(n-i+1)}{(n+1)^2(n+2)}$$

The least squares estimators (LSEs) of the unknown parameters can be obtained by minimizing the following quantity with respect to the unknown parameter

$$\sum_{i=1}^n \left[ F(X_{(i)}) - \frac{i}{n+1} \right]^2$$

Therefore, the least square estimates of the unknown parameters  $\alpha, \beta$ , and  $\lambda$  denoted by;  $\hat{\alpha}_2, \hat{\beta}_2$  and  $\hat{\lambda}_2$  respectively, can be obtained by minimizing

$$\sum_{i=1}^n \left[ \frac{1}{(e^\lambda - 1)} \left[ e^\lambda - e^{\lambda \left(1 + \frac{x_{(i)}}{\beta}\right)^{-\alpha}} \right] - \frac{i}{n+1} \right]^2, \quad (4.8)$$

with respect to  $\alpha, \beta$  and  $\lambda$

The weighted least squares estimators (WLSEs) of the unknown parameters can be obtained by minimizing

$$\sum_{i=1}^n w_i \left[ F(X_{(i)}) - \frac{i}{n+1} \right]^2$$

with respect to the unknown parameters. Therefore, in the case of LP distribution the weighted least squares estimators can be obtained by minimizing

$$\sum_{i=1}^n w_i \left[ F(X_{(i)}) - E(F(X_{(i)})) \right]^2,$$

with respect to the unknown parameters, where  $w_i = \frac{1}{\text{var}(F(X_{(i)}))} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ . Therefore, the weighted least squares estimators  $\hat{\alpha}_3, \hat{\beta}_3$  and  $\hat{\lambda}_3$  of  $\alpha, \beta$  and  $\lambda$  of Lomax-Poisson distribution can be obtained by minimizing,



$$\sum_{i=1}^n w_i \left[ \frac{1}{(e^\lambda - 1)} \left[ e^\lambda - e^{\lambda \left( 1 + \frac{x_{(i)}}{\beta} \right)^{-\alpha}} \right] - \frac{i}{n+1} \right]^2 \quad (4.9)$$

with respect to  $\alpha, \beta$  and  $\lambda$ .

## 5- Simulation Study

It is very difficult to compare the theoretical performance of the different estimators proposed in the previous Sections. Therefore, an extensive simulation study will be carried out to compare the performances of the different estimators, mainly, with respect to their biases and MSE's, for different sample sizes and for different parametric values. The simulation procedures are described through the following algorithm

**Step (1):** A random sample  $x_1, x_2, \dots, x_n$  of sizes  $n = 10$  (10) 50 are selected, these random samples are generated from the Lomax- Poisson distribution by using the following transformation  $x_{i,j} = \beta \left[ \frac{1}{\lambda} \ln \left[ e^\lambda - u_{i,j} (e^\lambda - 1) \right]^{-\frac{1}{\alpha}} - 1 \right]$ ,  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, n$  and  $u_{i,j}$  are random sample from uniform (0,1).

**Step (2):** Different values of the parameters will be selected as,  $\alpha = 0.5(0.5)2$ ,  $\beta = 1$ , and  $\lambda = 0.5(0.5)2$ .

**Step (3):** The maximum likelihood estimators  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{\lambda}$  will be obtained by solving the nonlinear equations (4.2), (4.3) and (4.4). The moment estimators  $\hat{\alpha}_1$ ,  $\hat{\beta}_1$  and  $\hat{\lambda}_1$  can be obtained by solving the non-linear equations (4.5), (4.6) and (4.7). Also, The LSE's and WLSE's of  $\alpha, \beta$ , and  $\lambda$  can be obtained by minimizing equations (4.8) and (4.9) with respect to  $\alpha, \beta$ , and  $\lambda$  respectively.

**Step (4):** Steps from 1 to 4 will be repeated 1000 times for each sample size and for the selected sets of parameters. Then, the biases and MSEs of different estimators of unknown parameters are computed.

Simulation results are reported in Tables 1 to 4. From these Tables, the following conclusions can be observed on the properties of estimated parameters from the Lomax Poisson distribution

- 1- The MSEs for the MLEs of both parameters  $\alpha$  and  $\lambda$  are smaller than the MSEs for the other methods of  $\alpha$  and  $\lambda$  in almost all the cases.
- 2 - The MSEs for the MEs of  $\beta$  is smaller than the MSEs for the other methods in almost all the cases.
- 3- For all methods of estimation, it is clear that biases and MSEs decrease as sample sizes increase.
- 4- For fixed value of  $\lambda$  and as the value of shape parameter  $\alpha$  increase, the biases and MSEs for estimators based on maximum likelihood method are increase.

- 5- For fixed value of  $\alpha$  and as the value of shape parameter  $\lambda$  increase, the MSEs for moment estimators are approximately constant.
- 6- The biases and MSEs for the LSEs and WLSEs are approximately constant, when  $\lambda$  is increasing, for all the values of  $\alpha$  in almost all the cases.
- 7- The MSEs for MLEs, for all parameters values, are the smallest among the other estimators.

## 6-Conclusions

In this paper, a new distribution called the Lomax Poisson distribution with three parameters will be introduced. The plots of PDF, CDF, reliability and hazard functions are presented to show the flexibility of the new distribution. Some mathematical properties such as the  $r$ th moment and moment generating function are derived. Estimation problem of unknown parameters for Lomax Poisson distribution based on different methods will be provided. Maximum likelihood, moments, least squares and weighted least squares methods of estimation are considered. The biases and MSEs will be calculated, to compare different estimators, via simulation.

In general it is observed that the biases and MSEs of all parameters decrease as sample size increases. Furthermore the MLEs  $\hat{\alpha}$  and  $\hat{\beta}$  are better than the other estimators relative to their biases and MSEs. While the moment estimates  $\hat{\beta}_1$  are better than the other estimators relative to their biases and MSEs. This study revealed that the estimators based on maximum likelihood method perform the best among rest estimators, in approximately, most situations.

**Table (1):** Results of simulation study of Biases and MSEs of estimates for different values of parameters  $(\alpha, \beta, \lambda)$  for the Lomax- Poisson distribution.

| Sample Size | Method | Properties | Set of parameters                          |        |        |                                          |        |        |                                            |        |        |                                          |        |        |
|-------------|--------|------------|--------------------------------------------|--------|--------|------------------------------------------|--------|--------|--------------------------------------------|--------|--------|------------------------------------------|--------|--------|
|             |        |            | $(\alpha = 0.5, \beta = 1, \lambda = 0.5)$ |        |        | $(\alpha = 0.5, \beta = 1, \lambda = 1)$ |        |        | $(\alpha = 0.5, \beta = 1, \lambda = 1.5)$ |        |        | $(\alpha = 0.5, \beta = 1, \lambda = 2)$ |        |        |
| n=10        | ML     | Bias       | 0.013                                      | 0.166  | 0.054  | 0.049                                    | 0.075  | 0.041  | 0.077                                      | -0.016 | 0.023  | 0.122                                    | -0.074 | 0.017  |
|             |        | MSE        | 0.012                                      | 0.181  | 0.012  | 0.017                                    | 0.127  | 0.009  | 0.024                                      | 0.088  | 0.004  | 0.037                                    | 0.072  | 0.003  |
|             | MM     | Bias       | -0.180                                     | 0.026  | -0.052 | -0.200                                   | -0.017 | -0.082 | -0.308                                     | -0.016 | -0.018 | -0.197                                   | -0.001 | -0.027 |
|             |        | MSE        | 0.034                                      | 0.008  | 0.033  | 0.04                                     | 0.533  | 0.714  | 0.163                                      | 0.022  | 0.032  | 0.039                                    | 0.001  | 0.068  |
|             | LS     | Bias       | 0.083                                      | 0.131  | -0.490 | 0.108                                    | -0.015 | -0.200 | 0.168                                      | 0.117  | -0.496 | 0.187                                    | 0.102  | -0.497 |
|             |        | MSE        | 0.039                                      | 0.140  | 0.243  | 0.027                                    | 0.015  | 0.040  | 0.065                                      | 0.098  | 0.247  | 0.076                                    | 0.053  | 0.248  |
|             | WLS    | Bias       | 0.134                                      | 0.037  | 0.004  | 0.124                                    | 0.019  | -0.499 | 0.113                                      | 0.004  | 0.004  | 0.111                                    | 0.005  | -0.500 |
|             |        | MSE        | 0.029                                      | 0.050  | 0.247  | 0.028                                    | 0.061  | 0.249  | 0.027                                      | 0.020  | 0.025  | 0.032                                    | 0.016  | 0.250  |
| n=20        | ML     | Bias       | -0.022                                     | -0.001 | 0.007  | 0.011                                    | -0.081 | -0.006 | 0.033                                      | -0.141 | -0.014 | 0.061                                    | -0.195 | -0.019 |
|             |        | MSE        | 0.004                                      | 0.049  | 0.003  | 0.005                                    | 0.040  | 0.001  | 0.008                                      | 0.048  | 0.001  | 0.012                                    | 0.058  | 0.001  |
|             | MM     | Bias       | -0.190                                     | 0.012  | -0.018 | -0.192                                   | 0.009  | -0.130 | -0.195                                     | 0.005  | -0.086 | -0.196                                   | 0.004  | -0.077 |
|             |        | MSE        | 0.038                                      | 0.002  | 0.296  | 0.038                                    | 0.001  | 0.234  | 0.039                                      | 0.001  | 0.160  | 0.039                                    | 0.008  | 0.147  |
|             | LS     | Bias       | 0.085                                      | 0.139  | -0.486 | 0.011                                    | -0.057 | -0.020 | 0.715                                      | 0.338  | 0.011  | 0.232                                    | 0.105  | -0.494 |
|             |        | MSE        | 0.036                                      | 0.084  | 0.241  | 0.021                                    | 0.012  | 0.040  | 0.083                                      | 0.090  | 0.243  | 0.097                                    | 0.054  | 0.246  |
|             | WLS    | Bias       | 0.144                                      | -0.015 | -0.100 | 0.132                                    | -0.037 | -0.100 | 0.126                                      | -0.035 | -0.100 | 0.115                                    | -0.029 | -0.100 |
|             |        | MSE        | 0.027                                      | 0.034  | 0.009  | 0.030                                    | 0.023  | 0.010  | 0.031                                      | 0.019  | 0.010  | 0.030                                    | 0.013  | 0.010  |
| n=30        | ML     | Bias       | -0.031                                     | -0.066 | -0.015 | -0.005                                   | -0.122 | -0.024 | 0.016                                      | -0.189 | -0.034 | 0.044                                    | -0.222 | -0.035 |
|             |        | MSE        | 0.003                                      | 0.027  | 0.001  | 0.003                                    | 0.033  | 0.001  | 0.001                                      | 0.050  | 0.001  | 0.006                                    | 0.060  | 0.001  |
|             | MM     | Bias       | -0.185                                     | 0.014  | -0.192 | -0.188                                   | 0.009  | -0.142 | -0.192                                     | 0.009  | -0.132 | -0.194                                   | 0.007  | -0.106 |
|             |        | MSE        | 0.037                                      | 0.002  | 0.301  | 0.038                                    | 0.001  | 0.207  | 0.038                                      | 0.001  | 0.214  | 0.039                                    | 0.001  | 0.187  |
|             | LS     | Bias       | 0.123                                      | -0.064 | -0.200 | 0.038                                    | -0.074 | -0.200 | 0.105                                      | -0.052 | -0.200 | 0.088                                    | -0.032 | -0.200 |
|             |        | MSE        | 0.022                                      | 0.015  | 0.040  | 0.066                                    | 0.012  | 0.040  | 0.028                                      | 0.009  | 0.040  | 0.022                                    | 0.007  | 0.040  |
|             | WLS    | Bias       | 0.146                                      | -0.054 | -0.100 | 0.137                                    | -0.055 | -0.100 | 0.132                                      | -0.046 | -0.100 | 0.121                                    | -0.038 | -0.100 |
|             |        | MSE        | 0.027                                      | 0.079  | 0.009  | 0.028                                    | 0.034  | 0.010  | 0.033                                      | 0.015  | 0.010  | 0.033                                    | 0.013  | 0.010  |
| n=40        | ML     | Bias       | -0.040                                     | -0.081 | -0.024 | -0.015                                   | -0.143 | -0.035 | 0.007                                      | -0.200 | -0.043 | 0.034                                    | -0.242 | -0.046 |
|             |        | MSE        | 0.003                                      | 0.023  | 0.001  | 0.002                                    | 0.033  | 0.002  | 0.003                                      | 0.050  | 0.002  | 0.004                                    | 0.066  | 0.002  |
|             | MM     | Bias       | -0.184                                     | 0.015  | -0.206 | -0.188                                   | 0.012  | -0.170 | -0.191                                     | 0.012  | -0.144 | -0.193                                   | 0.007  | -0.102 |
|             |        | MSE        | 0.037                                      | 0.002  | 0.342  | 0.037                                    | 0.001  | 0.271  | 0.038                                      | 0.001  | 0.237  | 0.039                                    | 0.001  | 0.166  |
|             | LS     | Bias       | 0.121                                      | -0.080 | -0.200 | 0.114                                    | -0.079 | -0.200 | 0.111                                      | -0.057 | -0.200 | 0.095                                    | -0.034 | -0.200 |
|             |        | MSE        | 0.020                                      | 0.015  | 0.040  | 0.025                                    | 0.014  | 0.040  | 0.031                                      | 0.010  | 0.040  | 0.028                                    | 0.007  | 0.040  |
|             | WLS    | Bias       | 0.124                                      | -0.201 | -0.300 | 0.105                                    | -0.229 | -0.300 | 0.096                                      | -0.240 | -0.300 | 0.080                                    | -0.249 | -0.300 |
|             |        | MSE        | 0.023                                      | 0.112  | 0.090  | 0.020                                    | 0.062  | 0.090  | 0.024                                      | 0.067  | 0.090  | 0.028                                    | 0.072  | 0.090  |
| n=50        | ML     | Bias       | -0.040                                     | -0.094 | -0.031 | -0.015                                   | -0.143 | -0.035 | 0.008                                      | -0.208 | -0.050 | 0.025                                    | -0.249 | -0.054 |
|             |        | MSE        | 0.003                                      | 0.020  | 0.002  | 0.002                                    | 0.033  | 0.002  | 0.002                                      | 0.050  | 0.003  | 0.003                                    | 0.068  | 0.003  |
|             | MM     | Bias       | -0.182                                     | 0.019  | -0.262 | -0.182                                   | 0.014  | -0.201 | -0.191                                     | -0.011 | -0.153 | -0.191                                   | -0.013 | -0.132 |
|             |        | MSE        | 0.036                                      | 0.002  | 0.455  | 0.037                                    | 0.002  | 0.304  | 0.039                                      | 0.533  | 0.394  | 0.039                                    | 0.513  | 0.427  |
|             | LS     | Bias       | 0.121                                      | -0.085 | -0.200 | 0.038                                    | -0.086 | -0.200 | 0.105                                      | -0.064 | -0.200 | 0.096                                    | -0.037 | -0.200 |
|             |        | MSE        | 0.021                                      | 0.015  | 0.040  | 0.066                                    | 0.014  | 0.040  | 0.027                                      | 0.010  | 0.040  | 0.029                                    | 0.006  | 0.040  |
|             | WLS    | Bias       | 0.102                                      | -0.226 | -0.300 | 0.084                                    | -0.255 | -0.300 | 0.068                                      | -0.269 | -0.300 | 0.046                                    | -0.280 | -0.300 |
|             |        | MSE        | 0.021                                      | 0.075  | 0.090  | 0.019                                    | 0.079  | 0.090  | 0.020                                      | 0.084  | 0.090  | 0.020                                    | 0.089  | 0.090  |

Table (2): Results of simulation study of Biases and MSEs of estimates for different values of parameters ( $\alpha$ ,  $\beta$ ,  $\lambda$ ) for the Lomax- Poisson distribution.

| Sample Size | method | properties | Set of parameters                        |        |        |                                        |        |        |                                          |        |        |                                        |        |        |
|-------------|--------|------------|------------------------------------------|--------|--------|----------------------------------------|--------|--------|------------------------------------------|--------|--------|----------------------------------------|--------|--------|
|             |        |            | $(\alpha = 1, \beta = 1, \lambda = 0.5)$ |        |        | $(\alpha = 1, \beta = 1, \lambda = 1)$ |        |        | $(\alpha = 1, \beta = 1, \lambda = 1.5)$ |        |        | $(\alpha = 1, \beta = 1, \lambda = 2)$ |        |        |
| n=10        | ML     | Bias       | 0.216                                    | -0.066 | 0.026  | 0.241                                  | -0.121 | 0.019  | 0.268                                    | -0.178 | 0.011  | 0.300                                  | -0.224 | 0.003  |
|             |        | MSE        | 0.065                                    | 0.057  | 0.003  | 0.077                                  | 0.062  | 0.002  | 0.094                                    | 0.068  | 0.002  | 0.114                                  | 0.076  | 0.002  |
|             | MM     | Bias       | -0.199                                   | -0.006 | 0.016  | -0.199                                 | -0.007 | 0.022  | -0.199                                   | 0.007  | 0.028  | -0.199                                 | 0.008  | 0.033  |
|             |        | MSE        | 0.040                                    | 0.001  | 0.023  | 0.040                                  | 0.001  | 0.180  | 0.040                                    | 0.001  | 0.019  | 0.040                                  | 0.001  | 0.014  |
|             | LS     | Bias       | 0.006                                    | 0.004  | -0.200 | 0.570                                  | 0.038  | -0.200 | 0.049                                    | 0.071  | -0.200 | 0.047                                  | 0.106  | -0.200 |
|             |        | MSE        | 0.020                                    | 0.009  | 0.040  | 0.023                                  | 0.009  | 0.040  | 0.036                                    | 0.013  | 0.040  | 0.047                                  | 0.019  | 0.040  |
|             | WLS    | Bias       | 0.104                                    | -0.013 | -0.500 | 0.098                                  | 0.010  | -0.500 | 0.090                                    | 0.035  | -0.500 | 0.089                                  | 0.060  | -0.500 |
|             |        | MSE        | 0.039                                    | 0.013  | 0.250  | 0.054                                  | 0.012  | 0.250  | 0.063                                    | 0.011  | 0.250  | 0.100                                  | 0.013  | 0.250  |
| n=20        | ML     | Bias       | 0.145                                    | -0.173 | -0.005 | 0.163                                  | -0.228 | -0.013 | 0.185                                    | -0.271 | -0.019 | 0.211                                  | -0.302 | -0.025 |
|             |        | MSE        | 0.026                                    | 0.046  | 0.001  | 0.033                                  | 0.064  | 0.001  | 0.042                                    | 0.082  | 0.001  | 0.054                                  | 0.097  | 0.001  |
|             | MM     | Bias       | -0.199                                   | -0.005 | 0.022  | -0.199                                 | -0.006 | 0.025  | -0.199                                   | -0.006 | 0.029  | -0.199                                 | -0.007 | 0.036  |
|             |        | MSE        | 0.040                                    | 0.001  | 0.008  | 0.040                                  | 0.001  | 0.120  | 0.040                                    | 0.001  | 0.016  | 0.040                                  | 0.001  | 0.011  |
|             | LS     | Bias       | 0.065                                    | -0.019 | -0.200 | 0.056                                  | 0.018  | -0.200 | 0.054                                    | 0.057  | -0.200 | 0.053                                  | 0.090  | -0.200 |
|             |        | MSE        | 0.020                                    | 0.006  | 0.040  | 0.024                                  | 0.006  | 0.040  | 0.039                                    | 0.009  | 0.040  | 0.043                                  | 0.013  | 0.040  |
|             | WLS    | Bias       | 0.094                                    | -0.052 | -0.100 | 0.098                                  | -0.021 | -0.100 | 0.077                                    | -0.001 | -0.100 | 0.063                                  | 0.021  | -0.100 |
|             |        | MSE        | 0.032                                    | 0.012  | 0.010  | 0.041                                  | 0.009  | 0.010  | 0.036                                    | 0.007  | 0.010  | 0.035                                  | 0.005  | 0.010  |
| n=30        | ML     | Bias       | 0.012                                    | -0.205 | -0.021 | 0.133                                  | -0.258 | -0.025 | 0.146                                    | -0.299 | -0.027 | 0.165                                  | -0.329 | -0.029 |
|             |        | MSE        | 0.017                                    | 0.051  | 0.001  | 0.021                                  | 0.073  | 0.001  | 0.026                                    | 0.094  | 0.001  | 0.034                                  | 0.112  | 0.001  |
|             | MM     | Bias       | -0.199                                   | -0.004 | 0.012  | -0.199                                 | -0.004 | 0.018  | -0.199                                   | 0.006  | 0.025  | -0.199                                 | -0.006 | 0.029  |
|             |        | MSE        | 0.040                                    | 0.001  | 0.023  | 0.040                                  | 0.001  | 0.018  | 0.040                                    | 0.001  | 0.050  | 0.040                                  | 0.001  | 0.020  |
|             | LS     | Bias       | 0.072                                    | -0.024 | -0.200 | 0.038                                  | 0.015  | -0.200 | 0.038                                    | 0.052  | -0.200 | 0.058                                  | 0.087  | -0.200 |
|             |        | MSE        | 0.024                                    | 0.005  | 0.040  | 0.066                                  | 0.004  | 0.040  | 0.066                                    | 0.006  | 0.040  | 0.045                                  | 0.011  | 0.040  |
|             | WLS    | Bias       | 0.113                                    | -0.061 | -0.100 | 0.112                                  | -0.032 | -0.100 | 0.078                                    | -0.018 | -0.100 | 0.068                                  | 0.040  | -0.100 |
|             |        | MSE        | 0.044                                    | 0.013  | 0.010  | 0.051                                  | 0.010  | 0.010  | 0.029                                    | 0.005  | 0.010  | 0.030                                  | 0.009  | 0.010  |
| n=40        | ML     | Bias       | 0.107                                    | -0.220 | -0.032 | 0.117                                  | -0.271 | -0.035 | 0.125                                    | -0.312 | -0.034 | 0.136                                  | -0.343 | -0.033 |
|             |        | MSE        | 0.013                                    | 0.054  | 0.001  | 0.016                                  | 0.078  | 0.001  | 0.018                                    | 0.101  | 0.001  | 0.022                                  | 0.120  | 0.001  |
|             | MM     | Bias       | -0.199                                   | -0.004 | 0.012  | -0.199                                 | -0.004 | 0.013  | -0.199                                   | -4.830 | 0.021  | -0.199                                 | 0.006  | 0.036  |
|             |        | MSE        | 0.040                                    | 0.001  | 0.017  | 0.040                                  | 0.001  | 0.024  | 0.040                                    | 0.001  | 0.018  | 0.039                                  | 0.001  | 0.003  |
|             | LS     | Bias       | 0.012                                    | -0.028 | -0.200 | 0.038                                  | 0.006  | -0.200 | 0.038                                    | 0.046  | -0.200 | 0.053                                  | 0.086  | -0.200 |
|             |        | MSE        | 0.016                                    | 0.004  | 0.040  | 0.066                                  | 0.003  | 0.040  | 0.066                                    | 0.005  | 0.040  | 0.036                                  | 0.010  | 0.040  |
|             | WLS    | Bias       | 0.050                                    | -0.269 | -0.300 | 0.046                                  | -0.264 | -0.300 | 0.044                                    | -0.248 | -0.300 | 0.062                                  | -0.217 | -0.300 |
|             |        | MSE        | 0.026                                    | 0.079  | 0.090  | 0.039                                  | 0.077  | 0.090  | 0.056                                    | 0.070  | 0.090  | 0.091                                  | 0.056  | 0.090  |
| n=50        | ML     | Bias       | 0.101                                    | -0.233 | -0.039 | 0.111                                  | -0.281 | -0.042 | 0.115                                    | -0.321 | -0.042 | 0.120                                  | -0.353 | -0.038 |
|             |        | MSE        | 0.011                                    | 0.059  | 0.002  | 0.014                                  | 0.083  | 0.002  | 0.015                                    | 0.106  | 0.001  | 0.017                                  | 0.126  | 0.002  |
|             | MM     | Bias       | -0.199                                   | -0.003 | 0.005  | -0.199                                 | -0.004 | 0.013  | -0.199                                   | -0.006 | 0.028  | -0.199                                 | -0.005 | 0.032  |
|             |        | MSE        | 0.040                                    | 0.001  | 0.024  | 0.040                                  | 0.001  | 0.019  | 0.040                                    | 0.002  | 0.005  | 0.039                                  | 0.001  | 0.001  |
|             | LS     | Bias       | 0.038                                    | -0.029 | -0.200 | 0.038                                  | 0.007  | -0.200 | 0.049                                    | 0.044  | -0.200 | 0.038                                  | 0.085  | -0.200 |
|             |        | MSE        | 0.066                                    | 0.004  | 0.040  | 0.066                                  | 0.002  | 0.040  | 0.024                                    | 0.005  | 0.040  | 0.066                                  | 0.009  | 0.040  |
|             | WLS    | Bias       | 0.027                                    | -0.285 | -0.300 | 0.006                                  | -0.292 | -0.300 | 0.013                                    | -0.271 | -0.300 | 0.007                                  | -0.247 | -0.300 |
|             |        | MSE        | 0.016                                    | 0.087  | 0.090  | 0.029                                  | 0.093  | 0.090  | 0.044                                    | 0.082  | 0.090  | 0.038                                  | 0.068  | 0.090  |

**Table (3):** Results of simulation study of Biases and MSEs of estimates for different values of parameters ( $\alpha$ ,  $\beta$ ,  $\lambda$ ) for the Lomax- Poisson distribution

| SmpleSize | Method | Properties | Set of parameters                          |        |        |                                          |        |        |                                            |        |        |                                          |        |        |
|-----------|--------|------------|--------------------------------------------|--------|--------|------------------------------------------|--------|--------|--------------------------------------------|--------|--------|------------------------------------------|--------|--------|
|           |        |            | $(\alpha = 1.5, \beta = 1, \lambda = 0.5)$ |        |        | $(\alpha = 1.5, \beta = 1, \lambda = 1)$ |        |        | $(\alpha = 1.5, \beta = 1, \lambda = 1.5)$ |        |        | $(\alpha = 1.5, \beta = 1, \lambda = 2)$ |        |        |
| n=10      | ML     | Bias       | 0.315                                      | -0.168 | 0.017  | 0.347                                    | -0.229 | 0.005  | 0.367                                      | -0.264 | -0.004 | 0.414                                    | -0.296 | -0.014 |
|           |        | MSE        | 0.117                                      | 0.058  | 0.002  | 0.142                                    | 0.071  | 0.003  | 0.159                                      | 0.085  | 0.003  | 0.198                                    | 0.098  | 0.004  |
|           | MM     | Bias       | -0.202                                     | -0.012 | 0.040  | -0.199                                   | -0.009 | 0.028  | -0.201                                     | -0.010 | 0.062  | -0.199                                   | -0.009 | 0.028  |
|           |        | MSE        | 0.046                                      | 0.006  | 0.030  | 0.040                                    | 0.001  | 0.026  | 0.042                                      | 0.001  | 0.657  | 0.040                                    | 0.001  | 0.032  |
|           | LS     | Bias       | 0.078                                      | 0.033  | -0.500 | 0.039                                    | 0.113  | -0.200 | 0.006                                      | 0.026  | -0.500 | -0.014                                   | 0.028  | -0.500 |
|           |        | MSE        | 0.064                                      | 0.007  | 0.250  | 0.043                                    | 0.020  | 0.040  | 0.042                                      | 0.003  | 0.250  | 0.033                                    | 0.003  | 0.250  |
|           | WLS    | Bias       | 0.085                                      | 0.037  | -0.500 | 0.076                                    | 0.061  | -0.500 | 0.080                                      | 0.088  | -0.500 | 0.066                                    | 0.107  | -0.500 |
|           |        | MSE        | 0.072                                      | 0.011  | 0.250  | 0.086                                    | 0.012  | 0.250  | 0.107                                      | 0.017  | 0.250  | 0.122                                    | 0.020  | 0.250  |
| n=20      | ML     | Bias       | 0.225                                      | -0.254 | -0.013 | 0.254                                    | -0.294 | -0.023 | 0.271                                      | -0.323 | -0.033 | 0.309                                    | -0.348 | -0.043 |
|           |        | MSE        | 0.057                                      | 0.074  | 0.001  | 0.072                                    | 0.092  | 0.001  | 0.083                                      | 0.109  | 0.002  | 0.107                                    | 0.124  | 0.002  |
|           | MM     | Bias       | -0.199                                     | -0.008 | 0.039  | -0.199                                   | -0.008 | 0.038  | -0.199                                     | -0.008 | 0.047  | -0.203                                   | -0.011 | 0.538  |
|           |        | MSE        | 0.040                                      | 0.001  | 0.012  | 0.040                                    | 0.001  | 0.018  | 0.040                                      | 0.001  | 0.009  | 0.050                                    | 0.002  | 0.028  |
|           | LS     | Bias       | 0.071                                      | 0.030  | -0.500 | 0.037                                    | 0.097  | -0.200 | -0.006                                     | 0.025  | -0.500 | -0.032                                   | 0.024  | -0.500 |
|           |        | MSE        | 0.058                                      | 0.004  | 0.250  | 0.029                                    | 0.014  | 0.040  | 0.038                                      | 0.003  | 0.250  | 0.035                                    | 0.003  | 0.250  |
|           | WLS    | Bias       | 0.063                                      | -0.006 | -0.100 | 0.056                                    | 0.024  | -0.100 | 0.042                                      | 0.052  | -0.100 | 0.055                                    | 0.085  | -0.100 |
|           |        | MSE        | 0.028                                      | 0.005  | 0.010  | 0.033                                    | 0.005  | 0.010  | 0.031                                      | 0.007  | 0.010  | 0.075                                    | 0.012  | 0.010  |
| n=30      | ML     | Bias       | 0.187                                      | -0.291 | -0.023 | 0.206                                    | -0.321 | -0.029 | 0.228                                      | -0.350 | -0.041 | 0.264                                    | -0.364 | -0.050 |
|           |        | MSE        | 0.039                                      | 0.089  | 0.001  | 0.048                                    | 0.106  | 0.115  | 0.059                                      | 0.124  | 0.002  | 0.076                                    | 0.134  | 0.003  |
|           | MM     | Bias       | -0.199                                     | -0.006 | 0.040  | -0.199                                   | -0.008 | 0.047  | -0.199                                     | -0.006 | 0.047  | -0.199                                   | 0.006  | -0.047 |
|           |        | MSE        | 0.040                                      | 0.001  | 0.008  | 0.039                                    | 0.001  | 0.008  | 0.040                                      | 0.001  | 0.009  | 0.040                                    | 0.001  | 0.015  |
|           | LS     | Bias       | 0.046                                      | 0.057  | -0.200 | 0.038                                    | 0.100  | -0.200 | 0.050                                      | 0.132  | -0.200 | 0.038                                    | 0.165  | -0.200 |
|           |        | MSE        | 0.026                                      | 0.007  | 0.040  | 0.066                                    | 0.014  | 0.040  | 0.054                                      | 0.021  | 0.040  | 0.048                                    | 0.030  | 0.040  |
|           | WLS    | Bias       | 0.081                                      | -0.019 | -0.100 | 0.057                                    | 0.004  | -0.100 | 0.054                                      | 0.034  | -0.100 | 0.064                                    | 0.065  | -0.100 |
|           |        | MSE        | 0.042                                      | 0.006  | 0.010  | 0.031                                    | 0.004  | 0.010  | 0.035                                      | 0.005  | 0.010  | 0.076                                    | 0.008  | 0.010  |
| n=40      | ML     | Bias       | 0.163                                      | -0.303 | -0.028 | 0.173                                    | -0.337 | -0.031 | 0.195                                      | -0.361 | -0.040 | 0.228                                    | -0.378 | -0.051 |
|           |        | MSE        | 0.029                                      | 0.095  | 0.001  | 0.033                                    | 0.116  | 0.001  | 0.043                                      | 0.132  | 0.002  | 0.058                                    | 0.144  | 0.003  |
|           | MM     | Bias       | -0.199                                     | -0.007 | 0.037  | -0.199                                   | 0.008  | 0.048  | -0.199                                     | -0.009 | 0.049  | -0.199                                   | -0.009 | 0.049  |
|           |        | MSE        | 0.039                                      | 0.001  | 0.013  | 0.039                                    | 0.001  | 0.004  | 0.040                                      | 0.001  | 0.012  | 0.040                                    | 0.000  | 0.015  |
|           | LS     | Bias       | 0.035                                      | 0.053  | -0.200 | 0.038                                    | 0.093  | -0.200 | 0.040                                      | 0.129  | -0.200 | 0.038                                    | 0.163  | -0.200 |
|           |        | MSE        | 0.018                                      | 0.005  | 0.040  | 0.066                                    | 0.012  | 0.040  | 0.043                                      | 0.019  | 0.040  | 0.054                                    | 0.029  | 0.040  |
|           | WLS    | Bias       | 0.026                                      | -0.253 | -0.300 | 0.056                                    | -0.216 | -0.300 | 0.066                                      | -0.182 | -0.300 | 0.061                                    | -0.153 | -0.300 |
|           |        | MSE        | 0.061                                      | 0.072  | 0.090  | 0.099                                    | 0.053  | 0.090  | 0.111                                      | 0.040  | 0.090  | 0.107                                    | 0.029  | 0.090  |
| n=50      | ML     | Bias       | 0.151                                      | -0.312 | -0.033 | 0.150                                    | -0.349 | -0.034 | 0.165                                      | -0.373 | -0.037 | 0.184                                    | -0.391 | -0.044 |
|           |        | MSE        | 0.024                                      | 0.100  | 0.001  | 0.025                                    | 0.124  | 0.001  | 0.031                                      | 0.141  | 0.002  | 0.040                                    | 0.153  | 0.002  |
|           | MM     | Bias       | -0.199                                     | -0.006 | 0.041  | -0.198                                   | -0.007 | 0.040  | -0.198                                     | -0.008 | 0.044  | -0.198                                   | -0.008 | 0.049  |
|           |        | MSE        | 0.039                                      | 0.001  | 0.007  | 0.039                                    | 0.001  | 0.011  | 0.039                                      | 0.001  | 0.009  | 0.039                                    | 0.001  | 0.008  |
|           | LS     | Bias       | 0.045                                      | 0.052  | -0.200 | 0.043                                    | 0.091  | -0.200 | 0.043                                      | 0.128  | -0.200 | 0.037                                    | 0.162  | -0.200 |
|           |        | MSE        | 0.025                                      | 0.005  | 0.040  | 0.027                                    | 0.011  | 0.040  | 0.037                                      | 0.019  | 0.040  | 0.057                                    | 0.028  | 0.040  |
|           | WLS    | Bias       | -0.008                                     | -0.276 | -0.300 | 0.009                                    | -0.242 | -0.300 | 0.007                                      | -0.210 | -0.300 | 0.004                                    | -0.181 | -0.300 |
|           |        | MSE        | 0.048                                      | 0.083  | 0.090  | 0.067                                    | 0.066  | 0.090  | 0.054                                      | 0.049  | 0.090  | 0.053                                    | 0.037  | 0.090  |

Table (4): Results of simulation study of Biases and MSEs of estimates for different values of parameters ( $\alpha$ ,  $\beta$ ,  $\lambda$ ) for the Lomax- Poisson distribution.

| SampleSize | Method | Properties | Set of parameters                        |        |        |                                        |        |        |                                          |        |        |                                        |        |        |
|------------|--------|------------|------------------------------------------|--------|--------|----------------------------------------|--------|--------|------------------------------------------|--------|--------|----------------------------------------|--------|--------|
|            |        |            | $(\alpha = 2, \beta = 1, \lambda = 0.5)$ |        |        | $(\alpha = 2, \beta = 1, \lambda = 1)$ |        |        | $(\alpha = 2, \beta = 1, \lambda = 1.5)$ |        |        | $(\alpha = 2, \beta = 1, \lambda = 2)$ |        |        |
| n=10       | ML     | Bias       | 0.401                                    | -0.234 | 0.008  | 0.213                                  | -0.374 | -0.045 | 0.464                                    | -0.306 | 0.018  | 0.505                                  | -0.331 | -0.030 |
|            |        | MSE        | 0.178                                    | 0.071  | 0.003  | 0.050                                  | 0.141  | 0.002  | 0.240                                    | 0.102  | 0.004  | 0.284                                  | 0.115  | 0.003  |
|            | MM     | Bias       | -0.200                                   | -0.009 | 0.038  | -0.200                                 | -0.009 | 0.028  | -0.202                                   | -0.009 | 0.036  | -0.200                                 | -0.009 | 0.021  |
|            |        | MSE        | 0.040                                    | 0.001  | 0.019  | 0.040                                  | 0.003  | 0.034  | 0.045                                    | 0.001  | 0.029  | 0.040                                  | 0.001  | 0.041  |
|            | LS     | Bias       | 0.017                                    | 0.029  | -0.500 | 0.016                                  | 0.169  | -0.200 | -0.039                                   | 0.025  | -0.500 | -0.064                                 | 0.023  | -0.500 |
|            |        | MSE        | 0.040                                    | 0.003  | 0.250  | 0.037                                  | 0.034  | 0.040  | 0.033                                    | 0.003  | 0.250  | 0.034                                  | 0.003  | 0.250  |
|            | WLS    | Bias       | 0.063                                    | 0.076  | -0.500 | 0.066                                  | 0.102  | -0.500 | 0.030                                    | 0.107  | -0.500 | 0.022                                  | 0.108  | -0.500 |
|            |        | MSE        | 0.070                                    | 0.014  | 0.250  | 0.098                                  | 0.018  | 0.250  | 0.063                                    | 0.021  | 0.250  | 0.092                                  | 0.026  | 0.250  |
| n=20       | ML     | Bias       | 0.297                                    | -0.302 | -0.024 | 0.327                                  | -0.329 | -0.036 | 0.356                                    | -0.354 | -0.049 | 0.403                                  | -0.369 | -0.062 |
|            |        | MSE        | 0.094                                    | 0.096  | 0.001  | 0.115                                  | 0.111  | 0.002  | 0.138                                    | 0.127  | 0.003  | 0.178                                  | 0.138  | 0.005  |
|            | MM     | Bias       | -0.201                                   | -0.009 | 0.041  | -0.199                                 | -0.009 | 0.036  | -0.199                                   | -0.979 | 0.037  | -0.205                                 | -0.012 | 0.027  |
|            |        | MSE        | 0.044                                    | 0.001  | 0.023  | 0.040                                  | 0.001  | 0.021  | 0.040                                    | 0.001  | 0.026  | 0.079                                  | 0.008  | 0.033  |
|            | LS     | Bias       | -0.001                                   | 0.026  | -0.500 | 0.038                                  | 0.162  | -0.200 | -0.047                                   | 0.240  | -0.500 | -0.068                                 | 0.023  | -0.500 |
|            |        | MSE        | 0.039                                    | 0.003  | 0.250  | 0.066                                  | 0.030  | 0.040  | 0.032                                    | 0.003  | 0.250  | 0.032                                  | 0.003  | 0.250  |
|            | WLS    | Bias       | 0.045                                    | 0.043  | -0.100 | 0.042                                  | 0.075  | -0.100 | 0.040                                    | 0.106  | -0.100 | 0.035                                  | 0.136  | -0.100 |
|            |        | MSE        | 0.038                                    | 0.006  | 0.010  | 0.043                                  | 0.009  | 0.010  | 0.063                                    | 0.016  | 0.010  | 0.079                                  | 0.023  | 0.010  |
| n=30       | ML     | Bias       | 0.257                                    | -0.322 | -0.034 | 0.279                                  | -0.348 | -0.045 | 0.307                                    | -0.368 | -0.055 | 0.344                                  | -0.383 | -0.066 |
|            |        | MSE        | 0.070                                    | 0.106  | 0.001  | 0.082                                  | 0.123  | 0.002  | 0.099                                    | 0.137  | 0.003  | 0.126                                  | 0.148  | 0.004  |
|            | MM     | Bias       | -0.199                                   | -0.009 | 0.043  | -0.201                                 | -0.012 | 0.052  | -0.199                                   | -0.009 | 0.046  | -0.200                                 | -0.009 | 0.055  |
|            |        | MSE        | 0.040                                    | 0.002  | 0.024  | 0.044                                  | 0.013  | 0.082  | 0.040                                    | 0.001  | 0.014  | 0.042                                  | 0.001  | 0.382  |
|            | LS     | Bias       | 0.036                                    | 0.120  | -0.200 | 0.034                                  | 0.155  | -0.200 | 0.015                                    | 0.187  | -0.200 | -0.001                                 | 0.217  | -0.200 |
|            |        | MSE        | 0.043                                    | 0.017  | 0.040  | 0.059                                  | 0.027  | 0.040  | 0.037                                    | 0.037  | 0.040  | 0.036                                  | 0.049  | 0.040  |
|            | WLS    | Bias       | 0.051                                    | 0.023  | -0.100 | 0.050                                  | 0.054  | -0.100 | 0.056                                    | 0.086  | -0.100 | 0.050                                  | 0.117  | -0.100 |
|            |        | MSE        | 0.041                                    | 0.004  | 0.010  | 0.057                                  | 0.006  | 0.010  | 0.111                                    | 0.012  | 0.010  | 0.077                                  | 0.017  | 0.010  |
| n=40       | ML     | Bias       | 0.222                                    | -0.341 | -0.037 | 0.244                                  | -0.361 | -0.047 | 0.272                                    | -0.381 | -0.059 | 0.303                                  | -0.394 | -0.069 |
|            |        | MSE        | 0.053                                    | 0.118  | 0.002  | 0.063                                  | 0.131  | 0.003  | 0.079                                    | 0.146  | 0.004  | 0.096                                  | 0.157  | 0.005  |
|            | MM     | Bias       | -0.199                                   | -0.009 | 0.041  | -0.199                                 | -0.009 | 0.035  | -0.199                                   | -0.009 | 0.042  | -0.199                                 | -0.009 | 0.029  |
|            |        | MSE        | 0.040                                    | 0.001  | 0.018  | 0.039                                  | 0.001  | 0.025  | 0.040                                    | 0.001  | 0.022  | 0.039                                  | 0.001  | 0.020  |
|            | LS     | Bias       | 0.047                                    | 0.119  | -0.200 | 0.238                                  | 0.152  | -0.200 | 0.009                                    | 0.183  | -0.200 | -0.010                                 | 0.213  | -0.200 |
|            |        | MSE        | 0.052                                    | 0.017  | 0.040  | 0.066                                  | 0.025  | 0.040  | 0.037                                    | 0.036  | 0.040  | 0.025                                  | 0.047  | 0.040  |
|            | WLS    | Bias       | 0.041                                    | -0.197 | -0.300 | 0.046                                  | -0.165 | -0.300 | 0.044                                    | 0.134  | -0.300 | 0.038                                  | -0.106 | -0.300 |
|            |        | MSE        | 0.087                                    | 0.045  | 0.090  | 0.088                                  | 0.032  | 0.090  | 0.093                                    | 0.023  | 0.090  | 0.084                                  | 0.016  | 0.090  |
| n=50       | ML     | Bias       | 0.187                                    | -0.353 | -0.035 | 0.207                                  | -0.349 | -0.044 | 0.234                                    | -0.390 | -0.055 | 0.269                                  | -0.402 | -0.067 |
|            |        | MSE        | 0.038                                    | 0.126  | 0.001  | 0.048                                  | 0.123  | 0.002  | 0.060                                    | 0.153  | 0.003  | 0.076                                  | 0.162  | 0.005  |
|            | MM     | Bias       | -0.199                                   | -0.009 | 0.546  | -0.203                                 | -0.014 | 0.049  | -0.210                                   | -0.020 | 0.053  | -0.205                                 | -0.013 | 0.034  |
|            |        | MSE        | 0.040                                    | 0.001  | 0.014  | 0.057                                  | 0.017  | 0.026  | 0.169                                    | 0.122  | 0.017  | 0.079                                  | 0.016  | 0.023  |
|            | LS     | Bias       | 0.041                                    | 0.116  | -0.200 | 0.038                                  | 0.153  | -0.200 | 0.017                                    | 0.184  | -0.200 | -0.009                                 | 0.213  | -0.200 |
|            |        | MSE        | 0.041                                    | 0.016  | 0.040  | 0.066                                  | 0.025  | 0.040  | 0.038                                    | 0.035  | 0.040  | 0.018                                  | 0.047  | 0.040  |
|            | WLS    | Bias       | 0.008                                    | -0.220 | -0.300 | 0.016                                  | -0.185 | -0.300 | 0.004                                    | -0.160 | -0.300 | 0.005                                  | -0.132 | -0.300 |
|            |        | MSE        | 0.045                                    | 0.053  | 0.090  | 0.087                                  | 0.039  | 0.090  | 0.076                                    | 0.030  | 0.090  | 0.091                                  | 0.022  | 0.090  |

### References

Adamidis, K., and Loukas, S., (1998). A life time distribution with decreasing failure rate. *Statistics and Probability Letters* 79, 2493-2500.

Adamidis, K., Dimitrakopoulou, T., and Loukas, S.,(2005). On a generalization of the exponential-geometric distribution. *Statistics and Probability Letters*, 73, 259-269.

Chahkandi, M., and Ganjali, M., (2009). On Some lifetime distributions with decreasing failure rate. *Computational Statistics and Data Analysis*, 53, 4433-4440.

Louzada, F. , Roman, M. and Cancho, V. G. (2011). The complementary exponential geometric distribution: Model, properties, and a comparison with its counterpart. *Computational Statistics and Data Analysis* 55, 2516-2524

Hemmati, F. Khorram, E. and Rezakhah, S. (2011). A new three- parameter ageing distribution. *Statistical Planning and Inference*, 141, 2266-2275.

Johnson,N.L., Kotz, S. and Balakrishnan, N. (1995) *Continuous Univariate Distribution*, vol.2, 2<sup>nd</sup> edition, New york, Wiely.

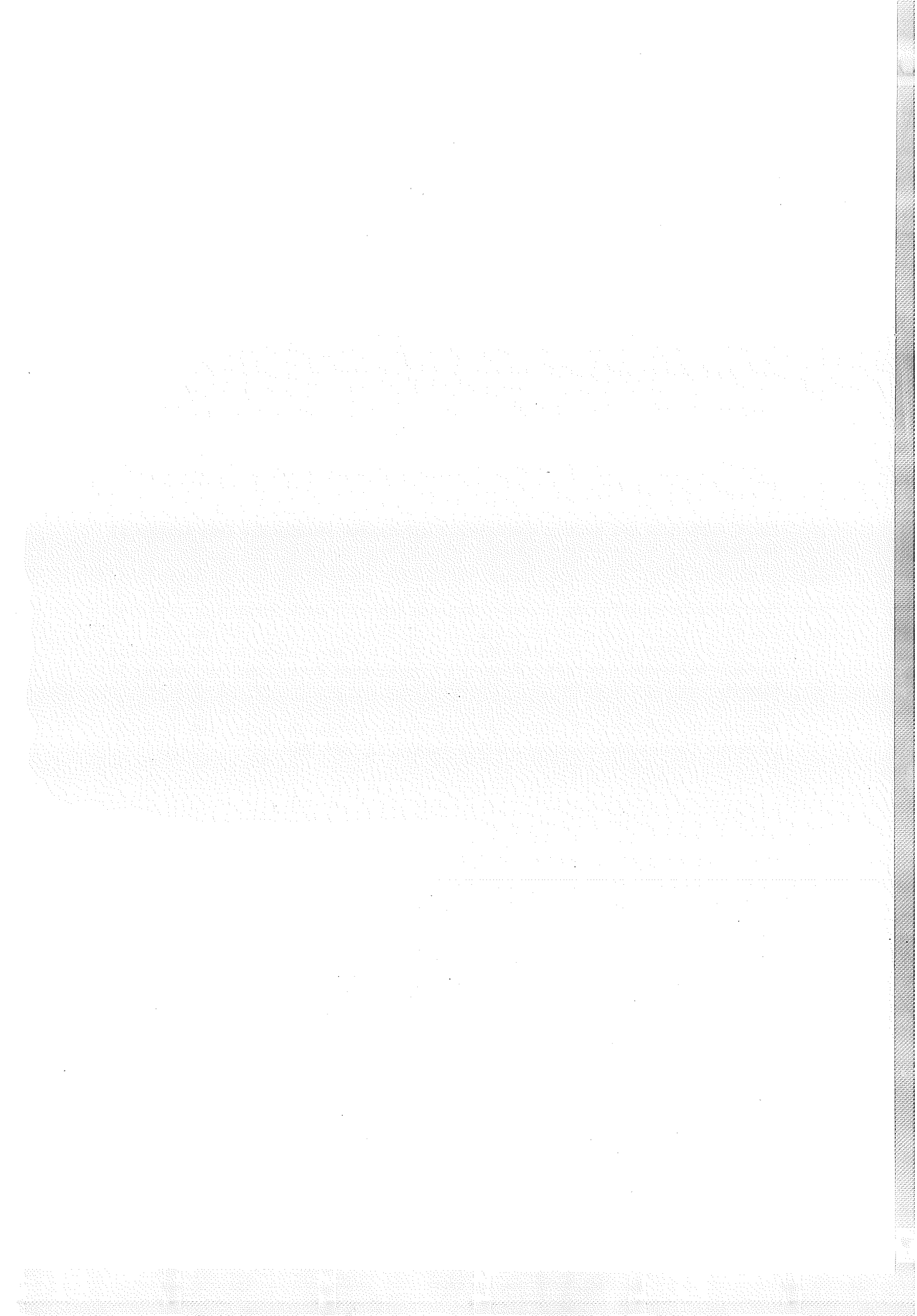
Kus, C., (2007). A new life time distribution. *Computational Statistics and Data Analysis* 51, 4497-4509.

Rezaei S., and Tahmasbi R., (2012). A new Lifetime Distribution with Increasing Failure rate: Exponential Truncated Poisson. *Journal of Basic and Applied Scientific Research*, 2,, 1749-1762.

Silva, R. B., Souza W. B. and Cordeiro, G. M. (2010). A new distribution with decreasing, increasing and upside-down bathtub failure rate. *Computational Statistics and Data Analysis* 54, 935-944.

Tahmasbi, R., and Rezaei, S., (2008). A two-Parameter lifetime distribution with decreasing failure rate. *Computational Statistics and Data Analysis*, 52, 3889- 3901.







## Maximum Likelihood Estimation of the Parameters for Exponentiated Weibull Poisson Distribution Based on Generalized Order Statistics

Abd-Elfattah, A. M.,<sup>1</sup>

Amal S. Hassan<sup>1</sup>

and

Said, G. Nassr<sup>2</sup>

### Abstract

This article, discusses maximum likelihood estimators of the unknown parameters for the exponentiated Weibull Poisson distribution based on generalized order statistics. Asymptotic variance and covariance matrix of the estimators are given. In addition, confidence interval for parameters of GOS is obtained. Progressively type-II censored as a special case of generalized order statistics are considered. Furthermore, a simulation study is included for illustrating the precision and variations of maximum likelihood estimators for different sample sizes via Mathcad (14) statistical package.

**Key words:** Generalized order statistics; Exponentiated Weibull Poisson distribution; Maximum likelihood method; Fisher information matrix; Progressively type-II censored.

### 1. Introduction

Order statistics and record values play an important role in statistics and its applications, reliability theory and life testing. Their distributional and stochastic properties have been studied extensively in the literature. However, they can be considered as special cases of generalized order statistics (GOS) that have been introduced and extensively studied by Kamps (1995). It represented can be considered as enable a unified approach to several models of ordered random variables such as ordinary order statistics, record values, sequential order statistics and progressively type II censored order statistics.

The statistical properties and the estimation problems based on GOS for some life time distribution are studied by several researchers. Ahsanallah (1996) studied the distribution properties of GOS for a uniform distribution. Habibullah and Ahsanullah (2000) obtained the estimates of the parameters of Pareto II distribution based on GOS. Their estimators based on order statistics and record values are shown to be special cases of their estimators. Ahsanallah (2000), studied some distributional properties of GOS for the two parameters exponential distribution. Jaheen (2002), considered the prediction of future for a general class of distribution. Jaheen (2005) estimating the parameters of the Burr type XII distribution based on GOS and upper record using maximum likelihood estimate (MLE), Bayesian and approximate Bayes.

Mailinowska *et. al.* (2006) derived the minimum variance linear unbiased estimators (MVLUE) and MLE based on n-selected GOS. Aboelenen (2010) discussed Bayesian and non-Bayesian estimation based on GOS for Weibull distribution. He obtained the estimators of the

---

<sup>1</sup> Institute of Statistical Studies & Research, Cairo University, EGYPT.

<sup>2</sup> PhD. student of Mathematical Statistics, Institute of Statistical Studies & Research, Cairo University, EGYPT.

parameters and confidence intervals for progressively censoring type II and record values. Burkschat (2010) derived the best linear unbiased estimators (BLUEs) and best equivariant estimators in location and scale families of GOS for generalized Pareto distribution.

Recently, Abo-Elfotouh, *et. al.* (2011) obtained the estimators of the unknown parameters of the Weibull extension model based on GOS using maximum likelihood and Bayesian methods. Ateya *et. al.* (2011) used the maximum likelihood and Bayes methods to obtain the estimators of a truncated type I generalized logistic distribution. They compared these methods by computing the mean squared errors of the estimates in both cases considering order statistics and upper record values cases. Also, they used Bayesian prediction intervals for the future GOS based on one sample scheme. Abd-Ellah (2012) obtained the estimators of the unknown parameters of the inverse Weibull distribution based on GOS using maximum likelihood and Bayesian methods. He examined Bayes estimates under various losses such as the balance squared error and balance LINEX loss functions.

This article can be organized as follows. In Section 2 the exponentiated Weibull Poisson (EWP) distribution is introduced as a lifetime model and the test method is also described. Section 3 presents point and interval estimates of parameters for the EWP based on generalized order statistics using maximum likelihood method. In section 4 the approximate asymptotic variances and covariances matrix are investigated. Section 5 explains the simulation study for illustrating the theoretical results. Finally, conclusions are included in Section 6. Tables are displayed in the appendix.

## 2. The Model and Concept of GOS

The exponentiated Weibull Poisson is a new distribution with four parameters presented by Mahmoudi *et. al.* (2012). This distribution obtained by compounding exponentiated Weibull and Poisson distributions. This distribution contains several lifetime sub-models such as: generalized exponentiated Poisson, complementary Weibull Poisson, complementary exponential Poisson, exponential Rayleigh Poisson and Raleigh Poisson distributions.

If failure time  $X$  follows EWP distribution, then its probability density function (pdf) is given by

$$f(x; \alpha, \lambda, \theta, \beta) = \frac{\alpha \lambda \theta \beta^\lambda x^{\lambda-1}}{e^\theta - 1} e^{-(\beta x)^\lambda} \left(1 - e^{-(\beta x)^\lambda}\right)^{\alpha-1} \exp\left[\theta \left(1 - e^{-(\beta x)^\lambda}\right)^\alpha\right]; \quad (2.1)$$

$$x > 0, \alpha, \lambda, \theta, \beta > 0$$

The distribution function is given by

$$F(x; \alpha, \lambda, \theta, \beta) = \frac{\exp\left[\theta \left(1 - e^{-(\beta x)^\lambda}\right)^\alpha\right] - 1}{e^\theta - 1}; \quad x > 0, \alpha, \lambda, \theta, \beta > 0 \quad (2.2)$$

Mahmoudi *et. al.* (2012) obtained several properties of the new distribution such as reliability and failure rate functions, quantiles and moments. They used the maximum likelihood method for estimating the parameters of the new model.

The concept of GOS has been introduced by Kamps (1995). Its enable a unified approach to several models of ordered random variables as ordinary order statistics, sequential order statistics, order statistics with non integral sample size, record values and progressively type II censored order statistics.

The random variable  $X(1, n, m, k), X(2, n, m, k), \dots, X(n, n, m, k)$  are called GOS if their joint probability density function is given by

$$\begin{aligned} f_{X(1, n, m, k), \dots, X(n, n, m, k)}(x_{(1)}, x_{(2)}, \dots, x_{(n)}) \\ = C_{n-1} \left( \prod_{i=1}^{n-1} [\bar{F}(x_{(i)})]^{m_i} f(x_{(i)}) \right) [\bar{F}(x_{(n)})]^{k-1} f(x_{(n)}) \end{aligned} \quad (2.3)$$

$$F^{-1}(0) < x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)} < F^{-1}(1)$$

where,  $C_{n-1} = k \prod_{j=1}^{n-1} \gamma_j$ ,  $\gamma_j = k + n - j + \prod_{i=j}^{n-1} m_i$  and  $\bar{F}(x) = 1 - F(x)$

In general, it is not easy to find a natural interpretation of generalized order statistics in terms of observed random samples. So, an interesting special case such as the progressive type II censored have been used. This model is the most application general models of ordered random values and is useful in reliability and life time studies.

### 3. Maximum Likelihood Estimation

The maximum likelihood is one of the most important and widely methods, used in statistics. The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. Furthermore, maximum likelihood estimators are consistent and asymptotically normally distributed. In this section point and interval estimation for the parameters of EWP distribution based on GOS are evaluated using maximum likelihood method.

Suppose that  $X(1, n, m, k), X(2, n, m, k), \dots, X(n, n, m, k)$  are  $n$  random sample of GOS drawn from EWP distribution whose pdf is given by (2.1), based on this set of GOS the likelihood function is

$$\begin{aligned} L(\underline{x}; \alpha, \lambda, \theta, \beta) = C_{n-1} (\alpha)^n (\lambda)^n (\theta)^n (\beta)^{n\lambda} (e^\theta - 1)^{-n} \left[ \prod_{i=1}^n x_{(i)}^{\lambda-1} \right] \left[ \exp \left( \theta \sum_{i=1}^n (E_i)^\alpha \right) \right] \\ \left[ \exp \left( - \sum_{i=1}^n (\beta x_{(i)})^\lambda \right) \right] \left[ \prod_{i=1}^n (E_i)^{\alpha-1} \right] \left[ \prod_{i=1}^{n-1} \left( 1 - \frac{e^{\theta(E_i)^\alpha} - 1}{e^\theta - 1} \right)^{m_i} \right] \left( 1 - \frac{e^{\theta(E_n)^\alpha} - 1}{e^\theta - 1} \right)^{(k-1)} \end{aligned} \quad (3.1)$$

where,  $E_i = \left( 1 - e^{-(\beta x_{(i)})^\lambda} \right)$  and  $E_n = \left( 1 - e^{-(\beta x_{(n)})^\lambda} \right)$

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Therefore, the logarithm of the likelihood function is

$$\ln L = \ln C_{n-1} + n \ln \alpha + n \ln \lambda + n \ln \theta + n \lambda \ln \beta - n \ln(e^\theta - 1) + (\lambda - 1) \sum_{i=1}^n \ln x_{(i)} - \sum_{i=1}^n (\beta x_{(i)})^\lambda + (\alpha - 1) \sum_{i=1}^n \ln(E_i) + \theta \sum_{i=1}^n (E_i)^\alpha + \sum_{i=1}^{n-1} m_i \ln \left( 1 - \frac{e^{\theta(E_i)^\alpha} - 1}{e^\theta - 1} \right) + (k - 1) \ln \left( 1 - \frac{e^{\theta(E_n)^\alpha} - 1}{e^\theta - 1} \right) \quad (3.2)$$

MLEs of  $\alpha, \lambda, \theta$  and  $\beta$  are solutions to the system of equations obtained by letting the first partial derivatives of the total log likelihood be zero with respect to  $\alpha, \lambda, \theta$  and  $\beta$ , respectively. Therefore, the system of equations is as follows:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln(E_i) + \theta \sum_{i=1}^n (E_i)^\alpha \ln(E_i) + \theta \sum_{i=1}^{n-1} m_i e^{\theta(E_i)^\alpha} (E_i)^\alpha \ln(E_i) (E_{i+1})^{-1} + \theta (k - 1) e^{\theta(E_n)^\alpha} (E_n)^\alpha \ln(E_n) (E_{1n})^{-1}, \quad (3.3)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} + n \ln \beta + \sum_{i=1}^n \ln x_{(i)} - \sum_{i=1}^n (T_i) + (\alpha - 1) \sum_{i=1}^n e^{-(\beta x_{(i)})^\lambda} (T_i) (E_i)^{-1} + \theta \alpha \sum_{i=1}^n (E_i)^{\alpha-1} e^{-(\beta x_{(i)})^\lambda} (T_i) + \alpha \theta \sum_{i=1}^{n-1} m_i e^{\theta(E_i)^\alpha} (E_i)^{\alpha-1} e^{-(\beta x_{(i)})^\lambda} (T_i) (E_{i+1})^{-1} + \alpha \theta (k - 1) e^{\theta(E_n)^\alpha} (E_n)^{\alpha-1} e^{-(\beta x_{(n)})^\lambda} (T_n) (E_{1n})^{-1}, \quad (3.4)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + n e^\theta (1 - e^\theta)^{-1} + \sum_{i=1}^n (E_i)^\alpha + \sum_{i=1}^{n-1} m_i \left[ e^\theta - e^{\theta(E_i)^\alpha} (E_i)^\alpha - e^\theta e^{\theta(E_i)^\alpha} + e^\theta e^{\theta(E_i)^\alpha} (E_i)^\alpha \right] \left[ (e^\theta - 1) (E_{i+1}) \right]^{-1} + (k - 1) \left[ e^\theta - e^{\theta(E_n)^\alpha} (E_n)^\alpha - e^\theta e^{\theta(E_n)^\alpha} + e^\theta e^{\theta(E_n)^\alpha} (E_n)^\alpha \right] \left[ (e^\theta - 1) (E_{1n}) \right]^{-1}, \quad (3.5)$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta} &= \frac{n\lambda}{\beta} - \lambda \sum_{i=1}^n x_{(i)} (\beta x_{(i)})^{\lambda-1} + \lambda(\alpha-1) \sum_{i=1}^n x_{(i)} (\beta x_{(i)})^{\lambda-1} e^{-(\beta x_{(i)})^\lambda} (E_i)^{-1} \\ &+ \theta \alpha \lambda \sum_{i=1}^n x_{(i)} (\beta x_{(i)})^{\lambda-1} e^{-(\beta x_{(i)})^\lambda} (E_i)^{\alpha-1} \\ &+ \alpha \theta \lambda \sum_{i=1}^{n-1} m_i (x_{(i)}) (\beta x_{(i)})^{\lambda-1} e^{\theta(E_i)^\alpha} e^{-(\beta x_{(i)})^\lambda} (E_i)^{\alpha-1} (E_{ii})^{-1} \\ &+ \alpha \theta \lambda (k-1) (x_{(n)}) (\beta x_{(n)})^{\lambda-1} e^{\theta(E_n)^\alpha} e^{-(\beta x_{(n)})^\lambda} (E_n)^{\alpha-1} (E_{nn})^{-1}, \end{aligned} \quad (3.6)$$

where,  $E_{ii} = [e^{\theta(E_i)^\alpha} - e^\theta]$ ,  $E_{nn} = [e^{\theta(E_n)^\alpha} - e^\theta]$ ,  $T_i = (\beta x_{(i)})^\lambda \ln(\beta x_{(i)})$ , and  $T_n = (\beta x_{(n)})^\lambda \ln(\beta x_{(n)})$ .

The estimators  $\hat{\alpha}$ ,  $\hat{\beta}$ ,  $\hat{\lambda}$  and  $\hat{\theta}$  for  $\alpha$ ,  $\beta$ ,  $\lambda$  and  $\theta$  can be obtained by setting (3.3) to (3.6) equal zero. Consequently, the system equations are reduced into four nonlinear equations as follows:

$$\frac{n}{\hat{\alpha}} + \sum_{i=1}^n \ln(E_i) + \hat{\theta} \sum_{i=1}^n (E_i)^{\hat{\alpha}} \ln(E_i) + \hat{\theta} \sum_{i=1}^{n-1} m_i W_i \ln(E_i) + \hat{\theta} (k-1) W_n \ln(E_n) = 0 \quad (3.7)$$

$$\begin{aligned} \frac{n}{\hat{\lambda}} + n \ln \hat{\beta} + \sum_{i=1}^n \ln x_{(i)} - \sum_{i=1}^n (T_i) + (\hat{\alpha}-1) \sum_{i=1}^n (A_i) + \hat{\theta} \hat{\alpha} \sum_{i=1}^n (A_i) (E_i)^{\hat{\alpha}} \\ + \hat{\alpha} \hat{\theta} \sum_{i=1}^{n-1} (m_i) (W_1) + \hat{\alpha} \hat{\theta} (k-1) (W_n) = 0 \end{aligned} \quad (3.8)$$

$$\frac{n}{\hat{\theta}} + n e^{\hat{\theta}} (1 - e^{\hat{\theta}})^{-1} + \sum_{i=1}^n (E_i)^{\hat{\alpha}} + \sum_{i=1}^{n-1} (m_i) (W_2) + (k-1) (W_n) = 0 \quad (3.9)$$

and

$$\begin{aligned} \frac{n\hat{\lambda}}{\hat{\beta}} - \hat{\lambda} \sum_{i=1}^n (A_1) + \hat{\lambda} (\hat{\alpha}-1) \sum_{i=1}^n (A_2) + \hat{\theta} \hat{\alpha} \hat{\lambda} \sum_{i=1}^n (A_2) (E_i)^{\hat{\alpha}} \\ + \hat{\alpha} \hat{\theta} \hat{\lambda} \sum_{i=1}^{n-1} m_i (A_2) (W_1) + \hat{\alpha} \hat{\theta} \hat{\lambda} (k-1) (A_2) (W_n) = 0 \end{aligned} \quad (3.10)$$

where,  $W_i = e^{\hat{\theta}(E_i)^{\hat{\alpha}}} (E_i)^{\hat{\alpha}} (E_{ii})^{-1}$ ,  $W_n = e^{\hat{\theta}(E_n)^{\hat{\alpha}}} (E_n)^{\hat{\alpha}} (E_{nn})^{-1}$ ,  $W_1 = (W_i)(A_i)$ ,

$A_i = e^{-(\hat{\beta} x_{(i)})^{\hat{\lambda}}} (T_i) (E_i)^{-1}$ ,  $A_n = e^{-(\hat{\beta} x_{(n)})^{\hat{\lambda}}} (T_n) (E_n)^{-1}$ ,  $W_1 = W_n (A_n)$ ,  $A_1 = x_{(i)} (\hat{\beta} x_{(i)})^{\hat{\lambda}-1}$ ,

$W_2 = [e^{\hat{\theta}} - e^{\hat{\theta}(E_i)^{\hat{\alpha}}} (E_i)^{\hat{\alpha}} - e^{\hat{\theta}} e^{\hat{\theta}(E_i)^{\hat{\alpha}}} + e^{\hat{\theta}} e^{\hat{\theta}(E_i)^{\hat{\alpha}}} (E_i)^{\hat{\alpha}}] [(e^{\hat{\theta}} - 1) (E_{ii})]^{-1}$ ,  $A_1 = x_{(n)} (\hat{\beta} x_{(n)})^{\hat{\lambda}-1}$

$$W_{2n} = \left[ e^{\hat{\theta}} - e^{\hat{\theta}(E_n)^{\hat{\alpha}}} (E_n)^{\hat{\alpha}} - e^{\hat{\theta}} e^{\hat{\theta}(E_n)^{\hat{\alpha}}} + e^{\hat{\theta}} e^{\hat{\theta}(E_n)^{\hat{\alpha}}} (E_n)^{\hat{\alpha}} \right] \left[ (e^{\hat{\theta}} - 1)(E_{in}) \right]^{-1},$$

$$A_{2i} = (A_{1i}) e^{-(\hat{\beta} x_{(i)})^{\hat{\lambda}}} (E_i)^{-1}, \text{ and } A_{2n} = (A_{1n}) e^{-(\hat{\beta} x_{(n)})^{\hat{\lambda}}} (E_n)^{-1}.$$

Since the closed form solution to nonlinear equations (3.7), (3.8), (3.9) and (3.10) are very hard to obtain. Therefore, an iterative procedure is applied to solve these equations numerically using Mathcad (14) statistical package.

In the following subsection, the progressively type II censored sample will be considered

### 3.1 Maximum Likelihood Estimation Based on Progressively Type II Censored Data

A progressively type II censored sample is observed as follows:  $n$  units are placed on a life testing experiment and only  $m \leq n$  are completely observed until failure. The censoring occurs progressively in  $m$  stages. The  $m$  stages are failure times of  $m$  completely observed units. At the time of the first failure (the first stage),  $R_1$  on  $(n-1)$  surviving units are randomly withdrawn from the experiment,  $R_2$  on  $(n-R_1-2)$  surviving units are withdrawn at the time of the second failure (the second stage) and so on. Finally, at the time of the  $m^{\text{th}}$  failure (the  $m^{\text{th}}$  stage), all the remaining  $(R_m - n - m - R_1 - \dots - R_{m-1})$  surviving units are withdrawn.

Burkschat *et al* (2007) referred this to as progressively type II censored scheme  $(R_1, R_2, \dots, R_m)$  then, the  $m$  completely observed failure times denoted by  $X_{i:m:n}^{(R_1, \dots, R_m)}$ ,  $i = 1, 2, \dots, m$ . The progressively type II censored sample  $X_{1:n:N}^{(R_1, \dots, R_m)}, \dots, X_{n:n:N}^{(R_1, \dots, R_m)}$ , with censoring scheme  $\tilde{R} = (R_1, R_2, \dots, R_n)$ , and  $R_i \in \mathbb{N}, 1 \leq i \leq n$  is special case of the generalized order statistics with the parameter  $m_i = R_i, i = 1, 2, \dots, n-1$  and  $k = \gamma_n = R_n + 1$ . Therefore, the log likelihood of progressively type II censored data in equation (3.2), is given by

$$\ln L = \ln C + n \ln \alpha + n \ln \lambda + n \ln \theta + n \ln \beta - (n) \ln(e^{\theta} - 1) + (\lambda - 1) \sum_{i=1}^n \ln x_{(i)} - \sum_{i=1}^n (\beta x_{(i)})^{\lambda} + (\alpha - 1) \sum_{i=1}^n \ln(E_i) + \theta \sum_{i=1}^n (E_i)^{\alpha} - \sum_{i=1}^n R_i \ln \left( \frac{e^{\theta(E_i)^{\alpha}} - 1}{e^{\theta} - 1} \right) \quad (3.11)$$

Maximum likelihood estimators of  $\alpha, \lambda, \theta$  and  $\beta$  are solutions to the system of equations obtained by letting the first partial derivatives of the total log likelihood be equal zero with respect to  $\alpha, \lambda, \theta$  and  $\beta$ , respectively. Therefore, the system of equations is as follows:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\hat{\alpha}_1} + \sum_{i=1}^n \ln(E_i) + \hat{\theta}_1 \sum_{i=1}^n (E_i)^{\hat{\alpha}_1} \ln(E_i) + \hat{\theta}_1 \sum_{i=1}^n (R_i) (W_i) \ln(E_i) = 0 \quad (3.12)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\hat{\lambda}_1} + n \ln \hat{\beta}_1 + \sum_{i=1}^n \ln x_{(i)} - \sum_{i=1}^n (T_i) + (\hat{\alpha}_1 - 1) \sum_{i=1}^n (A_i) + \hat{\theta}_1 \hat{\alpha}_1 \sum_{i=1}^n (A_i) (E_i)^{\hat{\alpha}_1} + \hat{\alpha}_1 \hat{\theta}_1 \sum_{i=1}^n (R_i) (W_1) = 0 \quad (3.13)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\hat{\theta}_1} + n e^{\hat{\theta}_1} (1 - e^{\hat{\theta}_1})^{-1} + \sum_{i=1}^n (E_i)^{\hat{\alpha}_1} - \sum_{i=1}^n (R_i) (W_2) = 0 \quad (3.14)$$

and

$$\frac{\partial \ln L}{\partial \beta} = \frac{n \hat{\lambda}_1}{\hat{\beta}_1} - \hat{\lambda}_1 \sum_{i=1}^n (A1_i) + \hat{\lambda}_1 (\hat{\alpha}_1 - 1) \sum_{i=1}^n (A2_i) + \hat{\theta}_1 \hat{\alpha}_1 \hat{\lambda}_1 \sum_{i=1}^n (A2_i) (E_i)^{\hat{\alpha}_1} + \hat{\alpha}_1 \hat{\theta}_1 \hat{\lambda}_1 \sum_{i=1}^n (R_i) (A2_i) (W_i) = 0 \quad (3.15)$$

Since the closed form solution to nonlinear equations (3.12) to (3.15) is very hard to obtain. Newton-Raphson method is applied for solving this nonlinear equations simultaneously to obtain  $\hat{\alpha}_1$ ,  $\hat{\lambda}_1$ ,  $\hat{\theta}_1$  and  $\hat{\beta}_1$ .

### 3.2. Interval Estimates of GOS

If  $L_\phi = L_\phi(y_1, \dots, y_n)$  and  $U_\phi = U_\phi(y_1, \dots, y_n)$  are function of the sample data  $x_1, \dots, x_n$  then a confidence interval for a population parameter  $\phi$  is given by

$$p[L_\phi \leq \phi \leq U_\phi] = \gamma$$

where,  $L_\phi$  and  $U_\phi$  are the lower and upper confidence limits which enclose  $\phi$  with probability  $\gamma$ . The interval  $[L_\phi, U_\phi]$  is called a  $100\gamma\%$  confidence interval for  $\phi$ .

For large sample size, the maximum likelihood estimates, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the approximate  $100\gamma\%$  confidence limits for the maximum likelihood estimate  $\hat{\phi}$  of a population parameter  $\phi$  can be constructed, such that

$$p[-z \leq \frac{\hat{\phi} - \phi}{\sigma(\hat{\phi})} \leq z] = \gamma$$

where,  $z$  is the  $[\frac{100(1-\gamma)}{2}]$  standard normal percentile. Therefore, the approximate  $100\gamma\%$  confidence limits for a population parameter  $\phi$  can be obtained, such that

$$p[\hat{\phi} - z \sigma(\hat{\phi}) \leq \phi \leq \hat{\phi} + z \sigma(\hat{\phi})] \cong \gamma \quad (3.16)$$

Then, the approximate confidence limits for  $\alpha, \lambda, \theta$  and  $\beta$  will be constructed using (3.16) with confidence levels 95 % and 99 % (see Table 3).

#### 4. Asymptotic Variances and Covariance's of Estimates

The asymptotic variances and covariance matrix of maximum likelihood estimates are given by the elements of the inverse of Fisher information matrix  $I_{ij}(\underline{\phi}) \cong E \left\{ -\partial^2 \ln L / \partial \theta_i \partial \theta_j \right\}$ . Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain. Therefore, Fisher information matrix is given by  $I_{ij}(\underline{\phi}) \cong \left\{ -\partial^2 \ln L / \partial \theta_i \partial \theta_j \right\}$ , which is obtained by dropping the expectation on operation  $E$  and replaced  $\alpha, \lambda, \theta$  and  $\beta$  with  $\hat{\alpha}, \hat{\lambda}, \hat{\theta}$  and  $\hat{\beta}$  respectively [Cohen (1965)]. The asymptotic variance covariance matrix  $F$  for the maximum likelihood estimates can be written as follows

$$F = [I_{ij}(\underline{\phi})], \quad i, j = 1, 2, 3, 4 \quad \text{and} \quad (\underline{\phi}) = (\alpha, \lambda, \theta, \beta)$$

The second partial derivatives of the maximum likelihood function are given as the following:

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha^2} &= -\frac{n}{\alpha^2} + \theta \sum_{i=1}^n [\ln(E_i)]^2 (E_i)^\alpha \\ &\quad - \theta \sum_{i=1}^{n-1} m_i \left\{ e^{\theta(E_i)^\alpha} (E_i)^\alpha [\ln(E_i)]^2 \left( e^\theta - e^{\theta(E_i)^\alpha} + \theta e^\theta (E_i)^\alpha \right) \right\} (E_i)^{-2} \\ &\quad - \theta(k-1) \left\{ e^{\theta(E_n)^\alpha} (E_n)^\alpha [\ln(E_n)]^2 \left( e^\theta - e^{\theta(E_n)^\alpha} + \theta e^\theta (E_n)^\alpha \right) \right\} (E_n)^{-2} \end{aligned} \quad (4.1)$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \lambda^2} &= -\frac{n}{\lambda^2} - \sum_{i=1}^n (T_i) \ln(\beta x_{(i)}) - (\alpha-1) \sum_{i=1}^n e^{-(\beta x_{(i)})^\lambda} (T_i) \ln(\beta x_{(i)}) (A3_i) \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right)^{-2} \\ &\quad - \theta \alpha \sum_{i=1}^n e^{-(\beta x_{(i)})^\lambda} (T_i) \ln(\beta x_{(i)}) (E_i)^\alpha \left[ (A3_i) - \alpha e^{-(\beta x_{(i)})^\lambda} (\beta x_{(i)})^\lambda \right] \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right)^{-2} \\ &\quad - \theta \alpha \sum_{i=1}^{n-1} m_i (A4_i) [a_1 - a_2 + a_3 + a_4 - a_5 - a_6 - a_7 + a_8 + a_9] \\ &\quad - \theta \alpha (k-1) (A4_n) [a_{10} - a_{11} + a_{12} + a_{13} - a_{14} - a_{15} - a_{16} + a_{17} + a_{18}] \end{aligned} \quad (4.2)$$

$$\text{where, } a_1 = e^{2\theta(E_i)^\alpha} e^{-2(\beta x_{(i)})^\lambda}, a_2 = e^{2\theta(E_i)^\alpha} e^{-(\beta x_{(i)})^\lambda}, a_3 = a_2 (\beta x_{(i)})^\lambda, a_4 = e^\theta e^{\theta(E_i)^\alpha} e^{-(\beta x_{(i)})^\lambda},$$

$$a_5 = e^\theta e^{\theta(E_i)^\alpha} e^{-2(\beta x_{(i)})^\lambda}, a_6 = \alpha a_1 (\beta x_{(i)})^\lambda, a_7 = a_4 (\beta x_{(i)})^\lambda, a_8 = \alpha a_5 (\beta x_{(i)})^\lambda, a_9 = \theta a_8 (E_i)^\alpha,$$

$$a_{10} = e^{2\theta(E_n)^\alpha} e^{-2(\beta x_{(n)})^\lambda}, a_{11} = e^{2\theta(E_n)^\alpha} e^{-(\beta x_{(n)})^\lambda}, a_{12} = a_{11} (\beta x_{(n)})^\lambda, a_{13} = e^\theta e^{\theta(E_n)^\alpha} e^{-(\beta x_{(n)})^\lambda},$$

$$a_{14} = e^\theta e^{\theta(E_n)^\alpha} e^{-2(\beta x_{(n)})^\lambda}, a_{15} = \alpha a_{10} (\beta x_{(n)})^\lambda, a_{16} = a_{13} (\beta x_{(n)})^\lambda, a_{17} = \alpha a_{14} (\beta x_{(n)})^\lambda,$$



$$a_{13} = \theta a_{17} (E_n)^\alpha, \quad A3_i = \left[ e^{-(\beta x_{(i)})^\lambda} + (\beta x_{(i)})^\lambda - 1 \right],$$

$$A4_i = (E_i)^\alpha (T_i) \ln(\beta x_{(i)}) \left\{ \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right)^{-2} (E_i)^{-2} \right\}, \text{ and}$$

$$A4_n = (E_n)^\alpha (T_n) \ln(\beta x_{(n)}) \left\{ \left( e^{-(\beta x_{(n)})^\lambda} - 1 \right)^{-2} (E_n)^{-2} \right\}.$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \theta^2} = & -\frac{n}{\theta^2} + ne^\theta (e^\theta - 1)^{-2} + \sum_{i=1}^{n-1} m_i \left\{ \frac{b_1 - b_2 + b_3 - b_4 - b_5 - b_6 + b_7 + b_8 - b_9 + b_{10}}{(E_i)^2 (e^\theta - 1)^2} \right\} \\ & + (k-1) \left\{ (E_n)^{-2} (e^\theta - 1)^{-2} \right\} [b_1 - b_{11} + b_{12} - b_{13} - b_{14} - b_{15} + b_{16} + b_{17} - b_{18} + b_{19}] \end{aligned} \quad (4.3)$$

where,  $b_1 = e^{3\theta}$ ,  $b_2 = e^\theta e^{\theta(E_i)^\alpha}$ ,  $b_3 = e^\theta e^{2\theta(E_i)^\alpha}$ ,  $b_4 = b_1 e^{\theta(E_i)^\alpha}$ ,  $b_5 = 4e^{2\theta} e^{\theta(E_i)^\alpha} (E_i)^\alpha$ ,  
 $b_6 = e^\theta e^{\theta(E_i)^\alpha} (E_i)^{2\alpha}$ ,  $b_7 = 2b_1 e^{\theta(E_i)^\alpha} (E_i)^\alpha$ ,  $b_8 = 2e^{2\theta} e^{\theta(E_i)^\alpha} (E_i)^{2\alpha}$ ,  $b_9 = b_1 e^{\theta(E_i)^\alpha} (E_i)^{2\alpha}$ ,  
 $b_{10} = 2b_2 (E_i)^\alpha$ ,  $b_{11} = e^\theta e^{\theta(E_n)^\alpha}$ ,  $b_{12} = e^\theta e^{2\theta(E_n)^\alpha}$ ,  $b_{13} = b_1 e^{\theta(E_n)^\alpha}$ ,  $b_{14} = 4e^{2\theta} e^{\theta(E_n)^\alpha} (E_n)^\alpha$ ,  
 $b_{15} = e^\theta e^{\theta(E_n)^\alpha} (E_n)^{2\alpha}$ ,  $b_{16} = 2b_1 e^{\theta(E_n)^\alpha} (E_n)^\alpha$ ,  $b_{17} = 2e^{2\theta} e^{\theta(E_n)^\alpha} (E_n)^{2\alpha}$ ,  $b_{18} = b_1 e^{\theta(E_n)^\alpha} (E_n)^{2\alpha}$ ,  
and  $b_{19} = 2b_{11} (E_n)^\alpha$ .

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta^2} = & -\frac{n\lambda}{\beta^2} - \lambda(\lambda-1) \sum_{i=1}^n (x_{(i)})^2 (\beta x_{(i)})^{\lambda-2} - \lambda(\alpha-1) \sum_{i=1}^n (d_i) [A] - \alpha\theta\lambda(k-1)(A5_n)(D_2) \\ & + \theta\alpha\lambda \sum_{i=1}^n (E_i)^\alpha (d_i) \left[ \alpha\lambda e^{-(\beta x_{(i)})^\lambda} (\beta x_{(i)})^\lambda - A \right] - \alpha\theta\lambda \sum_{i=1}^{n-1} (A5_i)(D_1) \end{aligned} \quad (4.4)$$

where,  $A = \lambda(\beta x_{(i)})^\lambda - e^{-(\beta x_{(i)})^\lambda} - \lambda + \lambda e^{-(\beta x_{(i)})^\lambda} + 1$ ,  $a_2 = e^{2\theta(E_i)^\alpha} e^{-(\beta x_{(i)})^\lambda}$ ,

$$a_1 = e^{2\theta(E_i)^\alpha} e^{-2(\beta x_{(i)})^\lambda}, \quad d_1 = \lambda a_2, \quad d_2 = \lambda a_1 a_4 = e^\theta e^{\theta(E_i)^\alpha} e^{-(\beta x_{(i)})^\lambda}, \quad a_5 = e^\theta e^{\theta(E_i)^\alpha} e^{-2(\beta x_{(i)})^\lambda},$$

$$d_3 = \lambda a_3, \quad d_4 = \lambda a_4, \quad d_5 = \lambda a_5, \quad d_6 = \lambda a_6, \quad d_7 = \lambda a_7, \quad d_8 = \lambda a_8, \quad d_9 = \lambda a_9,$$

$$a_{11} = e^{2\theta(E_n)^\alpha} e^{-(\beta x_{(n)})^\lambda}, \quad a_{10} = e^{2\theta(E_n)^\alpha} e^{-2(\beta x_{(n)})^\lambda}, \quad d_{10} = \lambda a_{11}, \quad d_{11} = \lambda a_{10}, \quad a_{13} = e^\theta e^{\theta(E_n)^\alpha} e^{-(\beta x_{(n)})^\lambda},$$

$$a_{14} = e^\theta e^{\theta(E_n)^\alpha} e^{-2(\beta x_{(n)})^\lambda}, \quad d_{12} = \lambda a_{12}, \quad d_{13} = \lambda a_{13}, \quad d_{14} = \lambda a_{14}, \quad d_{15} = \lambda a_{15}, \quad d_{16} = \lambda a_{16}, \quad d_{17} = \lambda a_{17},$$

$$\begin{aligned}
 d_{18} &= \lambda a_{18}, \quad d_i = e^{-(\beta x_{(i)})^\lambda} (\beta x_{(i)})^\lambda \left\{ \beta^{-2} \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right)^{-2} \right\}, \\
 AS_i &= m_i (E_i)^\alpha (\beta x_{(i)})^\lambda \left\{ \beta^{-2} \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right)^{-2} (E_i)^{-2} \right\}, \\
 AS_n &= (E_n)^\alpha (\beta x_{(n)})^\lambda \left\{ \beta^{-2} \left( e^{-(\beta x_{(n)})^\lambda} - 1 \right)^{-2} (E_n)^{-2} \right\}, \\
 D_1 &= [a_2 - a_1 - d_1 + d_2 - a_4 + a_5 + d_3 + d_4 - d_5 - d_6 - d_7 + d_8 + d_9], \text{ and} \\
 D_2 &= [a_{11} - a_{10} - d_{10} + d_{11} - a_{13} + a_{14} + d_{12} + d_{13} - d_{14} - d_{15} - d_{16} + d_{17} + d_{18}]. \\
 \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} &= \sum_{i=1}^n e^{-(\beta x_{(i)})^\lambda} (T_i)(E_i)^{-1} + \theta \sum_{i=1}^n \left\{ e^{-(\beta x_{(i)})^\lambda} \ln(\beta x_{(i)}) [\alpha \ln(E_i) + 1] (E_i)^\alpha (\beta x_{(i)})^\lambda \right\} (E_i)^{-1} \\
 &\quad + \theta \sum_{i=1}^{n-1} m_i (T_i)(E_i)^\alpha \left\{ \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right) (E_i)^2 \right\}^{-1} [a_4 - a_2 - h_1 + h_2 + h_3] \\
 &\quad + \theta (k-1) (T_n)(E_n)^\alpha \left\{ \left( e^{-(\beta x_{(n)})^\lambda} - 1 \right) (E_n)^2 \right\}^{-1} [a_{13} - a_{11} - h_4 + h_5 + h_6]
 \end{aligned} \tag{4.5}$$

where,  $h_1 = \alpha a_2 \ln(E_i)$ ,  $h_2 = \alpha a_4 \ln(E_i)$ ,  $h_3 = \theta h_2 (E_i)^\alpha$ ,  $h_4 = \alpha a_{11} \ln(E_n)$ ,  $h_5 = \alpha a_{13} \ln(E_n)$ , and  $h_6 = \theta h_5 (E_n)^\alpha$ .

$$\begin{aligned}
 \frac{\partial^2 \ln L}{\partial \alpha \partial \theta} &= \sum_{i=1}^n (E_i)^\alpha \ln(E_i) + \sum_{i=1}^{n-1} \frac{m_i (E_i)^\alpha \ln(E_i)}{(1-e^\theta)(E_i)^2} [h_7 - h_8 - h_9 + h_{10} - h_{11} + h_{12} - h_{13} + h_{14}] \\
 &\quad + (k-1) (E_n)^\alpha \ln(E_n) \left[ (1-e^\theta)(E_n)^2 \right]^{-1} [h_{15} - h_{16} - h_{17} + h_{18} - h_{19} + h_{20} - h_{21} + h_{22}]
 \end{aligned} \tag{4.6}$$

where,  $h_7 = e^{2\theta(E_i)^\alpha}$ ,  $h_8 = e^\theta e^{\theta(E_i)^\alpha}$ ,  $h_9 = e^\theta h_7$ ,  $h_{10} = e^{2\theta} e^{\theta(E_i)^\alpha}$ ,  $h_{11} = \theta h_{10}$ ,  $h_{12} = \theta h_8$ ,

$h_{13} = h_{12} (E_i)^\alpha$ ,  $h_{14} = h_{11} (E_i)^\alpha$ ,  $h_{15} = e^{2\theta(E_n)^\alpha}$ ,  $h_{16} = e^\theta e^{\theta(E_n)^\alpha}$ ,  $h_{17} = e^\theta h_{15}$ ,  $h_{18} = e^{2\theta} e^{\theta(E_n)^\alpha}$ ,

$h_{19} = \theta h_{18}$ ,  $h_{20} = \theta h_{16}$ ,  $h_{21} = h_{20} (E_n)^\alpha$ , and  $h_{22} = h_{19} (E_n)^\alpha$ .

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} &= \lambda \sum_{i=1}^n (x_{(i)}) e^{-(\beta x_{(i)})^\lambda} (\beta x_{(i)})^{\lambda-1} (E_i)^{-1} \\ &\quad + \theta \lambda \sum_{i=1}^n e^{-(\beta x_{(i)})^\lambda} [\alpha \ln(E_i) + 1] (E_i)^\alpha (\beta x_{(i)})^\lambda \left\{ \beta - \beta e^{-(\beta x_{(i)})^\lambda} \right\}^{-1} \\ &\quad - \theta \lambda \sum_{i=1}^{n-1} m_i (E_i)^\alpha (\beta x_{(i)})^\lambda \left\{ \beta (E_i) (E_u)^2 \right\}^{-1} [a_4 - a_2 - h_1 + h_2 + h_3] \\ &\quad - \theta \lambda (k-1) (E_n)^\alpha (\beta x_{(n)})^\lambda \left\{ \beta (E_n) (E_{ln})^2 \right\}^{-1} [a_{13} - a_{11} - h_4 + h_5 + h_6] \end{aligned} \quad (4.7)$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \lambda \partial \theta} &= \alpha \sum_{i=1}^n e^{-(\beta x_{(i)})^\lambda} (T_i) (E_i)^{\alpha-1} + \alpha \sum_{i=1}^{n-1} m_i (L_i) [h_7 - h_8 - h_9 + h_{10} - h_{11} + h_{12} - h_{13} + h_{14}] \\ &\quad + \alpha (k-1) (L_2) [h_{15} - h_{16} - h_{17} + h_{18} - h_{19} + h_{20} - h_{21} + h_{22}] \end{aligned} \quad (4.8)$$

where,  $L_1 = e^{-(\beta x_{(i)})^\lambda} (T_i) (E_i)^\alpha \left[ (e^\theta - 1) \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right) (E_u)^2 \right]^{-1}$  and

$$L_2 = e^{-(\beta x_{(n)})^\lambda} (T_n) (E_n)^\alpha \left[ (e^\theta - 1) \left( e^{-(\beta x_{(n)})^\lambda} - 1 \right) (E_{ln})^2 \right]^{-1}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \lambda \partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^n [\lambda \ln(\beta x_{(i)}) + 1] (\beta x_{(i)})^\lambda (\beta)^{-1} - (\alpha-1) \sum_{i=1}^n e^{-(\beta x_{(i)})^\lambda} (\beta x_{(i)})^\lambda (L_3) \left\{ \beta \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right)^2 \right\}^{-1} \\ &\quad - \theta \alpha \sum_{i=1}^n e^{-(\beta x_{(i)})^\lambda} (\beta x_{(i)})^\lambda (E_i)^\alpha (L_4) \left\{ \beta \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right)^2 \right\}^{-1} \\ &\quad + \alpha \theta \sum_{i=1}^{n-1} m_i [S_1 - S_2 + S_3 + S_4 + S_5 - S_6] + \alpha \theta (k-1) [S_7 - S_8 + S_9 + S_{10} + S_{11} - S_{12}] \end{aligned} \quad (4.9)$$

where,  $L_3 = e^{-(\beta x_{(i)})^\lambda} - \lambda \ln(\beta x_{(i)}) + \lambda e^{-(\beta x_{(i)})^\lambda} \ln(\beta x_{(i)}) + \lambda (T_i) - 1$ ,  $L_4 = L_3 - \alpha \lambda e^{-(\beta x_{(i)})^\lambda} (T_i)$ ,

$$S_1 = \alpha \theta \lambda (E_i)^\alpha (T_i) (\beta x_{(i)})^\lambda e^{-2(\beta x_{(i)})^\lambda} e^{\theta(E_i)^\alpha} (E_i)^{\alpha-1} [\beta (E_i) (E_u)]^{-1},$$

$$S_2 = \lambda e^{\theta(E_i)^\alpha} (T_i) (\beta x_{(i)})^\lambda e^{-(\beta x_{(i)})^\lambda} (E_i)^{\alpha-1} [\beta (E_u)]^{-1}, S_3 = S_2 (\beta x_{(i)})^{-\lambda},$$

$$S_4 = e^{\theta(E_i)^\alpha} (\beta x_{(i)})^\lambda e^{-(\beta x_{(i)})^\lambda} (E_i)^{\alpha-1} [\beta (E_u)]^{-1},$$

$$S_5 = \lambda (\alpha-1) e^{\theta(E_i)^\alpha} (T_i) (\beta x_{(i)})^\lambda e^{-2(\beta x_{(i)})^\lambda} (E_i)^{\alpha-1} [\beta (E_i) (E_u)]^{-1},$$

$$\begin{aligned}
 S_6 &= \alpha \theta \lambda e^{2\theta(E_i)^\alpha} e^{-2(\beta x_{(i)})^\lambda} (T_i) (\beta x_{(i)})^\lambda (E_i)^{2\alpha-1} [\beta(E_i)(E_{ii})^2]^{-1}, \\
 S_7 &= \alpha \theta \lambda (E_n)^\alpha (T_n) (\beta x_{(n)})^\lambda e^{-2(\beta x_{(n)})^\lambda} e^{\theta(E_n)^\alpha} (E_n)^{\alpha-1} [\beta(E_n)(E_{nn})]^{-1}, \\
 S_8 &= \lambda e^{\theta(E_n)^\alpha} (T_n) (\beta x_{(n)})^\lambda e^{-(\beta x_{(n)})^\lambda} (E_n)^{\alpha-1} [\beta(E_{nn})]^{-1}, S_9 = S_8 (\beta x_{(n)})^{-\lambda}, \\
 S_{10} &= e^{\theta(E_n)^\alpha} (\beta x_{(n)})^\lambda e^{-(\beta x_{(n)})^\lambda} (E_n)^{\alpha-1} [\beta(E_{nn})]^{-1}, \\
 S_{11} &= \lambda(\alpha-1) e^{\theta(E_n)^\alpha} (T_n) (\beta x_{(n)})^\lambda e^{-2(\beta x_{(n)})^\lambda} (E_n)^{\alpha-1} [\beta(E_n)(E_{nn})]^{-1}, \text{ and} \\
 S_{12} &= \alpha \theta \lambda e^{2\theta(E_n)^\alpha} e^{-2(\beta x_{(n)})^\lambda} (T_n) (\beta x_{(n)})^\lambda (E_n)^{2\alpha-1} [\beta(E_n)(E_{nn})^2]^{-1} \\
 \frac{\partial^2 \ln L}{\partial \theta \partial \beta} &= \alpha \lambda \sum_{i=1}^n (x_{(i)}) e^{-(\beta x_{(i)})^\lambda} (E_i)^{\alpha-1} (\beta x_{(i)})^{\lambda-1} + \alpha \lambda \sum_{i=1}^{n-1} m_i [L_5 (h_7 - h_8 - h_9 + h_{10} - h_{11} + h_{12} - h_{13} + h_{14})] \\
 &\quad + \alpha \lambda (k-1) [L_6 (h_{15} - h_{16} - h_{17} + h_{18} - h_{19} + h_{20} - h_{21} + h_{22})]
 \end{aligned} \tag{4.10}$$

where,  $L_5 = e^{-(\beta x_{(i)})^\lambda} (E_i)^\alpha (\beta x_{(i)})^\lambda \left[ \beta(e^\theta - 1) \left( e^{-(\beta x_{(i)})^\lambda} - 1 \right) (E_{ii})^2 \right]^{-1}$ , and

$$L_6 = e^{-(\beta x_{(n)})^\lambda} (E_n)^\alpha (\beta x_{(n)})^\lambda \left[ \beta(e^\theta - 1) \left( e^{-(\beta x_{(n)})^\lambda} - 1 \right) (E_{nn})^2 \right]^{-1}$$

Consequently, the maximum likelihood estimators of  $\alpha, \lambda, \theta$  and  $\beta$  have an asymptotic variance covariance matrix defined by inverting the Fisher information matrix  $F$  and by substituting  $\hat{\alpha}$  for  $\alpha$ ,  $\hat{\lambda}$  for  $\lambda$ ,  $\hat{\theta}$  for  $\theta$  and  $\hat{\beta}$  for  $\beta$ .

## 5. Simulation Studies

In any estimation problems, it is required to study the properties of the derived estimators. The derived expressions for the estimators are too complicated to study analytically. Consequently, a simulation study will be set up, treating separately the sampling distribution of the estimators. Simulation studies have been performed using Mathcad (14) statistical package for illustrating the theoretical results. The performance of the resulting estimators of the unknown parameters has been considered in terms of their absolute relative bias (RABias), mean square error (MSE), and relative error (RE). Furthermore, the asymptotic variance and covariance matrix and two-sided confidence intervals of the unknown parameters are obtained. The Simulation procedures will described below:

**Step 1:** 1000 random samples of sizes 30, 40, 50, 80, 100 are generated from EWP distribution. The true parameters selected values are  $(\alpha=1, \lambda=1, \theta=1.5, \beta=0.75, k=1.05, m=0.05)$  and  $(\alpha=0.25, \lambda=1, \theta=1.45, \beta=0.6, k=1.025, m=0.04)$ .

**Step 2:** For each sample and for the two sets of parameters of distribution are estimated based on generalized order statistics.

**Step 3:** Newton-Raphson method was used for solving the four nonlinear likelihood for  $\alpha, \lambda, \theta$  and  $\beta$  given in (3.3), (3.4), (3.5) and (3.6), respectively to obtain MLEs based on GOS.

**Step 4:** Newton-Raphson method was used for solving the four nonlinear likelihood for  $\alpha, \lambda, \theta$  and  $\beta$  given in (3.12) to (3.15), respectively to obtain MLEs based on progressive type II censored.

**Step 5:** The RABias, MSE, and RE of the estimators for the unknown parameters for all sample sizes and for two sets of parameters are tabulated (see Table 1).

**Step 6:** The two sided confidence limit with confidence level  $\gamma=0.95$  and  $\gamma=0.99$  of the parameters are constructed based on GOS (see Tables 3).

Simulation results are summarized in Tables 1, 2 and 3. Table 1 gives the RABias, MSE, and RE of the estimators based on generalized order statistics. Also, Table 2 gives the RABias, MSE, and RE of the estimators under progressive type II censored. Table 3 gives the approximated confidence limits at 95 % and 99 % for the estimated parameters based on GOS.

From these Tables, the following observations can be made on the performance of estimated parameters of EWP lifetime distribution based on GOS and progressive type II censored samples:

1. The second case of parameters  $(\alpha=0.25, \lambda=1, \theta=1.45, \beta=0.6)$  when  $(k=1.025, m=0.04)$ , the maximum likelihood estimators under generalized order statistics have good statistical properties than the first set of parameters  $(\alpha=0.45, \lambda=1, \theta=2, \beta=1)$  when  $(k=1.05, m=0.05)$  for all sample sizes (see Table1).
2. The maximum likelihood estimators of first case of parameters  $(\alpha=0.45, \lambda=1, \theta=2, \beta=1)$  when  $(k=1.05, m=0.05)$  have good statistical properties than the second set of parameters  $(\alpha=0.25, \lambda=1, \theta=1.45, \beta=0.6)$  when  $(k=1.025, m=0.04)$  under progressive type II censored (see Table 2).
3. The interval of the estimators decreases when the sample size is increasing. Also, the interval of the estimator at  $\gamma=0.95$  is shorter than the interval of estimator at  $\gamma=0.99$  (see Table3).

## 6. Conclusion

This study deals with the GOS from EWP distribution. The maximum likelihood estimators of unknown parameters of EWP will be obtained. In addition, the interval estimates of

the unknown parameter will be obtained for large sample. In particular, the maximum likelihood estimators will be obtained based on progressive type II censored samples. Monte Carlo simulation will be performed to illustrate theoretical results. Accordingly, the second set of parameters have good statistical properties than the first set of parameters for all sample sizes under GOS. While the first set of parameters have good statistical properties than the second set of parameters for all sample sizes under progressive type II censored. Maximum likelihood estimators are consistent and asymptotically normally distributed. Regarding the interval of estimators, it can be noted that the interval of the estimators at  $\gamma = 0.95$  is shorter than the interval of estimator at  $\gamma = 0.99$ . Also, as the sample size increases the interval of the estimators decreases for the two confidence level.

## References

- 1) Abd-Ellah, A. H. (2012). "Bayesian and non-Bayesian estimation of the inverse Weibull model based on generalized order statistics". *Intelligent Information Management*, 4, 23 – 31.
- 2) Aboelenen, Z. A. (2010). "Inference for Weibull Distribution Under Generalized Order Statistics". *Mathematics and Computers in Simulation*, 8, 26 – 36.
- 3) Abo-Elfotouh, S., and Nassar, M. M. (2011). "Estimation for the Parameters of the Weibull Extension Model Based on Generalized Order Statistics". *International Journal of Contemporary Mathematical Sciences*, 6 (36), 1749 – 1760.
- 4) Ahsanullah, M. (1996). "Generalized Order Statistics from Two Parameter Uniform Distribution". *Communication in Statistics – Theory and Methods*, 25 (10), 2311 – 2318.
- 5) Ahsanullah, M. (2000). "Generalized Order Statistics from Exponential Distribution". *Journal of Statistical Planning and Inference*, 85, 85 – 91.
- 6) Ateya, S. F., and Ahmad, A. E. (2011). " Inference Based on Generalized Order Statistics Under Truncated Type I Generalized Logistic Distribution ". *Statistic*, 45 (4), 389 – 402.
- 7) Burkschat, M. (2010). " Linear Estimators and Predictors Based on Generalized Order Statistics from Generalized Pareto distributions". *Communication in Statistics – Theory and Methods*, 39 (2), 311 – 326.
- 8) Cohen, A. C. (1965). "Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and Censored Samples". *Technometrics* , 5, 579 – 588.
- 9) Habibullah, M., and Ahsanullah. M. (2000). "Estimation of Parameters of a Pareto Distribution by Generalized Order Statistics". *Communication in Statistics – Theory and Methods*, 29 (7), 1597 – 1609.
- 10) Jaheen, Z. F. (2002). "On Bayesian Prediction of Generalized Order Statistics". *Journal of Statistical Theory and Applications*, 1 (3), 191 – 204.
- 11) Jaheen, Z. F. (2005). "Estimation Based on Generalized Order Statistics from the Burr Model". *Communication in Statistics – Theory and Methods*, 34, 785 – 794.
- 12) Kamps, U. (1995). "A Concept of Generalized Order Statistics". *Journal of Statistical Planning and Inference*, 48, 1 – 23.

- 13) Mahmoudi, E. and Sepahdar, A. (2012). "Exponentiated Weibull-Poisson distribution: model, properties and applications". Mathematics and Computers in Simulation, 92, 76 – 97.
- 14) Mailinowska. I., Pawlas, P., and Szynal, D. (2006). "Estimation of Location and Scale Parameters for the Burr XII Using Generalized Order Statistics". Linear Algebra and its Applications, 417, 150 – 162.

### Appendix

**Table 1: The RABias, MSE and RE of the Estimators of the Parameters  $(\alpha, \lambda, \theta, \beta)$  for different sample sizes Based on Generalized Order Statistics.**

| $n$ | Parameters<br>$(\alpha, \lambda, \theta, \beta)$ | Case 1 $k = 1.05, m = 0.05$<br>(0.45, 1, 2, 1) |       |       | Case 2 $k = 1.025, m = 0.04$<br>(0.25, 1, 1.45, 0.6) |        |       |
|-----|--------------------------------------------------|------------------------------------------------|-------|-------|------------------------------------------------------|--------|-------|
|     |                                                  | RABias                                         | MSE   | RE    | RABias                                               | MSE    | RE    |
|     |                                                  |                                                |       |       |                                                      |        |       |
| 30  | $\alpha$                                         | 0.259                                          | 0.015 | 0.274 | 0.058                                                | 0.002  | 0.165 |
|     | $\lambda$                                        | 0.232                                          | 0.055 | 0.234 | 0.165                                                | 0.030  | 0.173 |
|     | $\theta$                                         | 0.198                                          | 0.227 | 0.238 | 0.512                                                | 1.117  | 0.729 |
|     | $\beta$                                          | 0.451                                          | 0.214 | 0.463 | 0.578                                                | 0.206  | 0.756 |
| 40  | $\alpha$                                         | 0.007                                          | 0.002 | 0.075 | 0.033                                                | 0.001  | 0.137 |
|     | $\lambda$                                        | 0.196                                          | 0.039 | 0.198 | 0.107                                                | 0.0130 | 0.116 |
|     | $\theta$                                         | 0.280                                          | 0.365 | 0.302 | 0.624                                                | 1.238  | 0.767 |
|     | $\beta$                                          | 0.212                                          | 0.053 | 0.230 | 0.579                                                | 0.184  | 0.716 |
| 50  | $\alpha$                                         | 0.061                                          | 0.002 | 0.090 | 0.050                                                | 0.001  | 0.142 |
|     | $\lambda$                                        | 0.187                                          | 0.036 | 0.189 | 0.131                                                | 0.019  | 0.138 |
|     | $\theta$                                         | 0.328                                          | 0.469 | 0.343 | 0.539                                                | 1.030  | 0.700 |
|     | $\beta$                                          | 0.152                                          | 0.029 | 0.171 | 0.432                                                | 0.130  | 0.601 |
| 80  | $\alpha$                                         | 0.115                                          | 0.003 | 0.125 | 0.020                                                | 0.001  | 0.111 |
|     | $\lambda$                                        | 0.079                                          | 0.007 | 0.081 | 0.050                                                | 0.004  | 0.062 |
|     | $\theta$                                         | 0.383                                          | 0.607 | 0.390 | 0.580                                                | 0.983  | 0.684 |
|     | $\beta$                                          | 0.105                                          | 0.014 | 0.119 | 0.409                                                | 0.101  | 0.530 |
| 100 | $\alpha$                                         | 0.097                                          | 0.002 | 0.104 | 0.028                                                | 0.001  | 0.093 |
|     | $\lambda$                                        | 0.016                                          | 0.001 | 0.022 | 0.010                                                | 0.001  | 0.031 |
|     | $\theta$                                         | 0.357                                          | 0.522 | 0.361 | 0.528                                                | 0.768  | 0.604 |
|     | $\beta$                                          | 0.113                                          | 0.015 | 0.122 | 0.308                                                | 0.0620 | 0.414 |

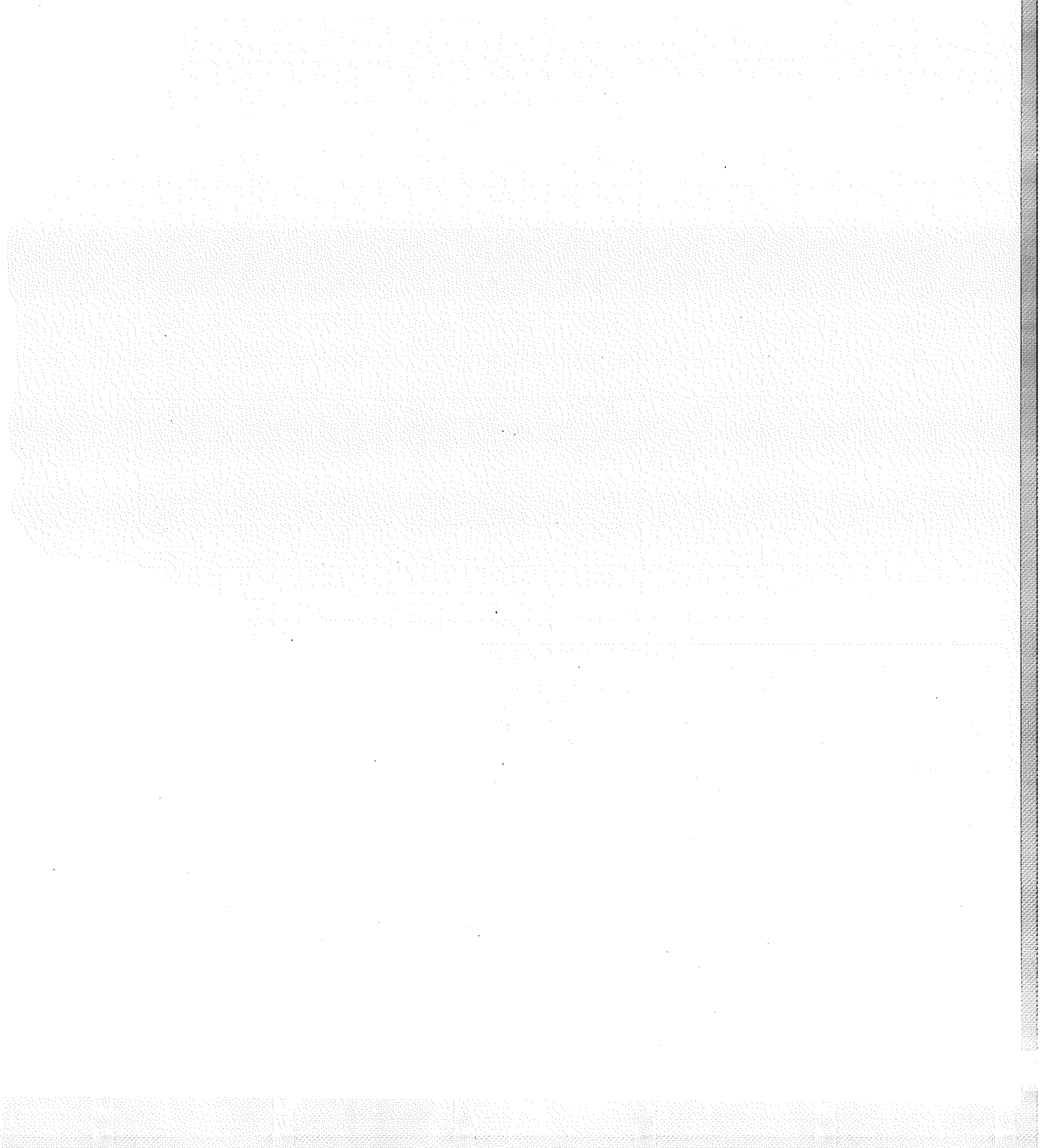
Table 2: The RABias, MSE and RE of the Estimators of the Parameters  $(\alpha, \lambda, \theta, \beta)$  for different sample sizes Based on Progressive Type II Censored.

| $n$ | Parameters<br>$(\alpha, \lambda, \theta, \beta)$ | Case 1 $k = 1.05, m = 0.05$<br>(0.45, 1, 2, 1) |       |       | Case 2 $k = 1.025, m = 0.04$<br>(0.25, 1, 1.45, 0.6) |       |       |
|-----|--------------------------------------------------|------------------------------------------------|-------|-------|------------------------------------------------------|-------|-------|
|     |                                                  | RABias                                         | MSE   | RE    | RABias                                               | MSE   | RE    |
|     |                                                  |                                                |       |       |                                                      |       |       |
| 30  | $\alpha$                                         | 0.564                                          | 0.023 | 0.611 | 0.437                                                | 0.035 | 0.750 |
|     | $\lambda$                                        | 0.803                                          | 1.051 | 1.025 | 1.509                                                | 2.606 | 1.614 |
|     | $\theta$                                         | 0.014                                          | 0.071 | 0.184 | 0.976                                                | 2.024 | 0.981 |
|     | $\beta$                                          | 0.965                                          | 0.336 | 0.966 | 0.369                                                | 0.117 | 0.569 |
| 40  | $\alpha$                                         | 0.538                                          | 0.020 | 0.568 | 0.419                                                | 0.020 | 0.570 |
|     | $\lambda$                                        | 0.854                                          | 0.977 | 0.999 | 1.562                                                | 2.835 | 1.684 |
|     | $\theta$                                         | 0.051                                          | 0.055 | 0.161 | 0.878                                                | 1.648 | 0.885 |
|     | $\beta$                                          | 0.965                                          | 0.335 | 0.965 | 0.311                                                | 0.082 | 0.478 |
| 50  | $\alpha$                                         | 0.520                                          | 0.018 | 0.544 | 0.428                                                | 0.026 | 0.642 |
|     | $\lambda$                                        | 0.800                                          | 0.855 | 0.925 | 1.469                                                | 2.492 | 1.579 |
|     | $\theta$                                         | 0.112                                          | 0.065 | 0.176 | 0.954                                                | 1.943 | 0.961 |
|     | $\beta$                                          | 0.965                                          | 0.336 | 0.966 | 0.425                                                | 0.116 | 0.567 |
| 80  | $\alpha$                                         | 0.507                                          | 0.017 | 0.529 | 0.388                                                | 0.021 | 0.583 |
|     | $\lambda$                                        | 0.848                                          | 0.911 | 0.204 | 1.074                                                | 1.967 | 1.403 |
|     | $\theta$                                         | 0.132                                          | 0.070 | 0.183 | 0.692                                                | 1.041 | 0.704 |
|     | $\beta$                                          | 0.968                                          | 0.339 | 0.970 | 0.308                                                | 0.057 | 0.398 |
| 100 | $\alpha$                                         | 0.490                                          | 0.017 | 0.515 | 0.133                                                | 0.007 | 0.358 |
|     | $\lambda$                                        | 0.816                                          | 0.858 | 0.926 | 0.929                                                | 1.160 | 1.077 |
|     | $\theta$                                         | 0.165                                          | 0.087 | 0.204 | 0.141                                                | 0.074 | 0.187 |
|     | $\beta$                                          | 0.965                                          | 0.337 | 0.968 | 0.037                                                | 0.034 | 0.308 |



Table 3: Confidence Bounds of the Parameters at Confidence level 0.95 and 0.99

| $n$ | Parameters<br>( $\alpha, \lambda, \theta, \beta$ ) | Case 1 $k = 1.05, m = 0.05$ |                |                | Case 2 $k = 1.025, m = 0.04$ |                |                |
|-----|----------------------------------------------------|-----------------------------|----------------|----------------|------------------------------|----------------|----------------|
|     |                                                    | (0.45, 1, 2, 1)             |                |                | (0.25, 1, 1.45, 0.6)         |                |                |
|     |                                                    | Standard deviation          | Lower Bound    | Upper Bound    | Standard deviation           | Lower Bound    | Upper Bound    |
| 30  | $\alpha$                                           | 0.039                       | 0.490<br>0.466 | 0.643<br>0.667 | 0.039                        | 0.189<br>0.165 | 0.340<br>0.364 |
|     | $\lambda$                                          | 0.034                       | 1.166<br>1.145 | 1.298<br>1.318 | 0.051                        | 1.065<br>1.033 | 1.266<br>1.298 |
|     | $\theta$                                           | 0.265                       | 1.878<br>1.714 | 2.915<br>3.079 | 0.752                        | 0.719<br>0.253 | 3.666<br>4.133 |
|     | $\beta$                                            | 0.102                       | 1.252<br>1.189 | 1.651<br>1.714 | 0.292                        | 0.374<br>0.193 | 1.519<br>1.700 |
| 40  | $\alpha$                                           | 0.034                       | 0.387<br>0.366 | 0.519<br>0.540 | 0.033                        | 0.193<br>0.172 | 0.323<br>0.344 |
|     | $\lambda$                                          | 0.029                       | 1.139<br>1.121 | 1.253<br>1.271 | 0.044                        | 1.020<br>0.992 | 1.194<br>1.222 |
|     | $\theta$                                           | 0.227                       | 2.116<br>1.975 | 3.005<br>3.145 | 0.647                        | 1.086<br>0.685 | 3.624<br>4.025 |
|     | $\beta$                                            | 0.088                       | 1.040<br>0.985 | 1.384<br>1.439 | 0.252                        | 0.453<br>0.297 | 1.442<br>1.598 |
| 50  | $\alpha$                                           | 0.030                       | 0.364<br>0.345 | 0.482<br>0.500 | 0.033                        | 0.172<br>0.152 | 0.303<br>0.324 |
|     | $\lambda$                                          | 0.026                       | 1.136<br>1.120 | 1.238<br>1.254 | 0.044                        | 1.044<br>1.016 | 1.218<br>1.246 |
|     | $\theta$                                           | 0.201                       | 2.261<br>2.137 | 3.049<br>3.173 | 0.647                        | 0.963<br>0.562 | 3.500<br>3.902 |
|     | $\beta$                                            | 0.079                       | 0.997<br>0.948 | 1.307<br>1.356 | 0.251                        | 0.367<br>0.211 | 1.351<br>1.506 |
| 80  | $\alpha$                                           | 0.021                       | 0.356<br>0.343 | 0.440<br>0.453 | 0.027                        | 0.192<br>0.175 | 0.298<br>0.315 |
|     | $\lambda$                                          | 0.018                       | 1.043<br>1.032 | 1.115<br>1.126 | 0.036                        | 0.979<br>0.956 | 1.121<br>1.144 |
|     | $\theta$                                           | 0.141                       | 2.490<br>2.403 | 3.043<br>3.130 | 0.526                        | 1.259<br>0.933 | 3.322<br>3.648 |
|     | $\beta$                                            | 0.056                       | 0.996<br>0.961 | 1.215<br>1.249 | 0.202                        | 0.450<br>0.324 | 1.242<br>1.367 |
| 100 | $\alpha$                                           | 0.017                       | 0.374<br>0.363 | 0.439<br>0.450 | 0.022                        | 0.199<br>0.186 | 0.286<br>0.300 |
|     | $\lambda$                                          | 0.015                       | 0.988<br>0.979 | 1.044<br>1.053 | 0.030                        | 0.952<br>0.934 | 1.069<br>1.087 |
|     | $\theta$                                           | 0.110                       | 2.497<br>2.429 | 2.930<br>2.998 | 0.427                        | 1.378<br>1.113 | 3.052<br>3.317 |
|     | $\beta$                                            | 0.044                       | 1.028<br>1.001 | 1.199<br>1.226 | 0.166                        | 0.459<br>0.356 | 1.111<br>1.214 |



## Multiple Step Stress Accelerated Life for Weibull Poisson Distribution with Type I Censoring

Amal S. Hassan<sup>1</sup>

Assar, S. M<sup>2</sup>

Shelbaia, A<sup>3</sup>

### Abstract

This article presents multiple step stress accelerated life tests based on type I censoring. It is assumed that the lifetime at design stress has Weibull Poisson distribution. The scale parameter of the Weibull Poisson failure time at constant levels is assumed to be inverse power law of the stress levels. Under the assumption of a cumulative exposure model, the maximum likelihood method is used to obtain the estimators of the model parameters. The optimal design of the accelerated life tests is considered according to A-optimality criterion to specify the optimal time of changing stress levels. Finally, the numerical studies are introduced to illustrate the proposed procedures.

**Keywords:** Cumulative Exposure Model; Lifetime distribution; Step Stress ALT; Optimal Test Plan.

### 1. Introduction

Accelerated life tests (ALTs) are used to estimate the lifetime of highly reliable products within a reasonable testing time. The test products are run at higher than usual levels of stress (which includes temperature, voltage, pressure, etc.) to obtain early failures. In ALT experiments, the researcher is often interested in the effects of varying stress levels on the lifetimes of experimental units.

One important way in ALT is step-stress accelerated life test (SSALT). There are mainly two types of SSALTs, a simple SSALT and a multiple-step SSALT. In the simple SSALT there is a single change of stress during the test. Nelson (1980) originally proposed the simple step stress ALT, in which only one change of stress occurs with a cumulative exposure (CE) model for Type-I and Type-II censored data. Miller and Nelson (1983) obtained the optimal simple step-stress ALT plans for the case where test products are assumed to have exponentially distribution. Xiong (1999) has studied an exponential CE model with a threshold parameter in the simple SSALT. Lu and Rudy (2002) have dealt with the Weibull CE model with the inverse power law in the simple SSALT. For more researches on step-stress ALTs, see for example, Xu and Fei (2007), Nelson (2008), and Wu et al. (2008).

---

1, 2- Institute of Statistical Studies and Research, Cairo University.

3- phd, student in Institute of Statistical Studies and Research, Cairo University.

On the other hand, in the multiple-step SSALT there are changes of stress level more than once. Khamis and Higgins (1996) considered the optimum three step SSALT for the exponentially distributed Type I censored data. Khamis (1997) proposed an optimal m-step SSALT design with multiple stress variables for three-step SSALT data. Abdel Ghaly, et al. (1998) addressed the situation where the time to failure was assumed to have a three parameter Pareto distribution. Yeo and Tang (1999) have investigated a three-step SSALT in an exponential CE model. McSorley, et al (2002) have shown the properties of the maximum likelihood (ML) estimators of parameters in the Weibull CE model with a log-linear function of stress on three-step SSALT data. Saxena, et al. (2013) dealt with the problem of designing an optimum step stress accelerated life test for Rayleigh distribution.

The modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields. Several distributions have been proposed in the literature to model lifetime data by compounding some useful life distributions. Lu and Shi (2012) introduced a new compounding distribution named the Weibull Poisson (WP) distribution. The failure rate function of WP distribution has various shapes; it can be increasing, decreasing and upside down bathtub shaped or unimodal. The property of various shapes of the failure rate function encourages practitioners to adopt the Weibull Poisson distribution (WP) for the inferences of life information and biological study. The probability density function (PDF) of Weibull Poisson distribution takes the following form

$$f(x) = \frac{\alpha\beta\lambda x^{\alpha-1}}{1 - e^{-\lambda}} \cdot e^{-\lambda - \beta x^{\alpha} + \lambda \exp(-\beta x^{\alpha})} \quad x > 0, \quad (1.1)$$

where  $\alpha > 0$  is the shape parameter,  $\beta > 0$  is scale parameter. The corresponding cumulative distribution function (CDF) is as follows:

$$F(x) = (e^{\lambda \exp(-\beta x^{\alpha})} - e^{\lambda}) (1 - e^{\lambda})^{-1} \quad x > 0. \quad (1.2)$$

Note that when  $\alpha = 1$ , the Weibull Poisson distribution reduces to an exponential Poisson and as  $\lambda \rightarrow 0$  the Weibull Poisson distribution reduces to a two parameter Weibull distribution.

This article deals with the problem of designing an optimum multi-step stress accelerated life test for Weibull Poisson distribution based on type I censoring. The scale parameter of the distribution is assumed to be an inverse power law function of stress level. Also the optimum time of changing stress levels will be obtained using A-optimality criterion which is based on the trace of the Fisher information matrix.

The layout of the article is as follows. The model and assumptions are described in details in Section 2. In Section 3 the ML method is introduced to estimate the point estimate of the model parameters under type I censoring data. In Section 4 the optimum time of changing stress levels will be presented. Finally the simulation technique will be presented in Section 5.

## 2. Assumptions and Test Procedure

During the multiple-step SSALT, specimens are subjected to successively higher levels of stress as follows. After a specimen is used at a normal level of stress, it is subjected to an initial level of stress for a predetermined time interval at the first stage in the test. If it does not fail, it is subjected to a higher level of stress for a predetermined time interval at the next stage. In analogy, it is repeatedly subjected to higher levels of stress until it fails. The other specimens are tested similarly. The pattern of stress levels and time intervals is the same for all specimens.

The model assumptions for multi-step stress ALT will be described as follows:

- i. There are multiple ( $k$ ) levels of high stress,  $V_j, j = 1, 2, \dots, k$  in the experiment, and  $V_0$  is the design stress that is the stress level under normal use conditions, where,  $V_0 < V_1 < \dots < V_k$ .
- ii. The failure times  $x_{ij}, i = 1, 2, \dots, n_j$  at stress levels  $V_j, j = 1, 2, \dots, k$  are assumed to be WP distribution at any stress; with probability density functions (pdf) defined in (1.1).
- iii. A random sample of  $N$  identical units is placed on test under the first stress  $V_1$  and run until time  $\tau_1$  and then stress is changed to  $V_2$  and run until time  $\tau_2$  and so on, the test is continued until all units are failed or it reached the censored time  $T$  (predetermined time).
- iv. The number of failures in the sample is  $\sum_{j=1}^k n_j$  where  $n_1$  is the number of failures from first step under the first stress level  $V_1$ ,  $n_2$  is the number of failures from second step under the second stress level  $V_2$  and  $n_k$  is the number of failures from last step under the last stress level  $V_k$ ,  $N$  are the total number of units run on the test.
- v. The lifetimes of the units at each stress level are identically independent distributed (i.i.d).
- vi. The WP scale parameter  $\beta_j, j = 1, 2, \dots, k$  of the underlying lifetime distribution in (2.1) is assumed to have an inverse power function of stress levels i.e,  $\beta_j = CS_j^p, j = 1, 2, \dots, k, C, p > 0, S_j = \frac{V^*}{V_j}, V^* = \prod_{j=1}^k V_j^{b_j}, b_j = \frac{n_j}{N}$ ,  $C$  is the constant of proportionality and  $p$  is the power of the applied stress.
- vii. The WP parameters  $\lambda$  and  $\alpha$  are constant, i.e. independent of stress.

To analyze the data from multi-step stress ALT, a model is needed to relate the distribution under step stress to the distribution under constant stress. The most commonly used model is cumulative exposure (CE) model proposed by Nelson (1980). The basic idea of the CE model starts from the fact that a step stress ALT model must explain the cumulative effect of the applied stresses; the CE model assumes that the remaining test units are failed according to the cumulative density function of current stress level regardless of the history on previous stress levels. According to the cumulative exposure model in Nelson (1990), the cumulative distribution function of  $X$  is given by

$$F(x) = \begin{cases} F_1(x) & 0 \leq x < \tau_1 \\ F_2((x - \tau_1) + u_1) & \tau_1 \leq x < \tau_2 \\ \vdots & \\ F_k((x - \tau_{k-1}) + u_{k-1}) & \tau_{k-1} \leq x < T \end{cases}$$

where

$$F_j((x_{ij} - \tau_{j-1}) + u_{j-1}) = \left( e^{\lambda \exp(-cs_j^p((x_{ij} - \tau_{j-1}) + u_{j-1})^\alpha)} - e^\lambda \right) (1 - e^\lambda)^{-1},$$

and  $u_{j-1}$  is determined by solving the equation  $F_j(u_{j-1}; \beta_j) = F_{j-1}(\tau_{j-1} - \tau_{j-2} + u_{j-2}; \beta_{j-1})$  by substituting the CDF of WP in the equation (1.1), then the general form of  $u_{j-1}$  will be as follows:

$$u_j = \left( \frac{\beta_j}{\beta_{j+1}} \right)^{1/\alpha} \cdot (\tau_j - \tau_{j-1} + u_{j-1})$$

note that  $u_0 = 0$  and  $\tau_0 = 0$  where  $\tau_j$  is the time of changing stress level, then the special cases of  $u_j$  will be as follows:

$$u_{j=1} = \left( \frac{\beta_1}{\beta_2} \right)^{1/\alpha} \cdot \tau_1$$

$$u_{j=2} = \left( \frac{\beta_2}{\beta_3} \right)^{1/\alpha} \cdot (\tau_2 - \tau_1 + u_1)$$

and the associated pdf will be as follows:

$$f(x) = \begin{cases} f_1(x) & 0 \leq x < \tau_1 \\ f_2((x - \tau_1) + u_1) & \tau_1 \leq x < \tau_2 \\ \vdots & \\ f_k((x - \tau_{k-1}) + u_{k-1}) & \tau_{k-1} \leq x < T \end{cases}$$

In the following subsection the likelihood function is formulated.

### 3. Likelihood Function for Multiple Step Stress ALT with Type I Censoring Data

According to type-I censoring, the test applied to N identical sample which will terminate when all units fail or censoring time T is reached. Let  $x_{i:N}, i = 1, \dots, N$  be independent and identically distributed Weibull Poisson random variable, then the likelihood function for ALT (multiple stresses) type I censored data is considered to have the following form (see Aly and Bleed (2013))

$$l = \prod_{j=1}^k \prod_{i=1}^{n_j} f_j(x_{ij} - \tau_{j-1} + u_{j-1}) \left(1 - F_j(T - \tau_{k-1} + u_{k-1})\right)^B$$

where  $B = N - \sum_{j=1}^k n_j$  is the number of surviving units. Then the likelihood function for three parameter WP distribution for multiple step stress ALT with type I censored is as follows:

$$l = \prod_{j=1}^k \prod_{i=1}^{n_j} \frac{\alpha C S_j^p \lambda (x^*)^{\alpha-1} e^{-\lambda - C S_j^p (x^*)^\alpha + \lambda \exp(-C S_j^p (x^*)^\alpha)}}{1 - e^{-\lambda}} \left[1 - \left(e^{\lambda \exp(-C S_k^p D^\alpha)} - e^\lambda\right) (1 - e^\lambda)^{-1}\right]^B$$

$$l = \prod_{j=1}^k \prod_{i=1}^{n_j} \frac{\alpha C S_j^p \lambda (x^*)^{\alpha-1} e^{-\lambda - C S_j^p (x^*)^\alpha + \lambda \exp(-C S_j^p (x^*)^\alpha)}}{1 - e^{-\lambda}} \left[1 - \frac{e^{\lambda \exp(-C S_k^p D^\alpha)} - e^\lambda}{e^{-\lambda} - 1}\right]^B$$

where  $x^* = x_{ij} - \tau_{j-1} + u_{j-1}$   $D = T - \tau_{k-1} + u_{k-1}$ ,

and  $u_{j=k-1} = \left(\frac{\beta_{k-1}}{\beta_k}\right)^{1/\alpha} \cdot (\tau_{k-1} - \tau_{k-2} + u_{k-2})$ .

The (maximum likelihood) ML estimators of the parameters are obtained by maximizing the logarithm of the likelihood function expressed in the following form:

$$\ln l = N \ln(\lambda \alpha) + p N \sum_{j=1}^k \ln(S_j) + N \ln C + (\alpha - 1) \sum_{j=1}^k \sum_{i=1}^{n_j} \ln(x^*) +$$

$$\lambda \sum_{j=1}^k \sum_{i=1}^{n_j} e^{-C S_j^p (x^*)^\alpha} - C \sum_{j=1}^k \sum_{i=1}^{n_j} S_j^p (x^*)^\alpha - N \ln(e^\lambda - 1) + B \ln \left[ \frac{1 - e^{\lambda \exp(-C S_k^p D^\alpha)}}{1 - e^\lambda} \right].$$

The first partial derivatives of the likelihood equation with respect to the parameters  $C, p, \lambda$  and  $\alpha$  ,respectively ,will be as the follows:

$$\frac{\partial \ln l}{\partial C} = \frac{N}{C} - \sum_{j=1}^k \sum_{i=1}^{n_j} (\lambda H G + H) + \left[ \frac{B}{M} \cdot \lambda S_k^p D^\alpha L \right], \quad (3.1)$$

$$\frac{\partial \ln l}{\partial p} = N \sum_{j=1}^k \ln(S_j) - \left[ \sum_{j=1}^k \sum_{i=1}^{n_j} (\lambda C H G \ln(S_j) + C H \ln(S_j)) \right] + \left[ \frac{B \cdot \lambda C S_k^p D^\alpha L \ln(S_k)}{M} \right], \quad (3.2)$$

$$\frac{\partial \ln l}{\partial \lambda} = \frac{N}{\lambda} + \sum_{j=1}^k \sum_{i=1}^{n_j} G - \frac{N e^\lambda}{e^\lambda - 1} + \left[ \frac{B}{M} \left( -L + \frac{1 - e^{\lambda \cdot e^{(-CS_k^p D^\alpha)}}}{e^{-\lambda} - 1} \right) \right] \quad (3.3)$$

and,

$$\frac{\partial \ln l}{\partial \alpha} = \frac{N}{\alpha} + \sum_{j=1}^k \sum_{i=1}^{n_j} [\ln(x^*) - \lambda C G H \ln(x^*) - C H \ln(x^*)] + \left[ \frac{B \cdot \lambda C S_k^p D^\alpha L \ln D}{M} \right] \quad (3.4)$$

where  $G = e^{-CS_j^p (x^*)^\alpha}$ ,  $H = S_j^p (x^*)^\alpha$ ,  $A = \frac{1 - e^{\lambda \cdot e^{(-CS_k^p D^\alpha)}}}{1 - e^\lambda}$ ,  $M = 1 - e^{\lambda \cdot e^{(-CS_k^p D^\alpha)}}$

and  $L = e^{\lambda e^{(-CS_k^p D^\alpha)} - CS_k^p D^\alpha}$ .

After equating the equations (3.1) to (3.4) with zero and solving it together, the likelihood estimates  $\hat{C}, \hat{p}, \hat{\lambda}$  and  $\hat{\alpha}$  will be obtained. Notice that the first derivative of equations (3.1) to (3.4) are nonlinear equations and their solutions will be obtained numerically. In addition, an estimates values  $\hat{\beta}_j$  for each stress will be obtained by substituting the estimates values of  $\hat{c}$  and  $\hat{p}$  in the inverse power law relationship  $\beta_j = CS_j^p$ .



#### 4. Optimum Test Plan

The purpose of this section is to explore the problem of determining the optimal test plan for multiple SSALT under type I censoring data assuming Weibull Poisson as a lifetime model. In general, the SSALT planning problem concerns of the determination of the type of stresses to be applied, number and magnitude of stress levels, the switch time, the total number of test units, optimum test duration and other measures.

An optimal test plan allows maximum possible information to be obtained from the tests by using the same number of test items. It provides certain objective, such as providing the most precise estimates to improve the quality of the statistical inference. In addition, this test reduces the total required time on test. A useful criterion for optimum plans is A optimality criterion which was mentioned in Hung, et al.(2013). A optimality criterion is based on Minimize the trace of the variance-covariance matrix of the MLE's of model parameters, then the optimal time of changing stress levels  $\tau_{j-1}$  can be obtained by solving the following equation:

$$\frac{\partial \text{tr } F}{\partial \tau_{j-1}} = 0 \quad , j = 2, \dots, k \quad (4.1)$$

where  $F$  is the Fisher information matrix for the MLE which can be obtained as follows:

$$\hat{F} = - \begin{bmatrix} \frac{\partial^2 \ln l}{\partial C^2} & \frac{\partial^2 \ln l}{\partial C \partial p} & \frac{\partial^2 \ln l}{\partial C \partial \lambda} & \frac{\partial^2 \ln l}{\partial C \partial \alpha} \\ \frac{\partial^2 \ln l}{\partial C \partial p} & \frac{\partial^2 \ln l}{\partial p^2} & \frac{\partial^2 \ln l}{\partial p \partial \lambda} & \frac{\partial^2 \ln l}{\partial p \partial \alpha} \\ \frac{\partial^2 \ln l}{\partial C \partial \lambda} & \frac{\partial^2 \ln l}{\partial p \partial \lambda} & \frac{\partial^2 \ln l}{\partial \lambda^2} & \frac{\partial^2 \ln l}{\partial \lambda \partial \alpha} \\ \frac{\partial^2 \ln l}{\partial C \partial \alpha} & \frac{\partial^2 \ln l}{\partial p \partial \alpha} & \frac{\partial^2 \ln l}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ln l}{\partial \alpha^2} \end{bmatrix} \downarrow (\hat{C}, \hat{p}, \hat{\lambda}, \hat{\alpha})$$

and

$$\text{tr } F = \frac{\partial^2 \ln l}{\partial C^2} + \frac{\partial^2 \ln l}{\partial p^2} + \frac{\partial^2 \ln l}{\partial \lambda^2} + \frac{\partial^2 \ln l}{\partial \alpha^2} \quad (4.2)$$

To obtain equation (4.1), it's important to apply the following steps:

Step (1): Obtaining the second partial derivatives of the log likelihood function with respect to parameters  $C, p, \lambda$  and  $\alpha$ , which is presented as follows:

$$\begin{aligned}\frac{\partial^2 \ln l}{\partial C^2} &= \frac{-N}{C^2} + \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} H(G)^2 - \frac{B \cdot (\lambda S_k^p D^\alpha L)^2}{M^2} \\ &\quad - \frac{B \lambda S_k^p D^\alpha L}{M} \{ \lambda e^{(-CS_k^p D^\alpha)} \cdot (S_k^p D^\alpha) + S_k^p D^\alpha \} \\ \frac{\partial^2 \ln l}{\partial p^2} &= \sum_{j=1}^k \sum_{i=1}^{n_j} \left( \lambda H(CG \ln(S_j))^2 + \lambda HCG(\ln(S_j))^2 - CG(\ln(S_j))^2 \right) - \frac{B(L \lambda CS_k^p D^\alpha \ln(S_k))^2}{M^2} \\ &\quad - \frac{B \lambda CS_k^p D^\alpha L \ln(S_k)}{M} \left( CS_k^p D^\alpha \lambda \ln(S_k) e^{(-CS_k^p D^\alpha)} + CS_k^p D^\alpha \ln(S_k) \right) \\ &\quad + \frac{B L \lambda CS_k^p D^\alpha (\ln(S_k))^2}{M} \\ \frac{\partial^2 \ln l}{\partial \lambda^2} &= \frac{-N}{\lambda^2} - \frac{(Ne^\lambda(e^\lambda - 1)) - N(e^\lambda)^2}{(e^\lambda - 1)^2} - \frac{B \cdot L}{M^2} \cdot \left( L + \frac{1 - e^{\lambda \cdot e^{(-CS_k^p D^\alpha)}}}{e^{-\lambda} - 1} \right) \\ &\quad + \frac{B}{M} \left[ \left( -Le^{(-CS_k^p D^\alpha)} \right) - \frac{L(e^{-\lambda} - 1) + (1 - e^{\lambda \cdot e^{(-CS_k^p D^\alpha)}}) e^{-\lambda}}{(e^{-\lambda} - 1)^2} \right] \\ \frac{\partial^2 \ln l}{\partial \alpha^2} &= \frac{-N}{\alpha^2} + \lambda \left[ \sum_{j=1}^k \sum_{i=1}^{n_j} G(CH \ln x^*)^2 - \sum_{j=1}^k \sum_{i=1}^{n_j} CGH(\ln x^*)^2 \right] - C \sum_{j=1}^k \sum_{i=1}^{n_j} H(\ln x^*)^2 \\ &\quad + \frac{B(\lambda CS_k^p D^\alpha L \ln D)^2}{M^2} + \frac{B L \lambda CS_k^p D^\alpha (\ln D)^2}{M} \\ &\quad + \frac{B \cdot CS_k^p D^\alpha \lambda L \ln D \{ -CS_k^p D^\alpha \lambda \ln D e^{(-CS_k^p D^\alpha)} - CS_k^p D^\alpha \ln D \}}{M}\end{aligned}$$

Step (2): Getting the first derivative with respect to  $\tau_{j-1}$  for equation (4.2) to obtain equation (4.1), which are as follows:

$$\frac{\partial f_{11}}{\partial \tau_{j-1}} = -\lambda \sum_{j=1}^k \sum_{i=1}^{n_j} 2\alpha S_j^{2p} (x^*)^{2\alpha-1} e^{-CS_j^p (x^*)^\alpha} + \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} \alpha CS_j^{3p} (x^*)^{3\alpha-1} e^{-CS_j^p (x^*)^\alpha},$$

$$\begin{aligned} \frac{\partial f_{22}}{\partial \tau_{j-1}} = & C\alpha \sum_{j=1}^k \sum_{i=1}^{n_j} S_j^p(x^*)^{\alpha-1} (\ln(S_j))^2 - \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} 2\alpha C^2 S_j^{2p}(x^*)^{2\alpha-1} \ln(S_j)^2 e^{-CS_j^p(x^*)^\alpha} \\ & + \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} \alpha C^3 S_j^{3p}(x^*)^{3\alpha-1} (\ln(S_j))^2 e^{-CS_j^p(x^*)^\alpha} \\ & + \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} \alpha C S_j^p(x^*)^{\alpha-1} (\ln(S_j))^2 e^{-CS_j^p(x^*)^\alpha} \\ & - \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} \alpha C^2 S_j^{2p}(x^*)^{2\alpha-1} (\ln(S_j))^2 e^{-CS_j^p(x^*)^\alpha}, \end{aligned}$$

$$\frac{\partial f_{33}}{\partial \tau_{j-1}} = 0,$$

$$\begin{aligned} \frac{\partial f_{44}}{\partial \tau_{j-1}} = & \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} \alpha(x^*)^{3\alpha-1} C^3 S_j^{3p}(\ln(x^*))^2 e^{-CS_j^p(x^*)^\alpha} \\ & + \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} -2(x^*)^{2\alpha-1} \alpha C^2 S_j^{2p}(\ln(x^*))^2 e^{-CS_j^p(x^*)^\alpha} \\ & + \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} 2 \frac{1}{x^*} C^2 S_j^{2p}(x^*)^{2\alpha} \ln(x^*) e^{-CS_j^p(x^*)^\alpha} \\ & + \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} \alpha C S_j^p(x^*)^{\alpha-1} (\ln(x^*))^2 e^{-CS_j^p(x^*)^\alpha} \\ & + \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} \frac{1}{x^*} 2 C S_j^p(x^*)^\alpha \ln(x^*) e^{-CS_j^p(x^*)^\alpha} \\ & - \lambda \sum_{j=1}^k \sum_{i=1}^{n_j} \alpha C^2 S_j^{2p}(x^*)^{2\alpha-1} (\ln(x^*))^2 e^{-CS_j^p(x^*)^\alpha} \\ & + C \sum_{j=1}^k \sum_{i=1}^{n_j} 2 \frac{1}{x^*} S_j^p(x^*)^\alpha \ln(x^*) + C \sum_{j=1}^k \sum_{i=1}^{n_j} \alpha S_j^p(x^*)^{\alpha-1} (\ln(x^*))^2 \end{aligned}$$

where  $f_{11} = \frac{\partial^2 \ln l}{\partial C^2}$ ,  $f_{22} = \frac{\partial^2 \ln l}{\partial p^2}$ ,  $f_{33} = \frac{\partial^2 \ln l}{\partial \lambda^2}$  and  $f_{44} = \frac{\partial^2 \ln l}{\partial \alpha^2}$ , After adding  $\frac{\partial f_{11}}{\partial \tau_{j-1}}$ ,  $\frac{\partial f_{22}}{\partial \tau_{j-1}}$ ,  $\frac{\partial f_{33}}{\partial \tau_{j-1}}$  and  $\frac{\partial f_{44}}{\partial \tau_{j-1}}$  to get equation (4.1), the optimum time of changing stress level  $\tau_{j-1}$  will be obtained by equating the equation by zero and solving it, notice that the resulting equation is nonlinear equation and its solution will be obtained numerically.

## 5. Simulation Study

To evaluate the performance of the ML method, two sample sizes  $N=30,50$  are generated from three parameter WP distribution, the mean square error (MSE) and the biases for the ML estimates are calculated for three step stress ALT, also optimum change time for each stress are calculated. The simulation procedures are described through the following steps:

- i. For given values of the parameters  $C = 0.5, \lambda = 1, p = (1.25, 1.5, 2)$  and  $\alpha = (1.25, 1.5, 2)$  and given values of the stress  $V_1 = 0.75, V_2 = 1$  and  $V_3 = 2$ , calculate  $\beta_j = CS_j^p$  for each stress level where  $j = 1, 2, 3$  ( $k = 3$ )
- ii. Generate random samples of size  $N = (30, 50)$  from uniform (0,1) distribution and obtain the order statistics  $(U_{1:N}, \dots, U_{N:N})$ .
- iii. For a given value of the first stress change time  $\tau_1 = 0.5$ , find  $n_1$  such that
 
$$U_{n_1:N} \leq \left( e^{\lambda \exp(-\beta_1 \tau_1^\alpha)} - e^\lambda \right) (1 - e^\lambda)^{-1} < U_{n_1+1:N}$$
- iv. For a given value of the second stress change time  $\tau_2 = 1$ , find  $n_2$  such that
 
$$U_{n_2:N-n_1} \leq \left( e^{\lambda \exp\left(-CS_2^p((\tau_2-\tau_1) + \left(\frac{-CS_1^p}{-CS_2^p}\right)^{1/\alpha} \cdot \tau_1)^\alpha\right)} - e^\lambda \right) (1 - e^\lambda)^{-1} < U_{n_2+1:N-n_1}$$
- v. For a given value of the censoring time  $T = 5$ , find  $n_3$  such that
 
$$U_{n_3:N-n_1-n_2} \leq \left( e^{\lambda \exp\left(-CS_3^p((T-\tau_2) + \left(\frac{-CS_2^p}{-CS_3^p}\right)^{1/\alpha} \cdot (\tau_2-\tau_1 + \left(\frac{-CS_1^p}{-CS_2^p}\right)^{1/\alpha} \cdot \tau_1))^\alpha\right)} - e^\lambda \right) (1 - e^\lambda)^{-1} < U_{n_3+1:N-n_1-n_2}$$
- vi. Then the order observations  $x_{1:N} \leq \dots \leq x_{n_1:N} \leq x_{n_1+1:N} \leq \dots \leq T$  are calculated from equation (1.2).
- vii. Based on  $n_1, n_2, n_3, \tau_1, \tau_2, T$  and the observed observations, MLE for the Parameters  $C, p, \lambda$  and  $\alpha$  can be obtained by solving the nonlinear equations from (3.1) to (3.4) numerically.
- viii. Repeat the pervious steps 100 times.
- ix. Compute the biases and the mean square error (MSE) for the ML estimates values.
- x. After getting the estimates values, the optimum time of changing stress will be obtained by solving equation (4.1) numerically. Simulation results are summarized in Tables from (1) to (3).

Table (1): The Bias and MSE for the ML Estimates of  $(C, p, \lambda, \alpha)$   $N = 30$ 

| $N = 30, C = 0.5, p = 2, \lambda = 1, \alpha = 2$       |         |         |           |          |           |           |           |
|---------------------------------------------------------|---------|---------|-----------|----------|-----------|-----------|-----------|
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | 0.0680  | -0.0330 | -0.1120   | -0.1060  | 0.2060    | 0.3360    | -0.0300   |
| MSE                                                     | 0.0390  | 0.5170  | 0.0130    | 0.0790   | 0.4040    | 0.1320    | 0.0031    |
| $N = 30, C = 0.5, p = 1.5, \lambda = 1, \alpha = 2$     |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | 0.1330  | 0.4530  | -0.0160   | -0.0510  | 0.5000    | 0.3840    | -0.0790   |
| MSE                                                     | 0.0480  | 0.5110  | 0.0003    | 0.0530   | 0.505     | 0.1640    | 0.0070    |
| $N = 30, C = 0.5, p = 2, \lambda = 1, \alpha = 1.25$    |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.1350 | -0.4800 | -0.0990   | 0.0280   | -0.2650   | 0.4010    | 0.0730    |
| MSE                                                     | 0.0350  | 0.4100  | 0.0098    | 0.0100   | 0.2500    | 0.1840    | 0.0087    |
| $N = 30, C = 0.5, p = 2, \lambda = 1, \alpha = 1.5$     |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.0970 | -0.3100 | 0.0001    | -0.2460  | 0.5410    | 0.5700    | 0.6890    |
| MSE                                                     | 0.413   | 0.4730  | 0.0001    | 0.0950   | 0.5620    | 0.6100    | 0.6700    |
| $N = 30, C = 0.5, p = 1.25, \lambda = 1, \alpha = 2$    |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | 0.0220  | 0.1120  | -0.0190   | -0.0160  | 0.4590    | -0.5130   | -0.0430   |
| MSE                                                     | 0.0180  | 0.1660  | 0.0043    | 0.0340   | 0.449     | 0.2860    | 0.0031    |
| $N = 30, C = 0.5, p = 1.5, \lambda = 1, \alpha = 1.25$  |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.0150 | -0.3060 | -0.0057   | -0.0680  | 0.3960    | 0.4390    | 0.0340    |
| MSE                                                     | 0.0190  | 0.3700  | 0.0001    | 0.0170   | 0.3560    | 0.2150    | 0.0050    |
| $N = 30, C = 0.5, p = 1.5, \lambda = 1, \alpha = 1.5$   |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | 0.196   | -0.349  | -0.004    | -0.207   | 0.534     | 0.547     | 0.545     |
| MSE                                                     | 0.21    | 0.585   | 0.0001    | 0.092    | 0.609     | 0.575     | 0.545     |
| $N = 30, C = 0.5, p = 1.25, \lambda = 1, \alpha = 1.5$  |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.0640 | -0.259  | -0.0990   | -0.0055  | 0.6660    | 0.5890    | -0.0050   |
| MSE                                                     | 0.0140  | 0.2240  | 0.0001    | 0.0150   | 0.6130    | 0.6000    | 0.0024    |
| $N = 30, C = 0.5, p = 1.25, \lambda = 1, \alpha = 1.25$ |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.061  | 0.0790  | -0.0120   | -0.0140  | 0.2410    | 0.3310    | 0.0480    |
| MSE                                                     | 0.0300  | 0.4220  | 0.0002    | 0.0330   | 0.3480    | 0.1310    | 0.0080    |

Table (2): The Bias and MSE for the ML Estimates of  $(C, p, \lambda, \alpha) N = 50$ 

| $N = 50, C = 0.5, p = 2, \lambda = 1, \alpha = 2$       |         |         |           |          |           |           |           |
|---------------------------------------------------------|---------|---------|-----------|----------|-----------|-----------|-----------|
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | 0.0540  | 0.0640  | -0.1110   | -0.0860  | 0.3400    | 0.2750    | -0.0370   |
| MSE                                                     | 0.0280  | 0.4680  | 0.0120    | 0.0620   | 0.425     | 0.1320    | 0.0029    |
| $N = 50, C = 0.5, p = 1.5, \lambda = 1, \alpha = 2$     |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | 0.1700  | 0.1300  | -0.0170   | -0.0160  | 0.5430    | 0.4290    | -0.053    |
| MSE                                                     | 0.053   | 0.143   | 0.0003    | 0.0260   | 0.5010    | 0.2020    | 0.0030    |
| $N = 50, C = 0.5, p = 2, \lambda = 1, \alpha = 1.25$    |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.1510 | -0.4260 | -0.0980   | 0.0330   | -0.2770   | 0.3840    | 0.0610    |
| MSE                                                     | 0.0330  | 0.3330  | 0.0097    | 0.0076   | 0.2170    | 0.165     | 0.0054    |
| $N = 50, C = 0.5, p = 2, \lambda = 1, \alpha = 1.5$     |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.0400 | -0.1690 | 0.0050    | -0.2270  | 0.5100    | 0.5400    | 0.3960    |
| MSE                                                     | 0.1820  | 0.1910  | 0.00007   | 0.066    | 0.5230    | 0.4000    | 0.4530    |
| $N = 50, C = 0.5, p = 1.25, \lambda = 1, \alpha = 2$    |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.0230 | 0.1280  | -0.0190   | -0.0330  | 0.3430    | 0.5300    | -0.0420   |
| MSE                                                     | 0.0120  | 0.0840  | 0.0003    | 0.0190   | 0.555     | 0.2940    | 0.0027    |
| $N = 50, C = 0.5, p = 1.5, \lambda = 1, \alpha = 1.25$  |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.0150 | -0.2150 | -0.0049   | -0.0730  | 0.4420    | 0.4490    | 0.0220    |
| MSE                                                     | 0.0099  | 0.2180  | 0.00005   | 0.0130   | 0.3290    | 0.2160    | 0.0033    |
| $N = 50, C = 0.5, p = 1.5, \lambda = 1, \alpha = 1.5$   |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | -0.0640 | -0.2040 | -0.0068   | -0.1430  | 0.5100    | 0.5200    | 0.4200    |
| MSE                                                     | 0.2570  | 0.2370  | 0.00005   | 0.0400   | 0.6000    | 0.5310    | 0.5760    |
| $N = 50, C = 0.5, p = 1.25, \lambda = 1, \alpha = 1.5$  |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | 0.046   | -0.196  | -0.011    | -0.029   | 0.416     | 0.425     | -0.004    |
| MSE                                                     | 0.009   | 0.191   | 0.0001    | 0.011    | 0.421     | 0.594     | 0.002     |
| $N = 50, C = 0.5, p = 1.25, \lambda = 1, \alpha = 1.25$ |         |         |           |          |           |           |           |
| Parameter                                               | C       | p       | $\lambda$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| Bias                                                    | 0.0680  | -0.1640 | -0.0120   | -0.0410  | 0.1430    | 0.3100    | 0.0570    |
| MSE                                                     | 0.0220  | 0.2880  | 0.0001    | 0.0200   | 0.1720    | 0.1110    | 0.0080    |

Table (3) The Values of the Optimum Time ( $\tau_1$ ) for  $N = (30, 50)$

| $C = 0.5, p = 2, \lambda = 1, \alpha = 2$       |       |       |       |          |
|-------------------------------------------------|-------|-------|-------|----------|
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 10    | 9     | 10    | 1.605    |
| 50                                              | 17    | 16    | 16    | 1.599    |
| $C = 0.5, p = 1.5, \lambda = 1, \alpha = 2$     |       |       |       |          |
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 8     | 9     | 11    | 1.404    |
| 50                                              | 14    | 16    | 18    | 1.305    |
| $C = 0.5, p = 2, \lambda = 1, \alpha = 1.25$    |       |       |       |          |
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 14    | 5     | 6     | 1.870    |
| 50                                              | 24    | 8     | 11    | 1.862    |
| $C = 0.5, p = 2, \lambda = 1, \alpha = 1.5$     |       |       |       |          |
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 13    | 6     | 8     | 1.743    |
| 50                                              | 22    | 11    | 14    | 1.721    |
| $C = 0.5, p = 1.25, \lambda = 1, \alpha = 2$    |       |       |       |          |
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 8     | 9     | 11    | 1.784    |
| 50                                              | 13    | 16    | 19    | 1.126    |
| $C = 0.5, p = 1.5, \lambda = 1, \alpha = 1.25$  |       |       |       |          |
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 10    | 5     | 11    | 1.675    |
| 50                                              | 21    | 9     | 14    | 1.656    |
| $C = 0.5, p = 1.5, \lambda = 1, \alpha = 1.5$   |       |       |       |          |
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 11    | 7     | 10    | 1.609    |
| 50                                              | 19    | 12    | 16    | 1.593    |
| $C = 0.5, p = 1.25, \lambda = 1, \alpha = 1.5$  |       |       |       |          |
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 10    | 7     | 11    | 1.581    |
| 50                                              | 17    | 12    | 18    | 1.581    |
| $C = 0.5, p = 1.25, \lambda = 1, \alpha = 1.25$ |       |       |       |          |
| N                                               | $n_1$ | $n_2$ | $n_3$ | $\tau_1$ |
| 30                                              | 11    | 6     | 9     | 1.585    |
| 50                                              | 19    | 10    | 16    | 1.588    |



## Conclusion

In this paper, the Maximum Likelihood (ML) method of estimation with type-I censoring are obtained and the data failure times at each stress level are assumed to follow the three parameter Weibull Poisson distribution, the parameters estimates values are compared using the biases and the mean square error criteria. In general, from table (1) to (2), notice that as  $N$  increases, the biases and the mean square error decrease especially for  $\hat{\lambda}$ ,  $\hat{C}$  and  $\hat{\beta}_3$  estimates. For fixed value of the parameter  $C$  and  $\lambda$ , The biases and MSE of  $\hat{p}$  are decreased when the value of the parameter  $p$  increase, also the biases and the MSE for  $\hat{\alpha}$  is decreased as the value of the parameter  $\alpha$  decrease, on the other hand, the biases and MSE for the parameter  $\lambda$  is the smallest in all cases.

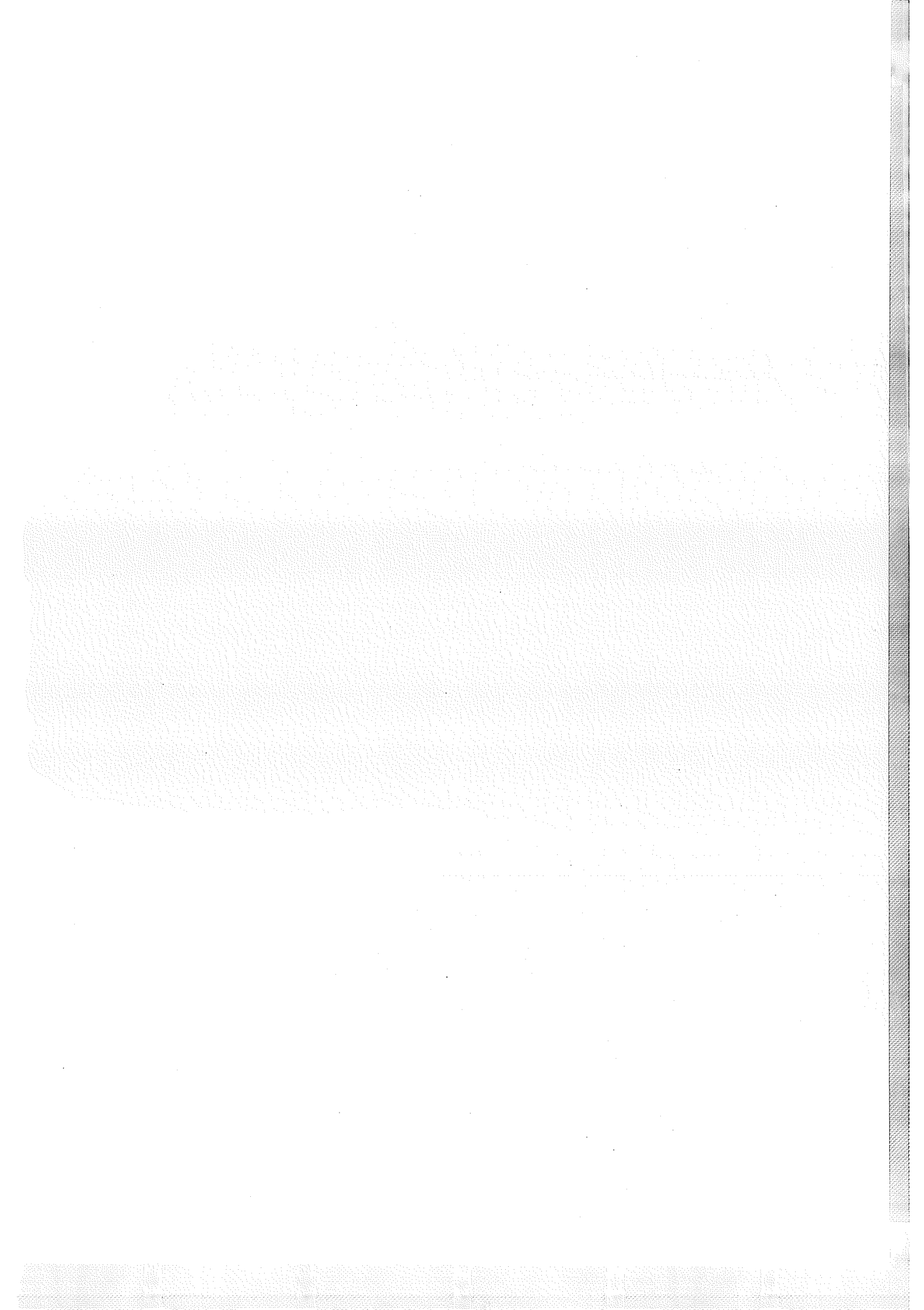
In additional, from table (3), the corresponding optimum time of changing stress levels are obtained numerically by the A-optimality which is based on the trace of the Fisher information matrix, notice that as  $N$  increases the values of  $\tau_1$  are similar for the same values of the parameters. Also numbers of failures for each step stress level  $n_1$ ,  $n_2$  and  $n_3$  are obtained.

## References

- Abdel-Ghaly,A.A., Attia, A.F. and Aly, H.M.(1998).Estimation of the parameters of pareto distribution and the reliability function using accelerated life testing with censoring. Communications in Statistics Simulation and Computation, 27, 469-484.
- Aly, H.M. and Bleed,S.O.(2013). Estimating and Planning Step Stress Accelerated Life Test for Generalized Logistic Distribution under Type I Censoring. International Journal of Applied Mathematics &Statistical Sciences, 2, 1-16.
- Hung,C.H.,Plante,R.D. and Tang, J.(2013). Statistical Equivalency and Optimality of Simple Step-Stress Accelerated Test Plans for the Exponential Distribution. Naval Research Logistics, 60,19-30.
- Khamis, I.H. and Higgins, J. J. (1996). Optimum Three Step Stress Tests. IEEE Transactions on Reliability, 45, 341–345.
- Khamis, I. H. (1997). Optimum M-step Design with K Stress Variables. Communications in Statistics-Simulation and Computation, 26, 1301–1313.
- Lu, M.-W. and Rudy, R.J. (2002). Step Stress Accelerated Test, International Journal of Materials and Product Technology, 17, 425–434.
- Lu,W. and Shi, D. (2012). A New Compounding Life Distribution the Weibull Poisson distribution, Journal of Applied Statistics, 39,21-38.
- McSorley, E.O., Lu, J.C. and Li, C.S. (2002). Performance of Parameter Estimates in Step Stress Accelerated Life Tests with Various Sample Sizes, IEEE Transactions on Reliability, 51, 271–277.
- Miller, R. and Nelson, W.B. (1983). Optimum Simple Step Stress Plans for Accelerated Life Testing. Transactions on Reliability, 32, 59-65.



- Nelson ,W.B.(1980). Accelerated Life Testing Step Stress Models and Data Analysis. .IEEE Transactions on Reliability, 29,103-108.
- Nelson, W. (1990). *Accelerated Life Testing: Statistical Models, Test Plan and Data Analysis*. New York: Wiley.
- Nelson, W.. (2008). Residuals and their Analysis for Accelerated Life Tests with Step and Varying Stress. IEEE Transactions on Reliability. 57, 360-368.
- Saxena, S., Zarrin, S., Kamal, M. and Ul-Islam,A. (2013). Optimum Step Stress Accelerated Life Testing for Rayleigh Distribution. Asian Journal of Mathematics and Statistics, submitted.
- Wu, S.J., Lin, Y.P., Chen, S.T., (2008). Optimal Step Stress Test under Type I progressive Group-Censoring with Random Removals. Statistical Planning. and Inference, 138, 817-826.
- Xiong, C. (1999). Step stress model with threshold parameter, Journal of Statistical Computation and Simulation, 63, 349-360.
- Xu, H.Y., Fei, H.L., (2007). Planning Step-Stress Accelerated Life Tests with Two Experimental Variables. IEEE Transactions on Reliability. 56, 569-579.
- Yeo, K.P. and Tang, L.C. (1999). Planning Step Stress Life Test with a Target Acceleration Factor. IEEE Transactions on Reliability, 48, 61-67.



## On The Chaotic Behaviour Of Nonlinear Time Series Models

A. Elhassanein<sup>1</sup> and A. Sh. Al-henawy<sup>2b</sup>

### Abstract

In this paper, the dynamical behaviour of the forced process feedback nonlinear autoregressive (FPFNAR) model is considered for different levels of noise intensity. The existence and stability of the equilibria of the deterministic system are studied. Numerical simulations are employed to show the model's complex dynamics by means of the largest Lyapunov exponents, bifurcations, fractal dimension, time series diagrams and phase portraits. The phenomenon of noise-induced intermittency near tangent bifurcation is discussed.

### Keywords

Forced PPNAR model; Bifurcations; Neimark-Sacker bifurcation; Noise-induced chaos

2010 MSC : 34K60; 34K23; 37M10

## 1. Introduction

The study of non-linear time series has attracted the attention of several researchers in recent years, and many classes of models have been proposed. A considerable number of studies have been conducted regarding nonlinear autoregressive models: bilinear autoregressive (AR) model [1], exponential AR (EXPAR) [2], threshold AR [3], nonlinear autoregressive moving average (ARMA) [4], autoregressive heteroscedastic (ARCH) [5] and extended approaches related to these models. The autoregressive coefficients are important for indicating the characteristics of the time series and this part of some nonlinear autoregressive models is often regarded as a function of the past data. Chen and Tsay [6] organized such models as a family in the functional-coefficient autoregressive (FCAR) model.

Specifically,  $x_t$  is said to follow a nonlinear autoregressive model of order  $n$  with general noise if there exist a function  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , such that

$$x_t = f(x_{t-1}, x_{t-2}, \dots, x_{t-n}, \varepsilon_t), t \in \mathbb{Z}$$

where  $\{\varepsilon_t\}$  is a sequence of independent identically distributed random variables ( $\varepsilon_t \sim i.i.d(0, \sigma_\varepsilon^2)$ ).

In this paper we discuss the dynamics of process feedback nonlinear autoregressive model for different levels of noise intensity. The model introduced in section 2. The stability of the free noise nonlinear system is analyzed in section 3. Numerical simulations are given in section 4, to discuss our analytical results and to show the effects of noise intensity on the dynamics of the model.

---

<sup>1</sup> Math. Dept., Faculty of Science, Damanhour University, Damanhour, Egypt.  
E-mail: el\_hassanein@yahoo.com; Tel.: +201064941983; Fax: +20453368757

<sup>2</sup> Math. Dept., Faculty of Science, Damietta University, New Damietta Egypt.  
E-mail: a\_henawy@yahoo.com.

Specifically,  $x_t$  is said to follow a nonlinear autoregressive model of order  $n$  with general noise if there exist a function  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , such that

$$x_t = f(x_{t-1}, x_{t-2}, \dots, x_{t-n}, \varepsilon_t), t \in \mathbb{Z}$$

where  $\{\varepsilon_t\}$  is a sequence of independent identically distributed random variables ( $\varepsilon_t \simeq i.i.d(0, \sigma_\varepsilon^2)$ ).

In this paper we discuss the dynamics of process feedback nonlinear autoregressive model for different levels of noise intensity. The model introduced in section 2. The stability of the free noise nonlinear system is analyzed in section 3. Numerical simulations are given in section 4, to discuss our analytical results and to show the effects of noise intensity on the dynamics of the model.

## 2 Forced Process Feedback Nonlinear Autoregressive (FPFNAR) Model

The PFNAR model was proposed by Katoa and Ozaki [7]. Let  $x_t (t = 1, 2, \dots, N)$  indicate the observed data. The PFNAR model is assumed to be the function of the past data that affect the current data point. For example, when  $k_1$  and  $k_2$  set two different delay parameters of the model, the PFNAR model is defined as

$$x_t = \sum_{m=1}^M f_t(x_{t-k_1}, x_{t-k_2})x_{t-m} + \varepsilon_t$$

where  $m$  and  $M$  are positive integers and indicate regressive orders, and  $\varepsilon_t$  is a sequence of  $i.i.d(0, \sigma_\varepsilon^2)$  random variables and independent of  $\{x_{t-i}, i > 0\}$ . The  $k_1$  and  $k_2$  time points do not need to be neighbors.

In this paper, we consider the FPFNAR model where  $k_1 = 1, k_2 = 2, M = 2$ , and  $f_t(x_{t-1}, x_{t-2}) = \alpha_m + \beta_m \exp(-x_{t-2}^2)$ , which has the form

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \beta_1 x_{t-1} \exp(-x_{t-2}^2) + \beta_2 x_{t-2} \exp(-x_{t-2}^2) + k\varepsilon_t \quad (1)$$

where  $k$  is a scalar parameter of the noise intensity.

## 3 The Skeleton

The notion of equilibrium points (fixed points) is central in the study of the dynamics of any physical system. In many applications in economics, biology, physics, engineering, etc., it is desirable that all solutions of a given system

tend to its equilibrium state. This is the subject of study of stability theory, a topic of great importance to scientists, engineers and economists [8-20].

The free noise system, where  $k = 0$ , is given by

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \beta_1 x_{t-1} \exp(-x_{t-2}^2) + \beta_2 x_{t-2} \exp(-x_{t-1}^2), \quad (2)$$

It is convenient to convert the nonlinear delayed system (2) to an equivalent first order system:

$$\begin{cases} y_t = z_{t-1} \\ z_t = \alpha_1 z_{t-1} + \alpha_2 y_{t-1} + (\beta_1 z_{t-1} + \beta_2 y_{t-1}) \exp(-y_{t-1}^2) \end{cases} \quad (3)$$

*Proposition 3.1.* The fixed points of the system (3) are  $E_0 = (0, 0)$ ,  $E_1 = (\sqrt{\ln(\frac{\beta_1 + \beta_2}{1 - \alpha_1 - \alpha_2})}, \sqrt{\ln(\frac{\beta_1 + \beta_2}{1 - \alpha_1 - \alpha_2})})$ , and,  $E_2 = (-\sqrt{\ln(\frac{\beta_1 + \beta_2}{1 - \alpha_1 - \alpha_2})}, -\sqrt{\ln(\frac{\beta_1 + \beta_2}{1 - \alpha_1 - \alpha_2})})$ , where (i)  $\alpha_1 + \alpha_2 > 1$  and  $\beta_1 + \beta_2 < 0$  or (ii)  $\alpha_1 + \alpha_2 < 1$  and  $\beta_1 + \beta_2 > 0$ .

*Proof.* The fixed points of the system (3) are obtained as the solution of the algebraic system:

$$\begin{cases} y = z \\ z = \alpha_1 z_{t-1} + \alpha_2 y_{t-1} + (\beta_1 z_{t-1} + \beta_2 y_{t-1}) \exp(-y_{t-1}^2). \end{cases}$$

Which is obtained by setting  $y_t = y_{t-1} = y$  and  $z_t = z_{t-1} = z$  in (3). Then we get the fixed point  $E_0 = (0, 0)$ . Let  $z \neq 0$  we get the equation

$$\exp(-y^2) = \frac{1 - \alpha_1 - \alpha_2}{\beta_1 + \beta_2}.$$

Assuming that (i) or (ii) is satisfied, we get  $E_j, j = 1, 2$ .

The local stability analysis of the system (3) can be studied by computing the variation matrix corresponding to each fixed point. The variation matrix of the system at state variable is given by

$$J(y_{t-1}, z_{t-1}) = \begin{pmatrix} 0 & 1 \\ \xi_{t-1} & \eta_{t-1} \end{pmatrix}, \quad (4)$$

where

$$\xi_{t-1} = \alpha_2 - (2\beta_1 y_{t-1} z_{t-1} + 2\beta_2 y_{t-1}^2 - \beta_2) \exp(-y_{t-1}^2),$$

and

$$\eta_{t-1} = \alpha_1 + \beta_1 \exp(-y_{t-1}^2).$$

*Proposition 3.2.* The fixed point  $E_0$  is locally asymptotically stable if

$$(a) \left| \frac{\alpha_1 + \beta_1}{2} + \sqrt{\left(\frac{\alpha_1 + \beta_1}{2}\right)^2 + \alpha_2 + \beta_2} \right| < 1,$$

and

$$(b) \left| \frac{\alpha_1 + \beta_1}{2} - \sqrt{\left(\frac{\alpha_1 + \beta_1}{2}\right)^2 + \alpha_2 + \beta_2} \right| < 1,$$

otherwise it is unstable fixed point.

*Proof.* In order to prove this result, we estimate the eigenvalues of Jacobian matrix at  $E_0$ . The Jacobian matrix for  $E_0$  is

$$J(E_0) = \begin{pmatrix} 0 & 1 \\ \alpha_2 + \beta_2 & \alpha_1 + \beta_1 \end{pmatrix}.$$

Hence the eigenvalues of  $J(E_0)$  are

$$\lambda_{1,2} = \frac{\alpha_1 + \beta_1}{2} \pm \sqrt{\left(\frac{\alpha_1 + \beta_1}{2}\right)^2 + \alpha_2 + \beta_2},$$

the proof is completed.

The Jacobian matrix (4) at  $E_j, j = 1, 2$ , is

$$J(E_{1,2}) = \begin{pmatrix} 0 & 1 \\ \xi & \eta \end{pmatrix},$$

where

$$\xi = \alpha_2 + 2(1 - \alpha_1 - \alpha_2) \ln\left(\frac{1 - \alpha_1 - \alpha_2}{\beta_1 + \beta_2}\right) + \beta_2\left(\frac{1 - \alpha_1 - \alpha_2}{\beta_1 + \beta_2}\right),$$

$$\eta = \alpha_1 + \beta_1\left(\frac{1 - \alpha_1 - \alpha_2}{\beta_1 + \beta_2}\right),$$

The characteristic equations are

$$P(\lambda) = \lambda^2 - \eta\lambda - \xi = 0.$$

If the eigenvalues of the Jacobian matrix of a fixed point are inside the unit circle of the complex plan, interior fixed point is local stable. Using Jury's conditions [12,21], we have necessary and sufficient condition for local stability of interior fixed point which are the necessary and sufficient condition for  $|\lambda| < 1$ .

$$(e) P(1) = 1 - \eta - \xi > 0,$$

$$(f) P(-1) = 1 + \eta - \xi > 0,$$

$$(g) |P(0)| = |\xi| < 1.$$

The inequality (e) is equivalent to  $\frac{1-\alpha_1-\alpha_2}{\beta_1+\beta_2} > 1$ , whenever (i) is satisfied and equivalent to  $0 < \frac{1-\alpha_1-\alpha_2}{\beta_1+\beta_2} < 1$ , whenever (ii) is satisfied, inequality (f) is equivalent to  $\alpha_2 - \alpha_1 + (2 \ln(\frac{1-\alpha_1-\alpha_2}{\beta_1+\beta_2}) + \frac{\beta_2-\beta_1}{\beta_2+\beta_1})(1 - \alpha_1 - \alpha_2) < 1$ , and inequality (g) is equivalent to  $|\alpha_2 + 2(1 - \alpha_1 - \alpha_2) \ln(\frac{1-\alpha_1-\alpha_2}{\beta_1+\beta_2}) + \beta_2(\frac{1-\alpha_1-\alpha_2}{\beta_1+\beta_2})| < 1$ , that determine the domains of the values of parameters for which  $E_{1,2}$  are stable.

## 4 Numerical Simulation

### 4.1 Deterministic System

The main purpose of this subsection is to investigate the qualitative behavior of the solution of the nonlinear system (3). To provide some numerical evidence for its chaotic behavior, we present various numerical results here to show the chaoticity including its bifurcation diagrams, Lyapunov exponents, sensitive dependence on initial conditions and fractal dimension. In Fig. 1(a) bifurcation diagram of the system (3) is plotted on the interval  $1.1 \leq \beta_2 \leq 3.5$  for initial point  $(y_0, z_0) = (0.1, 0.2)$  and other fixes parameters  $(\alpha_1, \alpha_2, \beta_1) = (0.1, -2, 1)$ . In Fig. 1(b), a plot of the Lyapunov exponent for attractors of the system (3) is presented. A positiveness of this exponent for  $\beta_2 > \beta_2^* \approx 2.48$  confirms the chaotic character of attractors in this parametrical zone (here, the value  $\beta_2^* \approx 2.48$  is a tangent bifurcation point). To show the sensitive dependence on initial conditions 400 observations of the system (3) are generated for initial points  $(0.1, 0.2)$  and  $(0.1, 0.2 + 0.00008)$  with parameters  $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (0.1, -2, 1, 3.5001)$  see Fig. 2. In Fig. 3 attractors are plotted for different values of  $\beta_2$  to show that the system (3) undergoes a Neimark-Sacker (discrete Hopf) bifurcation [14]. One of the commonly used characteristics for classifying and quantifying the chaoticity of a dynamical system is fractal dimensions, [21, 25]. Via simulation we get two Lyapunov exponents  $\lambda_1 = 0.2496 > 0 > \lambda_2 = -1.3316$  for parameters  $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (0.1, -2, 1, 3.5)$ , which means that  $d_L \approx 1 + \frac{0.2496}{1.3316} = 1.1874$ . Therefor the system (3) exhibits a fractal structure and its attractor has a fractal dimension which is chaotic behaviour. Strange attractor for  $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (0.1, -2, 1, 3.5001)$  is plotted in Fig. 4.

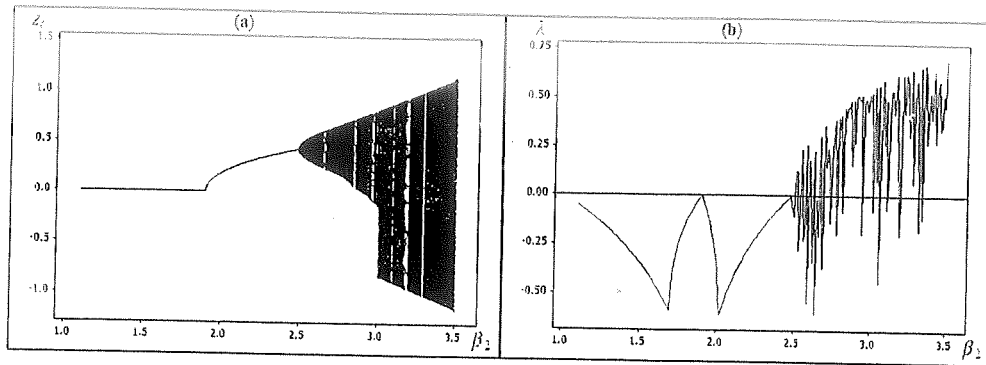


Fig. 1. (a) Bifurcation diagram of (3); (b) Lyapunov exponents.

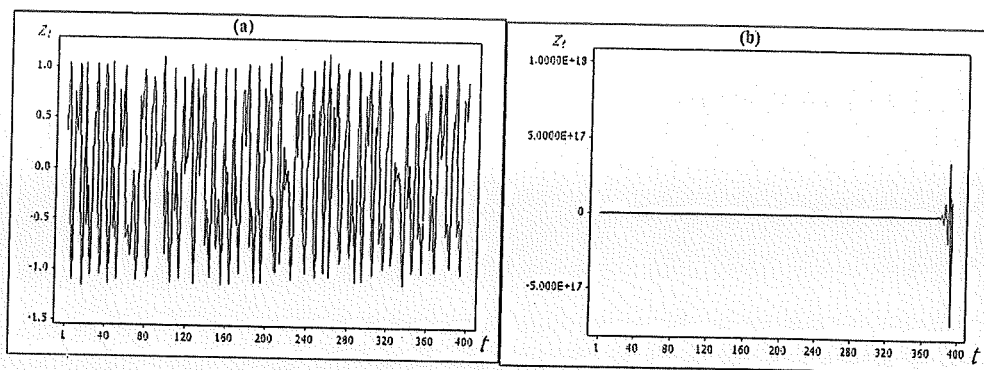


Fig. 2. Time series of the system (3) for: (a)  $(y_0, z_0) = (0.1, 0.2)$ ; (b)  $(y_0, z_0) = (0.1, 0.2 + 0.00008)$ .

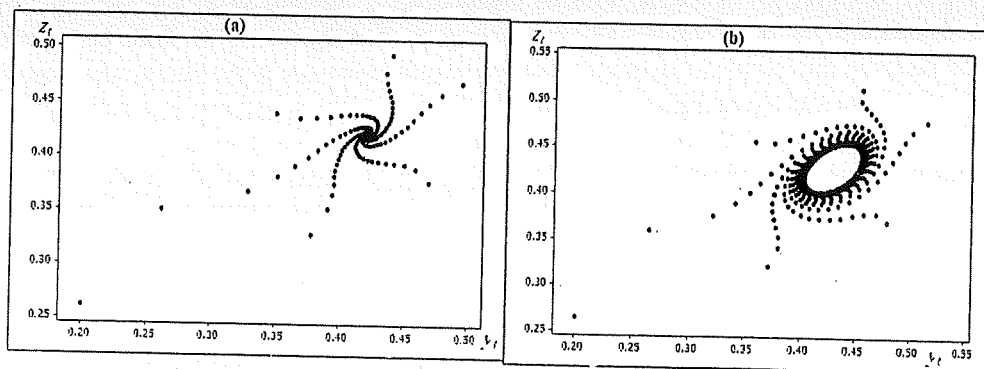


Fig. 3. Attractors of the system (3) with  $(\alpha_1, \alpha_2, \beta_1) = (0.1, -2, 1)$  for: (a)  $\beta_2 = 2.46$ ; (b)  $\beta_2 = 2.4899$ .



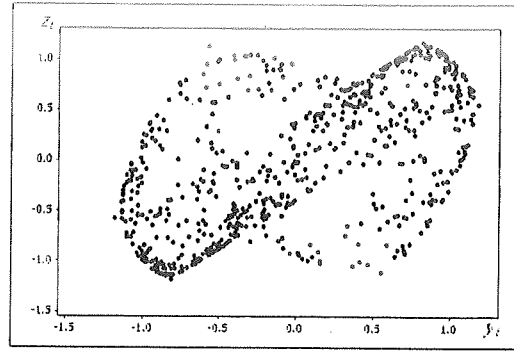


Fig. 4. Strange attractors of  $(y_t, z_t)$  for  $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (0.1, -2, 1, 3.5001)$ .

#### 4.2 Stochastic Model

In this subsection, we consider a stochastic system forced by additive noise (1), where  $\varepsilon_t$  is uncorrelated Gaussian random process with parameters  $E\varepsilon_t = 0$ ,  $E\varepsilon_t^2 = 1$ , and  $k$  is a scalar parameter of the noise intensity. We study a behavior of this stochastic model for different sets of parameters  $\beta_2$  and  $k$ . In Figs. 6 and 7, stochastic attractors and Lyapunov exponents of the system (1) are plotted on the interval  $1.8 \leq \beta_2 \leq 3.5$  for three values of the noise intensity  $k = 0.002, 0.005, 0.007$ . As we can see, noise deforms the deterministic attractor (compare Figs. 1(a) and 6). As noise intensity increases, a border between order and chaos moves to the left. The changes of the arrangement of attractors are accompanied by the changes in dynamical characteristics (compare Lyapunov exponents in Figs. 1(b) and 8). The most essential difference between stochastic and deterministic attractors is observed near the bifurcation point  $\beta_2^* \approx 2.48$ . The underlying reason is that in the vicinity of this bifurcation point attractors are highly sensitive to random disturbances. Consider in detail a behavior of stochastic system (1) near  $\beta_2 = 2.48$ . Compare the stochastic response of this system for two fixed values  $\beta_2 = 2.471$  and  $\beta_2 = 2.4799$ . In Figs. 8 and 9, time series are presented for different values of noise intensity. Consider  $\beta_2 = 2.471$ . For low noise  $k = 0.002$ , random states are concentrated near the stable deterministic equilibrium  $E_1$  (see Fig. 7(a)). For  $k = 0.007$  one can see stochastic oscillations of large amplitude. Indeed, as the noise intensity increases, the dispersion of random states near  $E_1$  grows and iterations with high probability jump over the unstable equilibrium  $E_0$ . After these oscillations, iterations come to the vicinity of the point  $E_1$  again and so on (see Fig. 10(b)). In this case, the stochastic model (1) exhibits a coexistence of two different dynamical regimes even if the system (3) has a stable equilibrium only. This type of dynamics of the system (1) can be determined as a noise-induced intermittency [26]. In Fig. 9, time series of the stochastic system (1) with  $\beta_2 = 2.4799$  for  $k = 0.002$  and  $k = 0.007$  are plotted. As

can be seen, noise-induced intermittency for this  $\beta_2 = 2.4799$  is observed for the lower noise intensity. For stochastic attractors and their dynamic characteristics, a dependence on noise level is illustrated in Figs. 10 and 11 for  $\beta_2 = 2.471$  and  $\beta_2 = 2.471$ . In Fig. 10, one can see a sharp growth of the size of the attractor as noise intensity exceeds some critical value. A change of the sign of Lyapunov exponent from minus to plus can be interpreted as a transition from regular to noise-induced chaotic regime (see Fig. 11).

Thus, the results presented here give us a qualitative description of noise-induced transitions from the regular regime to intermittency.

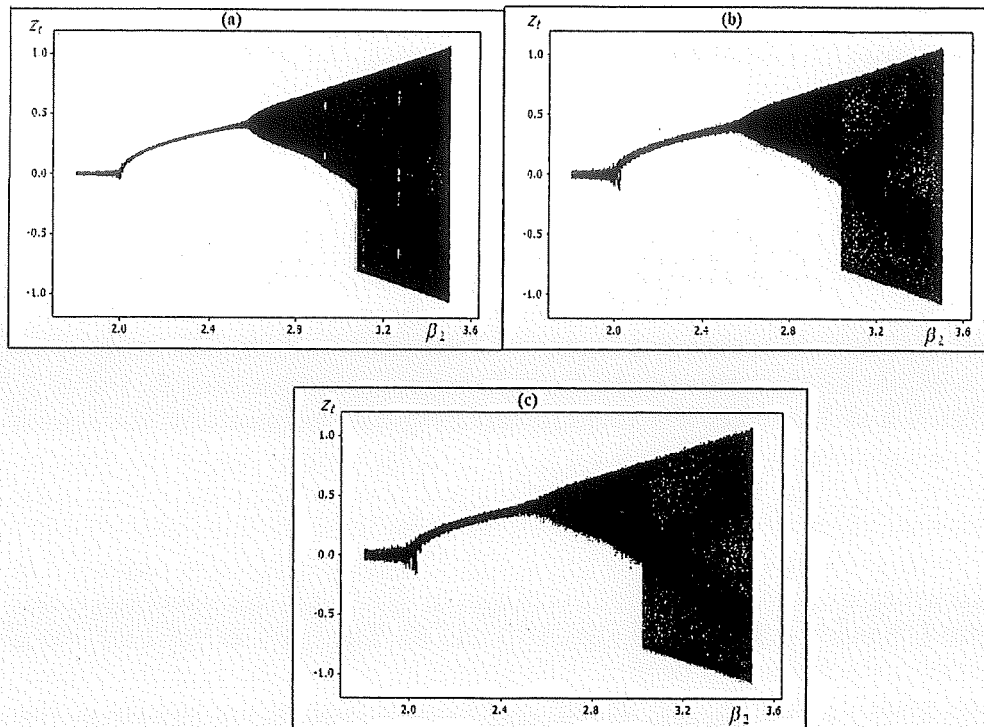
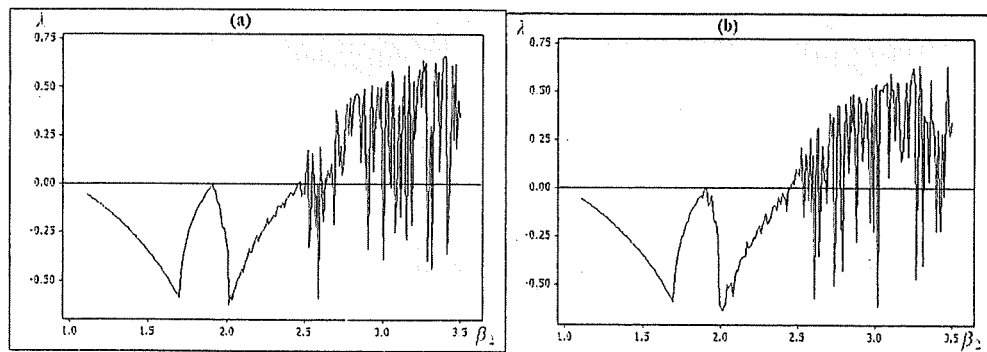


Fig. 6. Bifurcation diagrams of the stochastic model (1) with  $\beta_2 \in [1.8, 3.5]$ : (a)  $k = 0.002$ ; (b)  $k = 0.005$ ; (c)  $k = .007$ .



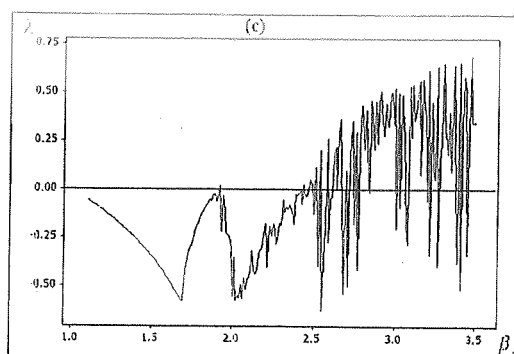


Fig. 7. Lyapunov exponents of the stochastic model (1) with  $\beta_2 \in [1.8, 3.5]$ : (a)  $k = 0.002$ ; (b)  $k = 0.005$ ; (c)  $k = .007$ .

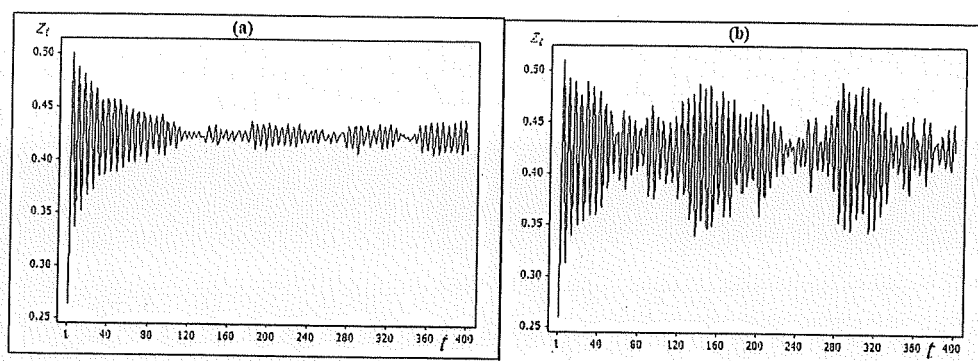


Fig. 8. Time series of the stochastic model (1) with  $\beta_2 = 2.471$  for (a)  $k = 0.002$ ; (b)  $k = 0.007$ .

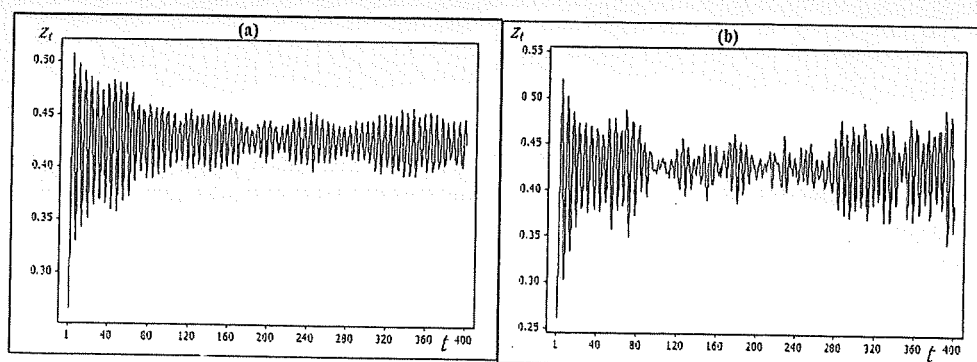


Fig. 9. Time series of the stochastic system (1) with  $\beta_2 = 2.4799$  for (a)  $k = 0.002$ ; (b)  $k = 0.007$ .

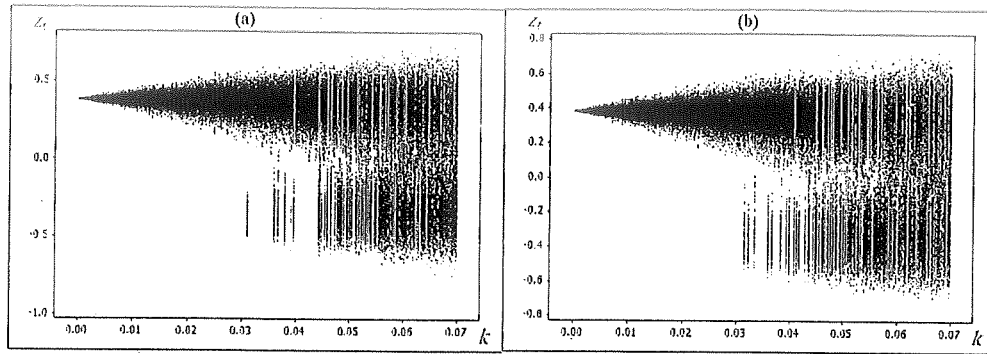


Fig. 10. Attractors of the stochastic system (1) for (a)  $\beta_2 = 2.471$ ; (b)  $\beta_2 = 2.4799$ .

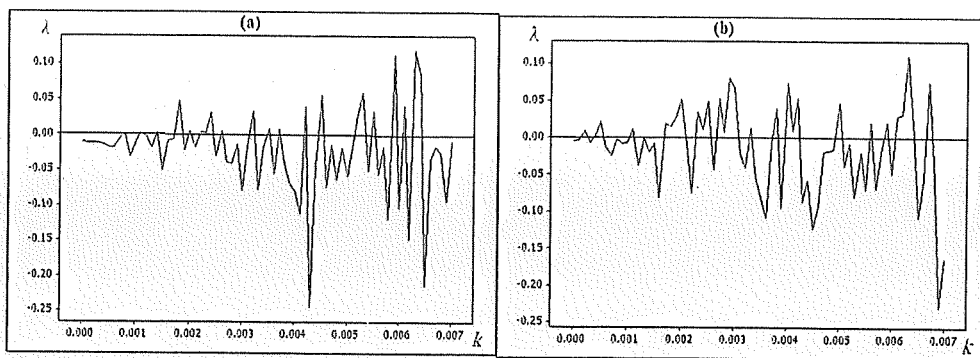


Fig. 11. Lyapunov exponents of the stochastic system (1) for (a)  $\beta_2 = 2.471$ ; (b)  $\beta_2 = 2.4799$ .

## 5 Conclusion

The real life systems rarely can be isolated from the surroundings even during strict experimental conditions. Consequently, the output represents the combined action of the intrinsic dynamics of the system and the input to the system from surroundings. Any variability in the output can therefore arise from the system input, from the intrinsic deterministic dynamics of the system, or from both. The current paper studies dynamics of the FPFNAR model for different levels of noise intensity. We discuss the noise-induced intermittency and chaotization observed near tangent bifurcation. The remarkable feature of the dynamics of the model considered here is that small noises generate large-amplitude chaotic oscillation. We show that this phenomenon can be explained by the high stochastic sensitivity of the equilibrium.

## References

- [1] C. W. Granger, A.P. Andersen, An introduction to bilinear time series models, Vanderhoek and Ruprecht, Gottingen, 1978.
- [2] V. Haggan, T. Ozaki, Modeling nonlinear random vibration using an amplitude-dependent autoregressive time series model, *Biometrika* 68 (1) (1981) 189–196.
- [3] H. Tong, K. S. Lim, Threshold autoregressions, limit cycles, and data, *J. Roy. Statist. Soc. B* 42 (1980) 245–92.
- [4] S. Chen, S.A Billings, Modeling and analysis of non-linear time series, *Internat. J. Control* 50 (6), (1989) 2151–2171.
- [5] R. F. Engle, Autoregressive conditional heteroscedasticity with estimates of the variance of U.K. inflation, *Econometrica* 55 (1982) 251–276.
- [6] R. Chen, R.S. Tsay, Functional-coefficient autoregressive models, *J. Amer. Statist. Assoc.* 88 (421) (1993) 298–308.
- [7] H. Katoa, T. Ozakib, Adding data process feedback to the nonlinear autoregressive model, *Signal Processing* 82 (2002) 1189–1204.
- [8] R. L. Devaney, An introduction to chaotic dynamical systems. Addison-Wesely, Reading, 1989.
- [9] D. Gulick, Encounters with Chaos, McGraw Hill, New York, 1992.
- [10] S. N. Elaydi, An Introduction to Difference Equations. third ed. Springer-Verlag Publishers, 2005.
- [11] T. Puu, Attractors, Bifurcations and Chaos: Nonlinear Phenomena in Economics, Springer, Berlin, 2000.
- [12] H. S. Steven, Nonlinear dynamics and chaos: with applications to physics, biology, chemistry and engineering, Perseus Books Publishing, L. L. C., 1994.
- [13] H.-W Lorenz, Nonlinear Dynamical Economics and Chaotic Motion. Lectures Notes in Economics and Mathematical Systems 334, Springer, Berlin, 1993.
- [14] J. L. Kaplan, Y.A Yorke, A regime observed in a fluid flow model of Lorenz, *Comm. Math. Phys.* 67, (1979) 93–108.
- [15] J. Zhang , Q. Da, Y. Wang, Analysis of nonlinear duopoly game with heterogeneous players, *Economic Modelling* 24 (2007) 138–148.
- [16] F. Tramontana, Heterogeneous duopoly with isoelastic demand function, *Economic Modelling* 27 (2010) 350–357.
- [17] B. Xin , T. Chen, On a master-slave Bertrand game model, *Economic Modelling* 28 (2011) 1864–1870.
- [18] H. Agiza, A. Elsadany, Nonlinear dynamics in the Cournot duopoly game with heterogeneous players. *Physica A*, (2003) 320, 512–524.

- [19] LU Yali, Dynamics of a Delayed Duopoly Game with Increasing Marginal Costs and Bounded Rationality Strategy, *Procedia Engineering*, (2011) 15, 4392 – 4396.
- [20] J. M. T. Thompson, H. B. Stewart, *Nonlinear Dynamics and Chaos*, second ed. John Wiley & Sons, Ltd. 2002.
- [21] S. Chatterjee, M. Yilmaz, Chaos, fractals and statistics, *Statistical Science* 7, (1992)49-68.
- [22] J. H. E. Cartwright, Nonlinear stiffness, Lyapunov exponents, and attractor dimension, *Phys. Lett. A* 264, (1999) 298–304.
- [23] I. Bashkirtseva, L. Ryashko, Stochastic sensitivity analysis of noise-induced intermittency and transition to chaos in one-dimensional discrete-time systems, *Physica A* 392, (2013) 295–306.

## Estimation of Stress-Strength Parameter for Rayleigh Distribution Based on Progressive Type-II Censoring

A. M. Abd-Elfattah <sup>1</sup>, M. H. Abu-Moussa <sup>2</sup>, M. M. El-Fahham <sup>3</sup>

### Abstract.

In this paper, the estimation of stress-strength parameter  $R = P(Y < X)$  is considered. When  $X, Y$  the strength and stress respectively are two independent random variables of Rayleigh distribution. The samples taken for  $X$  and  $Y$  are progressively censoring of type II. Maximum likelihood estimator (MLE) and uniformly minimum variance unbiased estimator (UMVUE) of  $R = P(Y < X)$  are obtained. The exact confidence interval of  $R$  based on MLE is obtained. The performance of the proposed estimators is compared using the computer simulation.

**Keywords:** Rayleigh distribution; progressive type-II censoring; stress-strength model; unbiased estimator; maximum-likelihood estimator; uniformly minimum variance unbiased estimator; confidence intervals.

## 1 Introduction

Rayleigh distribution was first considered by Lord Rayleigh[12]. It is used in modeling lifetime data and reliability. The Rayleigh distribution has the following distribution function for  $X > 0$ :

$$F(x) = 1 - e^{-x^2/2\sigma^2}; \quad \sigma > 0, x \geq 0 \quad (1.1)$$

---

<sup>1</sup>Department of Mathematical Statistics, Institute of Statistical Studies-Research Cairo University, Egypt, corresponding author

<sup>2</sup>Department of Mathematics, Faculty of Science, Cairo University, Giza-Egypt

<sup>3</sup>Department of Mathematics, Faculty of Science, Cairo University, Giza-Egypt

And the density function of Rayleigh for  $X > 0$  denoted by  $\text{Ray}(\sigma)$  is

$$f(x; \sigma) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}; \quad \sigma > 0 \quad (1.2)$$

Rayleigh distribution related to other distributions such as Weibull distribution  $\text{Rayleigh}(\sigma) = \text{Weibull}(2, \sigma\sqrt{2})$ , In life-testing experiments, one often encounters situations where it takes a substantial amount of time to obtain a reasonable number of failures necessary to carry out reliable inference, so censored samples are used for analyzing lifetime data. Among various censoring schemes, the Type II progressive censoring scheme has become very popular one in the last decade. It can be described as follows: let  $n$  units be placed on test at time zero with  $m$  failures to be observed. At the first failure a number  $r_1$  of the surviving units  $(n - 1)$  are randomly selected and removed from the experiment. At the second observed failure,  $r_2$  of the surviving units  $(n - r_1 - 2)$  are randomly selected and removed from the experiment, and so on until the  $m$ -th failure is observed. The all remaining surviving units  $r_m = n - m - r_1 - r_2 - \dots - r_{m-1}$  are removed. We denote to progressively Type II censoring with scheme  $(n, m, r_1, r_2, \dots, r_m)$ . Traditional Type II censoring scheme is included when  $(r_1 = r_2 = \dots = r_{m-1} = 0)$  and  $(r_m = n - m)$  and complete sampling scheme when  $(n = m)$  and  $(r_1 = \dots = r_{m-1} = r_m = 0)$ . Balakrishnan and Aggarwala[7] and Balakrishnan[6] present a study on different features of progressive censoring schemes.

In stress-strength model, the stress ( $Y$ ) and the strength ( $X$ ) are treated as random variables and the reliability of a component during a given period is taken to be the probability that its strength exceeds the stress during the entire interval, i.e. the reliability  $R$  of a component is  $R = P(Y < X)$ . For a particular situation, if we consider  $Y$  as the pres-



sure of a chamber generated by ignition of a solid propellant and  $X$  as the strength of the chamber. Then  $R$  represents the probability of successful firing of the engine. Stress-strength model can be used as a general measure of the difference between two populations and has applications in many area. For example comparing two treatments  $X$  and  $Y$ , then  $R = P(Y < X)$  is the measure of the response of treatment  $X$ . For other applications see Kotz et al.[11]. Many authors considered the problem of estimating the stress-strength parameter based on complete samples, it first considered by Birnbaum[8]. Johnson[10] present a good review on stress-strength model in reliability. Awad and Charraf [5] studied the case when  $X$  and  $Y$  are independent Burr random variables of type XII, they obtained maximum likelihood, uniformly minimum variance unbiased estimator (UMVUE) and Bayesian estimates of  $R$ . Fathipour et al. [9] obtained the estimation of  $R$  in the Generalized Rayleigh Distribution with Different Scale Parameters. Ahmed et al. [3] consider this problem when  $X$  and  $Y$  are two independent random variables have Burr Type X distribution. Based on censored samples Saraçoğlu et al.[14] obtained the estimation for  $R$  based on exponential distribution with type II progressive censoring. Abd-Elfattah et al.[1] get the estimation of  $R$  based on Burr type XII distribution with type II progressive censoring. Recently, Asgharzadeh et al.[4] discussed inference for the stress-strength reliability  $R$  when  $X$  and  $Y$  are two independent Weibull distributions with different scale parameters based on progressively censored samples, they obtained the MLE of  $R$  and different confidence intervals as Asymptotic confidence interval (C.I) and Bootstrap C.I.s. The Bayes estimator of  $R$  and the corresponding credible interval. For some of the recent references, the readers may refer to [2, 13].

In the present paper, the study the estimation of  $R = P(Y < X)$

when  $X$  and  $Y$  are two independent but not identically random variables belonging to Rayleigh distribution with parameters  $\sigma_1$  and  $\sigma_2$  respectively. In Section (2), maximum likelihood estimator of reliability  $R$  and exact confidence interval is obtained. Also UMVUE of  $R$  is obtained in sections (3). Simulation based on progressive censored of type-II are presented in Section(4) and the tables of results are founded in the appendix. Finally the concluding remarks given in section (5).

## 2 MLE of $R$

In this section the MLE of  $R$  is obtained. Let  $X, Y$  are two independent random variables such that  $X \sim Ray(\sigma_1)$  and  $Y \sim Ray(\sigma_2)$  then  $R$  is:

$$\begin{aligned} R &= P(Y < X) = \int_0^\infty \int_0^x f(x)f(y)dydx \\ &= \int_0^\infty \int_0^x \frac{x}{\sigma_1^2} e^{-x^2/2\sigma_1^2} \cdot \frac{y}{\sigma_2^2} e^{-y^2/2\sigma_2^2} dydx \\ &= \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \end{aligned} \quad (2.1)$$

Now to get MLE of  $R$  we first get the MLE of parameters  $\sigma_1$  and  $\sigma_2$ .

Let  $X_{1:m_1:n_1}, \dots, X_{m_1:m_1:n_1}$  be a progressive censored sample from  $Ray(\sigma_1)$  with progressive censoring scheme  $(n_1, m_1, r_1, \dots, r_{m_1})$ , and let  $Y_{1:m_2:n_2}, \dots, Y_{m_2:m_2:n_2}$  be a progressive censored sample from  $Ray(\sigma_2)$  with progressive censoring scheme  $(n_2, m_2, s_1, \dots, s_{m_2})$ , then the jointly likelihood function  $L(\sigma_1, \sigma_2)$  is

$$\begin{aligned} L(\sigma_1, \sigma_2) &= [k_1 \prod_{i=1}^{m_1} f(x_i)[1 - F(x_i)]^{r_i}] \cdot [k_2 \prod_{j=1}^{m_2} f(y_j)[1 - F(y_j)]^{s_j}] \\ &= \frac{k_1 k_2}{\sigma_1^{2m_1} \sigma_2^{2m_2}} (\prod_{i=1}^{m_1} x_i) (\prod_{j=1}^{m_2} y_j) \\ &\quad \times \exp \left( \frac{-1}{2\sigma_1^2} \sum_{i=1}^{m_1} (1 + r_i) x_i^2 \right) \exp \left( \frac{-1}{2\sigma_2^2} \sum_{j=1}^{m_2} (1 + s_j) y_j^2 \right) \end{aligned} \quad (2.2)$$

Where  $k_1$  and  $k_2$  are:

$$k_1 = n_1(n_1 - 1 - r_1)(n_1 - 2 - r_1 - r_2) \dots (n_1 - m_1 + 1 - r_1 - \dots - r_{m_1-1})$$

$$k_2 = n_2(n_2 - 1 - s_1)(n_2 - 2 - s_1 - s_2) \dots (n_2 - m_2 + 1 - s_1 - \dots - s_{m_2-1})$$

Now the jointly log-likelihood function  $\ell$  is:

$$\begin{aligned} \ell = & \ln(k_1 k_2) - 2m_1 \ln \sigma_1 - 2m_2 \ln \sigma_2 + \sum_{i=1}^{m_1} \ln x_i + \sum_{j=1}^{m_2} \ln y_j \\ & - \frac{1}{2\sigma_1^2} \sum_{i=1}^{m_1} (1 + r_i) x_i^2 - \frac{1}{2\sigma_2^2} \sum_{j=1}^{m_2} (1 + s_j) y_j^2 \end{aligned} \quad (2.4)$$

By differentiation on equation (2.4) with respect to  $\sigma_1$  and  $\sigma_2$ . Then we get:

$$\frac{\partial \ell}{\partial \sigma_1} = -\frac{2m_1}{\sigma_1} + \frac{1}{\sigma_1^3} \sum_{i=1}^{m_1} (1 + r_i) x_i^2 \quad (2.5)$$

$$\frac{\partial \ell}{\partial \sigma_2} = -\frac{2m_2}{\sigma_2} + \frac{1}{\sigma_2^3} \sum_{j=1}^{m_2} (1 + s_j) y_j^2 \quad (2.6)$$

by putting equations (2.5) and (2.6) equal to zero then we get:

$$\hat{\sigma}_1 = \left[ \frac{1}{2m_1} \sum_{i=1}^{m_1} (1 + r_i) x_i^2 \right]^{0.5} \quad (2.7)$$

$$\hat{\sigma}_2 = \left[ \frac{1}{2m_2} \sum_{j=1}^{m_2} (1 + s_j) y_j^2 \right]^{0.5} \quad (2.8)$$

Then MLE of R is

$$\hat{R} = \frac{1}{1 + \frac{m_1 E}{m_2 T}} = \frac{1}{1 + \frac{\sigma_2^2}{\sigma_1^2} \lambda} \quad (2.9)$$

where  $T = \sum_{i=1}^{m_1} (1 + r_i)x_i^2$ ,  $E = \sum_{j=1}^{m_2} (1 + s_j)y_j^2$  and  $\lambda = \frac{\sigma_1^2 m_1 E}{\sigma_2^2 m_2 T}$ . consider the following transformations:

$$\begin{aligned} Z_1 &= n_1 X_1^2 \\ Z_2 &= (n_1 - r_1 - 1)[X_2^2 - X_1^2] \end{aligned}$$

$$Z_{m_1} = (n_1 - r_1 - r_2 - \dots - r_{m_1-1} - m_1 + 1)[X_{m_1}^2 - X_{m_1-1}^2] \quad (2.10)$$

Balakrishnan & Aggarwala [7] show that  $Z_i^s$  are independent & identically distributed exponential random variables with mean  $\sigma_1$ , moreover

$$T = \sum_{i=1}^{m_1} Z_i = \sum_{i=1}^{m_1} (1 + r_i)T_i = \sum_{i=1}^{m_1} (1 + r_i)X_i^2 \sim \text{Gamma}(m_1, \sigma_1). \quad (2.11)$$

Similarly  $E$  has a gamma distribution with the shape parameter  $m_1$  and scale parameter  $\sigma_1$ . Hence,  $\lambda$  follow F-distribution with degrees of freedom  $2m_2$  and  $2m_1$ . Using the fact That:

$$\frac{R}{1-R} \times \frac{1-\hat{R}}{\hat{R}} \sim F_{2m_2, 2m_1} \quad (2.12)$$

Then the  $100(1 - \alpha)\%$  exact confidence interval of R is:

$$P \left( \frac{1}{1 + F_{2m_1, 2m_2, \alpha/2}(\frac{1}{\hat{R}} - 1)} < R < \frac{1}{1 + F_{2m_1, 2m_2, 1-\alpha/2}(\frac{1}{\hat{R}} - 1)} \right) = 1 - \alpha \quad (2.13)$$

Where  $\alpha$  is the level of significance and  $2m_1, 2m_2$  are the degree of freedom of F.

### 3 UMVUE of R

In this section the uniformly minimum variance unbiased estimator (UMVUE) is obtained for stress-strength parameter R. Let  $X_{1:m_1:n_1}, \dots, X_{m_1:m_1:n_1}$

be a progressive censored sample from  $Ray(\sigma_1)$  with progressive censoring scheme  $(n_1, m_1, r_1, \dots, r_{m_1})$ , and the progressive censored sample  $Y_{1:m_2:n_2}, \dots, Y_{m_2:m_2:n_2}$  from  $Ray(\sigma_2)$  with progressive censoring scheme  $(n_2, m_2, s_1, \dots, s_{m_2})$ , then from the jointly log-likelihood function of  $X, Y$  that given in equation (2.4), we obtained that  $T = \sum_{i=1}^{m_1} (1 + r_i) x_i^2$  is a sufficient statistics for  $\sigma_1$  and  $E = \sum_{j=1}^{m_2} (1 + s_j) y_j^2$  is a sufficient statistics for  $\sigma_2$ . From equation (2.11)

$$f_T(t) = \frac{1}{\sigma_1^{m_1} \Gamma(m_1)} t^{m_1-1} \exp\left(-\frac{t}{\sigma_1}\right), \quad 0 < t < \infty \quad (3.1)$$

**Lemma 3.1.** *The conditional pdf. of  $T_1 = X_1^2$  given  $T$  is:*

$$f_{T_1|T}(x) = \frac{f_{T_1,T}(x)}{f_T(t)} = n_1(m_1 - 1) \frac{(T - n_1 T_1)^{m_1-2}}{T^{m_1-1}}, \quad 0 < T_1 < T/n_1 \quad (3.2)$$

*Proof.* Let  $W = \sum_{i=2}^{m_1} Z_i$  then clearly  $W$  &  $Z_1$  are independent. Then the joint pdf. of  $T_1$  &  $T$ ,  $f_{T_1,T}(x)$  can be easily obtained from the jointly distribution of  $W$  &  $Z_1$  using the transformations  $Z_1 = n_1 T_1$  &  $W = T - Z_1 = T - T_1$  then

$$f_{W,Z_1} = f_W \cdot f_{Z_1} = \frac{1}{\sigma_1^{m_1} \Gamma(m_1 - 1)} W^{m_1-2} \exp\left(-\frac{W + Z_1}{\sigma_1}\right) \quad (3.3)$$

And

$$f_{T_1,T} = \frac{n_1}{\sigma_1^{m_1} \Gamma(m_1 - 1)} (T - n_1 T_1)^{m_1-2} \exp\left(-\frac{T}{\sigma_1}\right) \quad (3.4)$$

From equations (3.4), (3.1), we get the result.  $\square$

Similarly if  $E = \sum_{j=1}^{m_2} (1 + s_j) E_j$  where  $E_j = Y_j^2$  Then

$$f_{E_1|E}(y) = n_2(m_2 - 1) \frac{(E - n_2 E_1)^{m_2-2}}{E^{m_2-1}}, \quad 0 < E_1 < E/n_2 \quad (3.5)$$

**Lemma 3.2.** *The unbiased estimator of  $R$  is:*

$$\phi(T_1, E_1) = \begin{cases} 1 & \text{if } n_2 E_1 < n_1 T_1 \\ 0 & \text{if } n_2 E_1 \geq n_1 T_1 \end{cases} \quad (3.6)$$

Where  $E_1 = Y_1^2$  and  $T_1 = X_1^2$ .

Proof.

$$\begin{aligned} E(\phi) &= 1.P(n_2 E_1 < n_1 T_1) = P(Y_1 < [\frac{n_1 x_1^2}{n_2}]^{1/2}) = P(Y_1 < a) \\ &= \int_0^\infty \int_0^a f_{X_1}(x) f_{Y_1}(y) dy dx \end{aligned} \quad (3.7)$$

Where the distributions of order statistics  $X_1$  and  $Y_1$  are

$$\begin{aligned} f_{X_1}(x) &= n_1 \frac{x}{\sigma_1^2} e^{-n_1 x^2 / 2\sigma_1^2} \\ f_{Y_1}(y) &= n_2 \frac{y}{\sigma_2^2} e^{-n_2 y^2 / 2\sigma_2^2} \end{aligned} \quad (3.8)$$

Then by using equations(3.8) we get

$$E(\phi) = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = R \quad (3.9)$$

□

**Theorem 3.3.** Based on the sufficient statistics  $T$  and  $E$ , as defined before for  $\sigma_1$  and  $\sigma_2$  respectively and the unbiased statistics  $\phi$ , the UMVUE of  $R$ , say  $\tilde{R}$ , for  $m_1 \geq 2$  and  $m_2 \geq 2$  can be expressed as follows:

$$\tilde{R} = \begin{cases} 1 - \sum_{k=0}^{m_2-1} (-1)^k \left(\frac{T}{E}\right)^k \frac{\binom{m_2-1}{k}}{\binom{m_1+k-1}{k}} & \text{if } T < E \\ \sum_{k=0}^{m_1-1} (-1)^k \left(\frac{E}{T}\right)^k \frac{\binom{m_1-1}{k}}{\binom{m_2+k-1}{k}} & \text{if } T \geq E \end{cases} \quad (3.10)$$

Proof. For  $T < E$  using the Rao-Blackwell theorem

$$\tilde{R} = E(\phi(T_1, E_1) | T, E) = \int \int_A f_{(T_1|T)} f_{(E_1|E)} dE_1 dT_1, \quad (3.11)$$

Where  $A = \{(E_1, T_1) : 0 < T_1 < \frac{T}{n_1}, 0 < E_1 < \frac{E}{n_2} \text{ and } n_2 E_1 < n_1 T_1\}$  and  $f_{(T_1|T)}$  &  $f_{(E_1|E)}$  are defined in equations(3.2),(3.5) respectively. Then  $\tilde{R}$  becomes:

$$\begin{aligned} \tilde{R} &= \int_0^{T/n_1} \int_0^{n_1 T_1/n_2} n_1(m_1-1) \frac{(T - n_1 T_1)^{m_1-2}}{T^{m_1-1}} \cdot n_2(m_2-1) \frac{(E - n_2 E_1)^{m_2-2}}{E^{m_2-1}} dE_1 dT_1 \\ &= 1 - \int_0^{T/n_1} n_1(m_1-1) \frac{(T - n_1 T_1)^{m_1-2}}{T^{m_1-1}} \frac{(E - n_1 T_1)^{m_2-1}}{E^{m_2-1}} dT_1 \end{aligned} \quad (3.12)$$

let  $c = \frac{n_1 T_1}{T}$ , then  $\tilde{R}$  becomes:

$$\tilde{R} = 1 - \int_0^1 (m_1 - 1)(1 - c)^{m_1 - 2} \left(1 - c \frac{T}{E}\right)^{m_2 - 1} dc \quad (3.13)$$

Since the binomial expansion of  $(1 - c \frac{T}{E})^{m_2 - 1} = \sum_{k=0}^{m_2 - 1} (-1)^k \binom{m_2 - 1}{k} \left(\frac{cT}{E}\right)^k$ .

Then  $\tilde{R}$  is obtained as following:

$$\begin{aligned} \tilde{R} &= 1 - \sum_{k=0}^{m_2 - 1} (-1)^k \binom{m_2 - 1}{k} \left(\frac{T}{E}\right)^k \int_0^1 c^k (1 - c)^{m_1 - 2} dc \\ &= 1 - \sum_{k=0}^{m_2 - 1} (-1)^k \left(\frac{T}{E}\right)^k \frac{\binom{m_2 - 1}{k}}{\binom{m_1 + k - 1}{k}} \end{aligned} \quad (3.14)$$

Similarly if  $T \geq E$  then  $\tilde{R}$  becomes:

$$\tilde{R} = \sum_{k=0}^{m_1 - 1} (-1)^k \left(\frac{E}{T}\right)^k \frac{\binom{m_1 - 1}{k}}{\binom{m_2 + k - 1}{k}} \quad (3.15)$$

□

## 4 Simulation Study

Within this section, the Monte Carlo simulation is performed to check the performance of the different estimators of R under several types of progressive censoring schemes. Samples are generated under progressive type-II censoring with many different schemes for the (n-m) removed items. This schemes are described as follows:

Scheme I: complete sample (n=m) i.e there is no removed items. Scheme II:  $(r_1 = 0, \dots, r_{m-1} = 0, r_m = n - m)$ . Scheme III:  $(r_1 = n - m, \dots, r_{m-1} = 0, r_m = 0)$ . Scheme IV: The remaining items (n-m) are removed equally at each failure time. For example if n=10 and m=5 then scheme IV become  $(r_1 = 1, r_2 = 1, \dots, r_5 = 1)$ . Simulation is performed 1000 times with different sample sizes  $n_1, n_2 = 10, 20, 30$  and the number of failures  $m_1, m_2 = 5, 10, 15, 20, 30$  for X and Y. Different values of parameters

$(\sigma_1, \sigma_2) = (1, 1), (1, 2), (2, 1)$  for the distributions of  $X$  and  $Y$  are used. The average estimates of MLE for  $R$  and the average of MSE's are reported in Table 1. for all set of parameters. Also simulation is constructed 1000 times for UMVUE of  $R$  suggested in Section (3) at each set of values in table 2. All tables are founded in the appendix. We note that in such cases as the effective sample size increases the estimates of  $R$  become better.

## 5 Conclusion

We have presented some efficient estimators of the stress-strength parameter  $R$  using MLE and UMVUE methods. The methods are very efficient. We have found that, our estimates of  $R$  using progressive censoring schemes are very close to estimates in case of complete samples so this estimates are better to accelerate the life testing. This work gives a general estimates since the case when sample sizes equal the number of failures is a special case. The exact confidence intervals based on MLE of  $R$  are obtained. When  $n = m$  i.e in case of complete samples the biased is decreased. Also when  $(n_1, m_1) = (n_2, n_2)$  the estimates are good. The types of schemes of progressive censoring affect on the simulation results. Numerical results are presented which exhibit the performance of the proposed methods. From the mean square error (MSE) we found that UMVUE of  $R$  is more effective than MLE of  $R$  in many cases.

## References

- [1] Abd-Elfattah, A.M., Abu-Moussa, M. H., El-Fahham and M. M. Estimation of Stress-Strength Parameter for Burr type XII distribution Based on progressive type-II Censoring, ICNHBAS



conference, 2013.

- [2] Abd-Elfattah, A.M., and R.M. Mandouh, Estimation of  $\Pr Y < X$  in Lomax case, The 39th Annual Conference on Statistics, Computer Science and Operation Research, ISSR, Cairo University, Egypt, part 1, 156-166, 2004.
- [3] Ahmed, K. E., Fakhry, M. E. and Jaheen, Z. F., Empirical Bayes estimation of  $R = P(Y < X)$  and characterizations of Burr-type X model. Journal of Statistical Planning and Inference, 64(2), 297-308, 1997.
- [4] Asgharzadeh, A., Valiollahi, R., & Raqab, M. Z. Stress-strength reliability of Weibull distribution based on progressively censored samples. Sort: Statistics and Operations Research Transactions, 35(2), 103-124, 2011.
- [5] Awad, A.M. and Charraf, M.K., Estimation of  $P(Y < X)$  in the Burr case; A comparative study. Commun. Statist. Simul., 15(2), 389-403, 1986.
- [6] Balakrishnan, N. , "Progressive censoring methodology: an appraisal", (with discussions)", TEST, vol.16, no. 2, 211 - 296, 2007.
- [7] Balakrishnan, N. and Aggarwala, R., Progressive Censoring: Theory, Methods, and Applications. Birkhuser, Boston, 2000.
- [8] Birnbaum, Z.W., On a use of the Mann-Whitney statistic. Proceedings of the Third Berkeley Symposium Math. Statistic Probability I, 13-17, 1956.

- [9] Fathipour, P., Abolhasani, A. and Khamnei, H. J., Estimating  $R = P(Y < X)$  in the Generalized Rayleigh Distribution with Different Scale Parameters. Applied Mathematical Sciences, 87-92, Vol. 7, 2, 2013.
- [10] Johnson, R. A., Stress strength models for reliability. Handbook of statistics, P. R. Krishnaiah and C. R. Rao, Amsterdam, North-Holland, 1988.
- [11] Kotz, S., Lumelskii, Pensky, M., The stress-strength model and its generalization: theory and applications, World Scientific, Singapore, 2003.
- [12] [Lord Rayleigh, J.W.] Strutt, "Wave theory of light" , Moscow-Leningrad (1940) (In Russian; translated from English).
- [13] Panahi, H., and S. Asadi. "Estimation of  $R = P[Y < X]$  for two-parameter Burr Type XII Distribution." World Academy of Science, Engineering and Technology 72, 465-470, 2010.
- [14] Saraçoğlu, B, Kinaci, I and Kundu, D, on estimation of  $R = P(Y < X)$  for exponential distribution under progressive Type II censoring, Journal of Statistics Computation and Simulation, 729-744, vol. 82, 5, 2012.

## Appendix

Table 1: MLE  $\hat{R}_1$  and MSE for R at different sets of values of  $\sigma_1, \sigma_2$  with different scheme types r,s

| $(n_1, m_1)$ | $(n_2, m_2)$ | r    | s    | $(\sigma_1, \sigma_2) = (1, 1)$<br>R=0.5 |                       | $(\sigma_1, \sigma_2) = (1, 2)$<br>R=0.2 |                      | $(\sigma_1, \sigma_2) = (2, 1)$<br>R=0.8 |                      |
|--------------|--------------|------|------|------------------------------------------|-----------------------|------------------------------------------|----------------------|------------------------------------------|----------------------|
|              |              |      |      | $\hat{R}_1$                              | MSE                   | $\hat{R}_1$                              | MSE                  | $\hat{R}_1$                              | MSE                  |
| (10,10)      | (10,10)      | Comp | Comp | 0.499078                                 | $1.1 \cdot 10^{-2}$   | 0.21291                                  | $5.89 \cdot 10^{-3}$ | 0.78995                                  | $5.84 \cdot 10^{-3}$ |
| (10,5)       | (10,5)       | II   | II   | 0.501915                                 | $2.205 \cdot 10^{-2}$ | 0.21925                                  | $1.29 \cdot 10^{-2}$ | 0.779165                                 | $1.22 \cdot 10^{-2}$ |
|              |              | III  | III  | 0.498651                                 | $1.30 \cdot 10^{-2}$  | 0.21216                                  | $5.74 \cdot 10^{-3}$ | 0.78699                                  | $6.18 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.506229                                 | $1.34 \cdot 10^{-2}$  | 0.208308                                 | $6.16 \cdot 10^{-3}$ | 0.78757                                  | $6.36 \cdot 10^{-3}$ |
| (20,20)      | (10,10)      | Comp | Comp | 0.50448                                  | $7.95 \cdot 10^{-3}$  | 0.21169                                  | $4.33 \cdot 10^{-3}$ | 0.79852                                  | $3.86 \cdot 10^{-3}$ |
| (10,10)      | (20,20)      | Comp | Comp | 0.49126                                  | $8.92 \cdot 10^{-3}$  | 0.19994                                  | $3.72 \cdot 10^{-3}$ | 0.78893                                  | $4.45 \cdot 10^{-3}$ |
| (20,10)      | (10,5)       | II   | II   | 0.516308                                 | $1.79 \cdot 10^{-2}$  | 0.22161                                  | $9.23 \cdot 10^{-3}$ | 0.79655                                  | $7.60 \cdot 10^{-3}$ |
|              |              | III  | III  | 0.512684                                 | $9.64 \cdot 10^{-3}$  | 0.21439                                  | $4.98 \cdot 10^{-3}$ | 0.79974                                  | $4.12 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.527223                                 | $1.10 \cdot 10^{-2}$  | 0.22960                                  | $6.47 \cdot 10^{-3}$ | 0.81033                                  | $4.22 \cdot 10^{-3}$ |
| (10,5)       | (20,10)      | II   | II   | 0.483445                                 | $1.81 \cdot 10^{-2}$  | 0.201719                                 | $7.44 \cdot 10^{-3}$ | 0.780636                                 | $8.89 \cdot 10^{-3}$ |
|              |              | III  | III  | 0.486932                                 | $1.03 \cdot 10^{-2}$  | 0.19692                                  | $4.26 \cdot 10^{-3}$ | 0.78366                                  | $4.66 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.467997                                 | $1.09 \cdot 10^{-2}$  | 0.19005                                  | $4.27 \cdot 10^{-3}$ | 0.773707                                 | $6.23 \cdot 10^{-3}$ |
| (20,10)      | (20,10)      | II   | II   | 0.502365                                 | $1.18 \cdot 10^{-2}$  | 0.20661                                  | $5.19 \cdot 10^{-3}$ | 0.78873                                  | $5.14 \cdot 10^{-3}$ |
|              |              | III  | III  | 0.500906                                 | $6.55 \cdot 10^{-3}$  | 0.20345                                  | $3.04 \cdot 10^{-3}$ | 0.79276                                  | $2.81 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.499608                                 | $6.44 \cdot 10^{-3}$  | 0.20333                                  | $2.77 \cdot 10^{-3}$ | 0.79387                                  | $3.31 \cdot 10^{-3}$ |
| (20,20)      | (20,20)      | Comp | Comp | 0.49969                                  | $6.08 \cdot 10^{-3}$  | 0.20485                                  | $2.85 \cdot 10^{-3}$ | 0.79693                                  | $2.53 \cdot 10^{-3}$ |
| (30,15)      | (30,15)      | II   | II   | 0.49626                                  | $8.27 \cdot 10^{-3}$  | 0.20828                                  | $3.79 \cdot 10^{-3}$ | 0.795464                                 | $3.42 \cdot 10^{-3}$ |
|              |              | III  | III  | 0.50307                                  | $4.16 \cdot 10^{-3}$  | 0.20735                                  | $2.01 \cdot 10^{-3}$ | 0.798036                                 | $1.69 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.498426                                 | $4.58 \cdot 10^{-3}$  | 0.20148                                  | $1.91 \cdot 10^{-3}$ | 0.79720                                  | $1.68 \cdot 10^{-3}$ |
| (30,30)      | (30,30)      | Comp | Comp | 0.50059                                  | $3.94 \cdot 10^{-3}$  | 0.20296                                  | $1.79 \cdot 10^{-3}$ | 0.79717                                  | $1.67 \cdot 10^{-3}$ |

Table 2: UMVUE  $\hat{R}_2$  and MSE for R at different sets of values of  $\sigma_1, \sigma_2$  with different scheme types r,s

| $(n_1, m_1)$ | $(n_2, m_2)$ | r    | s    | $(\sigma_1, \sigma_2) = (1, 1)$<br>R=0.5 |                      | $(\sigma_1, \sigma_2) = (1, 2)$<br>R=0.2 |                      | $(\sigma_1, \sigma_2) = (2, 1)$<br>R=0.8 |                      |
|--------------|--------------|------|------|------------------------------------------|----------------------|------------------------------------------|----------------------|------------------------------------------|----------------------|
|              |              |      |      | $\hat{R}_2$                              | MSE                  | $\hat{R}_2$                              | MSE                  | $\hat{R}_2$                              | MSE                  |
| (10,10)      | (10,10)      | Comp | Comp | 0.508084                                 | $1.35 \cdot 10^{-2}$ | 0.1979                                   | $5.29 \cdot 10^{-3}$ | 0.80086                                  | $5.38 \cdot 10^{-3}$ |
| (10,5)       | (10,5)       | II   | II   | 0.495515                                 | $2.82 \cdot 10^{-2}$ | 0.196772                                 | $1.13 \cdot 10^{-2}$ | 0.802039                                 | $1.27 \cdot 10^{-2}$ |
|              |              | III  | III  | 0.504539                                 | $1.52 \cdot 10^{-2}$ | 0.189784                                 | $6.19 \cdot 10^{-3}$ | 0.811118                                 | $6.29 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.49617                                  | $1.66 \cdot 10^{-2}$ | 0.19043                                  | $5.48 \cdot 10^{-3}$ | 0.810735                                 | $6.38 \cdot 10^{-3}$ |
| (20,20)      | (10,10)      | Comp | Comp | 0.49997                                  | $8.62 \cdot 10^{-3}$ | 0.20129                                  | $4.41 \cdot 10^{-3}$ | 0.80175                                  | $3.78 \cdot 10^{-3}$ |
| (10,10)      | (20,20)      | Comp | Comp | 0.50002                                  | $1.00 \cdot 10^{-2}$ | 0.19802                                  | $3.78 \cdot 10^{-3}$ | 0.80355                                  | $3.89 \cdot 10^{-3}$ |
| (20,10)      | (10,5)       | II   | II   | 0.50059                                  | $2.15 \cdot 10^{-2}$ | 0.20457                                  | $9.18 \cdot 10^{-3}$ | 0.80413                                  | $7.97 \cdot 10^{-3}$ |
|              |              | III  | III  | 0.50359                                  | $1.15 \cdot 10^{-2}$ | 0.19251                                  | $4.62 \cdot 10^{-3}$ | 0.80694                                  | $4.18 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.52110                                  | $1.21 \cdot 10^{-2}$ | 0.20198                                  | $4.81 \cdot 10^{-3}$ | 0.81298                                  | $4.13 \cdot 10^{-3}$ |
| (10,5)       | (20,10)      | II   | II   | 0.49943                                  | $2.05 \cdot 10^{-2}$ | 0.20083                                  | $8.82 \cdot 10^{-3}$ | 0.80030                                  | $9.25 \cdot 10^{-3}$ |
|              |              | III  | III  | 0.496728                                 | $1.15 \cdot 10^{-2}$ | 0.19241                                  | $4.18 \cdot 10^{-3}$ | 0.80489                                  | $4.46 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.47983                                  | $1.24 \cdot 10^{-2}$ | 0.18361                                  | $4.48 \cdot 10^{-3}$ | 0.80132                                  | $5.29 \cdot 10^{-3}$ |
| (20,10)      | (20,10)      | II   | II   | 0.50182                                  | $1.29 \cdot 10^{-2}$ | 0.20285                                  | $5.59 \cdot 10^{-3}$ | 0.79938                                  | $5.39 \cdot 10^{-3}$ |
|              |              | III  | III  | 0.49939                                  | $7.09 \cdot 10^{-3}$ | 0.19574                                  | $2.93 \cdot 10^{-3}$ | 0.80537                                  | $3.09 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.50416                                  | $6.80 \cdot 10^{-3}$ | 0.19113                                  | $2.64 \cdot 10^{-3}$ | 0.80283                                  | $2.91 \cdot 10^{-3}$ |
| (20,20)      | (20,20)      | Comp | Comp | 0.50026                                  | $6.29 \cdot 10^{-3}$ | 0.20091                                  | $2.67 \cdot 10^{-3}$ | 0.79727                                  | $2.74 \cdot 10^{-3}$ |
| (30,15)      | (30,15)      | II   | II   | 0.50353                                  | $8.53 \cdot 10^{-3}$ | 0.19523                                  | $3.54 \cdot 10^{-3}$ | 0.79645                                  | $3.56 \cdot 10^{-3}$ |
|              |              | III  | III  | 0.49911                                  | $4.64 \cdot 10^{-3}$ | 0.19582                                  | $1.72 \cdot 10^{-3}$ | 0.80565                                  | $1.88 \cdot 10^{-3}$ |
|              |              | IV   | IV   | 0.50236                                  | $4.88 \cdot 10^{-3}$ | 0.19517                                  | $1.84 \cdot 10^{-3}$ | 0.80332                                  | $1.89 \cdot 10^{-3}$ |
| (30,30)      | (30,30)      | Comp | Comp | 0.49885                                  | $4.23 \cdot 10^{-3}$ | 0.20195                                  | $1.68 \cdot 10^{-3}$ | 0.80127                                  | $1.72 \cdot 10^{-3}$ |

## On Markov chain-gambler's ruin problem with ties allowed

M. A. El-Shehawey<sup>\*</sup>, M. E. El-Tantawy<sup>\*\*</sup>, and Gh. A. Al-Shreef<sup>\*\*\*</sup>

### Abstract.

A Markov chain- gambler's ruin problem in the case that the probabilities of winning/ losing a particular game depend on the amount of the current fortune with ties allowed is considered. Closed-form formulas for the solution of the MC-GR problem and the expected duration of the game in the presence of partially absorbing- partially segregating- partially reflecting boundaries are deduced, and some very simple closed forms which immediately lead to exact and explicit formulas for some special cases are given.

**Key words:** Gambler's ruin; Markov chain-random walk; probability of ruin partially; the expected duration of the game.

### 1. Introduction

Markov chain-gambler's ruin (MC-GR) problem is typically represented by Markov chain-random walk (MC-RW) on the set of integers. It has been long considered. It can be used in various disciplines: in statistics to analyze sequential test procedures, in computer science to estimate the size of the World Wide Web using randomized algorithms, in ecology to describe individual animal movements and population dynamics, in physics as a simplified model of Brownian motion, in economics to model share prices and their derivatives, in medicine and biology where absorbing barriers give a natural model for a wide variety of phenomena (cf. Yamamoto (2012) and Uem (2013)). There are numerous applications of MC-GR problem, and has attracted some attention in the seventies in the context of cancer growth modeling (cf. Bell (1976) and Beyer and Waterman (1977)). In particular, there was significant interest in estimating the expected time for a cancerous clone of cells to reach tumor size from a single way-ward cell, rather than dying off before reaching tumor size. The analysis can help to determine the appearance of the first wayward cancerous cell, and perhaps to identify reasons for the cancerous growth (cf. El-Shehawey (1994) and references therein). We now give a brief historical review of the use of barriers in a MC-RW. Weesakul (1961) discussed MC-RW restricted between a reflecting and an absorbing barrier. Using generating functions he obtains explicit expressions for the probability of absorption. Lehner (1963) studies MC-RW with a partially reflecting barrier using combinatorial methods. Dua, Khadilkar and Sen (1976) find the bivariate generating functions of the probabilities of a particle reaching a certain state under different conditions. Percus (1985) considers an asymmetric MC-RW, with one or two boundaries. Using generating functions the

---

Department of Mathematics, Damietta University, Faculty of Science, P. O. Box 6,  
New Damietta, Egypt.

E-mails: \* el\_shehawey@du.edu.eg

\*\* mitantawi@yahoo.com

\*\*\* gh\_alshreef@du.edu.eg

probability distribution of being at position  $j$  after  $n$  steps is obtained. El-Shehawey (2000) obtains absorption probabilities at the boundaries for a MC-RW between one or two partially absorbing boundaries as well as the conditional mean for the number of steps before stopping given the absorption at a specified barrier, using conditional probabilities. The MC-GR problem was essentially formulated as follows: Two players play a series of single games in which one unit changes hands until one player goes bankrupt. Suppose that a one player (refer as gambler) starts with an initial capital of  $i$  units, against the other (refer as the opponent) has  $N - i$  units,  $0 \leq i \leq N$ . In each single game the gambler either wins one unit with probability  $p$  from his adversary, or loses it with probability  $q = 1 - p$  to his adversary. The games go on for a while until either the gambler loses all his units or wins all his opponent's units. The actual capital possessed by the gambler is thus represented by a MC-RW that is,  $\{X_n, n = 0, 1, \dots\}$  whose state space is  $S = \{0, 1, \dots, N\}$  with absorbing barriers at 0 and  $N$ , and transition probabilities  $p_{i,i+1} = p = 1 - p_{i,i-1}$  for  $i = 1, 2, \dots, N - 1$ . Absorption being interpreted as the ruin of one or the other of the gamblers.

### 1.1 Statement of the general MC-GR Problem

We consider a generalization of the MC-GR problem by introducing the following assumptions:

- 1- The initial capital of the gambler is  $j$  units,  $0 \leq j \leq N$ .
- 2- The ultimate fortune which the gambler wants to accumulate is  $N$  units.
- 3- We allow ties in the single games (the possibility of staying with the same fortune when no unit is changed).
- 4- The probabilities of winning and losing depend on the current fortune.
- 5- The probabilities of winning, losing, and neither winning nor losing one unit, respectively, are  $p_i, q_i$ , and  $r_i = 1 - p_i - q_i, 1 \leq i \leq N - 1$ , instead of  $p$  and  $q = 1 - p$  as above.
- 6- If the gambler fortune drops down to 0 unit; at the next game the gambler goes broke (ruined) with probability  $1 - \delta_0 - \rho_0, 0 \leq \delta_0, \rho_0 \leq 1$ , win the right to spin again with probability  $\delta_0$ , or starts again with one unit capital with probability  $\rho_0$ .
- 7- At the boundary  $N$ , the gambler reaches his objective; at the next game his adversary may go broke with probability  $1 - \delta_N - \rho_N, 0 \leq \delta_N, \rho_N \leq 1$ , win the right to spin again with probability  $\delta_N$ , or start again with capital of  $N - 1$  units with probability  $\rho_N$ .

One may think of this generalization as a general MC-RW with transition probabilities  $p_{i,i+1} = p_i, p_{i,i-1} = q_i$  and  $p_{i,i} = 1 - (p_i + q_i), i \in \{1, 2, \dots, N - 1\}$ ; with the following modifications at the boundaries. When the process reaches the boundary state  $b, (b = 0 \text{ or } N)$ , it is absorbed with probability  $1 - \delta_b - \rho_b$ , reflects (to the state  $b + 1$  or  $b - 1$ , respectively) with probability  $\rho_b$ , or remains at the boundary state  $b$  with probability  $\delta_b$ .

Physically, this corresponds to the situation where upon reaching the boundary state  $b$ , ( $b = 0$  or  $N$ ), the particle is either remains at the boundary state  $b$  with probability  $\delta_b$ , or lost from the system with probability  $1 - \delta_b - \rho_b$ , or turned back to the state  $b + 1$  or  $b - 1$  of the system with probability  $\rho_b$ , and reduces to the classical MC-RW problem with ties allowed.

We also generalize the case of spatial homogeneity by assuming that

$$q_j = \frac{q}{2\gamma j + 1}, p_j = \frac{p}{2\gamma j + 1} \text{ and } r_j = 1 - \frac{p + q}{2\gamma j + 1}, \text{ for } j \in S \setminus \{0, N\} \quad (1.1)$$

where  $\gamma$  is non-negative constant. Notice that the spatial homogeneity is the particular case when  $\gamma = 0$ .

We will concern with giving closed form formulas for the solutions of the general MC-GR problem, i.e., we compute the probability of ruin problem and the expected duration with "winning partially". This problem is a generalization to previous works (cf. Feller (1968), p. 345, Srinivasan, and Mehata (1978), p. 90, Percus (1985), El-Shehawey (2000) and El-Shehawey and El-Shreef (2009)), it is apparently not covered by the literature.

## 2. Closed form solution of the general MC-GR problem

Let  $P_0(j; p_j, q_j; \delta_0, \rho_0, \delta_N, \rho_N) \equiv P_0(j)$  " $P_N(j; p_j, q_j; \delta_0, \rho_0, \delta_N, \rho_N) \equiv P_N(j)$ ",  $j \in S$  be the probability of ruin partially "winning partially", for a gambler starting with  $j$  units,  $j \in S$ , and playing against an adversary with initial capital  $N - j$  units. The probabilities of winning, losing, and neither winning nor losing one unit, respectively, are  $p_i, q_i$ , and  $r_i = 1 - p_i - q_i, 1 \leq i \leq N - 1$ , in the presence of partially segregating- partially reflecting- partially absorbing barrier with probabilities  $\delta_0, \rho_0$  and  $1 - \delta_0 - \rho_0$  at  $0$ ;  $\delta_N, \rho_N$  and  $1 - \delta_N - \rho_N$  at  $N$ , respectively. The probability  $P_0(j)$  " $P_N(j)$ " satisfies the following difference equation

$$P_0(j) = p_j P_0(j + 1) + r_j P_0(j) + q_j P_0(j - 1), \quad j \in S \setminus \{0, N\}, \quad (2.1)$$

with the boundary conditions

$$P_0(0) = \sigma_0 P_0(1) + (1 - \sigma_0), \quad (2.2)$$

$$P_0(N) = \sigma_N P_0(N - 1), \quad (2.3)$$

and

$$P_N(j) = p_j P_N(j + 1) + r_j P_N(j) + q_j P_N(j - 1), \quad j \in S \setminus \{0, N\}, \quad (2.4)$$

subject to

$$P_N(0) = \sigma_0 P_N(1), \quad (2.5)$$

$$P_N(N) = \sigma_N P_N(N - 1) + (1 - \sigma_N). \quad (2.6)$$

where  $\sigma_h = \rho_h / (1 - \delta_h)$ ,  $h = 0, N$ .

Notice that, contrary to the classical MC-GR problem, here allow ties in the single games (the possibility of staying with the same fortune when no unit is changed), the probabilities of winning, losing, and neither winning nor losing depend on the current fortune, also at the boundaries 0 and  $N$ , it is possible to start, to stay, to leave or to reach.

**2.1. Main result.** The main purpose of this contribution is to find closed form formula for the solution for the general MC-GR problem. Solving the difference equation (2.1) with (2.2) and (2.3) (or (2.4) with (2.5) and (2.6)) systematically, the result is formulated in the next theorem:

**Theorem 2.1.** The probability  $P_0(j; p_j, q_j; \delta_0, \rho_0, \delta_N, \rho_N) \equiv P_0(j)$  takes the form

$$P_0(j) = 1 - \frac{(1 - \sigma_N) \left[ \sigma_0 / (1 - \sigma_0) + \sum_{m=1}^j \nu_{m-1} \right]}{(1 - \sigma_N) \sigma_0 / (1 - \sigma_0) + \sum_{m=1}^N \nu_{m-1} - \sigma_N \sum_{m=1}^{N-1} \nu_{m-1}}, \quad j \in S \quad (2.7)$$

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; \delta_0, \rho_0, \delta_N, \rho_N) \\ = \frac{(1 - \sigma_N) \left[ \sigma_0 / (1 - \sigma_0) + \sum_{m=1}^j \nu_{m-1} \right]}{(1 - \sigma_N) \sigma_0 / (1 - \sigma_0) + \sum_{m=1}^N \nu_{m-1} - \sigma_N \sum_{m=1}^{N-1} \nu_{m-1}}, \quad j \in S \quad (2.8)$$

where for  $j = 0$  the sum in the numerator is to be interpreted as 0,  $\sigma_h = \rho_h / (1 - \delta_h)$ ,

$h = 0, N$ ,  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

Notice that the complementary probability,  $P_N(j)$ , may be obtained from (2.7) by interchanging  $p_j, \delta_0, \rho_0$  and  $q_j, \delta_N, \rho_N$  respectively, and replacing  $j$  by  $N - j$ ,  $j \in S$ .

Clearly,  $P_N(j) \equiv P_N(j; p_j, q_j; \delta_0, \rho_0, \delta_N, \rho_N) = 1 - P_0(j)$ .

### 3. Some special consequences

Many interesting special cases can be derived from theorem 2.1 through an appropriate choice of the initial fortune/ winning and/or losing probabilities/ boundary probabilities, some cases are listed below

#### 3.1. MC-GR with various initial fortune

In this subsection the probabilities of winning, losing, and neither winning nor losing one unit, respectively, are  $p_i, q_i$ , and  $r_i = 1 - p_i - q_i$ ,  $1 \leq i \leq N - 1$ , and at the boundaries 0 and  $N$  there are partially segregating-partially reflecting-partially absorbing barriers with respective probabilities  $\delta_0, \rho_0$  and  $1 - \delta_0 - \rho_0$  at 0;  $\delta_N, \rho_N$  and  $1 - \delta_N - \rho_N$  at  $N$ .



(i) If the gambler starts from the origin (i.e., without initial capital), the probability of winning partially the game is given by

$$P_N(0) \equiv P_N(0; p_j, q_j; \delta_0, \rho_0, \delta_N, \rho_N) \\ = \frac{\frac{\rho_0}{1 - \delta_0 - \rho_0}}{\frac{\rho_0}{1 - \delta_0 - \rho_0} + \frac{1 - \delta_N}{1 - \delta_N - \rho_N} \left( \sum_{m=1}^N \nu_{m-1} - \frac{\rho_N}{1 - \delta_N} \sum_{m=1}^{N-1} \nu_{m-1} \right)}, \quad (3.1)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

(ii) If the gambler starts from the boundary  $N$  (i.e., with initial capital is  $N$  against an opponent without initial capital), then the probability of ruin partially is

$$P_0(N) \equiv P_0(N; p_j, q_j; \delta_0, \rho_0, \delta_N, \rho_N) \\ = \frac{\frac{\rho_N}{1 - \delta_N - \rho_N} \nu_{N-1}}{\frac{\rho_0}{1 - \delta_0 - \rho_0} + \left( \frac{1 - \delta_N}{1 - \delta_N - \rho_N} \right) \nu_{N-1} + \sum_{m=1}^{N-1} \nu_{m-1}}, \quad (3.2)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

(iii) If the gambler and his opponent's start at the middle, (i.e.,  $j = s$  and  $N = 2s$ ), then the probability of ruin partially is

$$P_0(s) \equiv P_0(s; p_j, q_j; \delta_0, \rho_0, \delta_{2s}, \rho_{2s}) \\ = 1 - \frac{\frac{\rho_0}{1 - \delta_0 - \rho_0} - \sum_{m=1}^s \nu_{m-1}}{\frac{\rho_0}{1 - \delta_0 - \rho_0} + \sum_{m=1}^{2s} \nu_{m-1} + \frac{\rho_{2s}}{1 - \delta_{2s} - \rho_{2s}} \nu_{2s-1}}, \quad (3.3)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

We note that the case with  $j = s$  and  $N = 2s$  corresponds to the popular winning-a-series-by- $s$ -games scheme in sports (often combined with a requirement on the minimum number of games to be won) (cf. Samuels (1975), Siegrist (1989) and Lengyel (2009)).

### 3.2. MC-GR with various boundary probabilities

In this subsection the initial capital of the gambler is  $j$  units,  $0 \leq j \leq N$ , and the probabilities of winning, losing, and neither winning nor losing one unit, respectively, are  $p_i, q_i$ , and  $r_i = 1 - p_i - q_i, 1 \leq i \leq N - 1$ .

(i) For the case  $0 < \rho_0 < 1$ ,  $\delta_0 = 0$  and  $0 < \rho_N < 1$ ,  $\delta_N = 0$ , the two barriers at 0 and  $N$  are partially absorbing- partially reflecting, that is when the gambler is reduced to penury, at the next game the gambler goes broke or start again with one unit capital with, respective probabilities  $1-\rho_0$  and  $\rho_0$ . When the gambler reaches his objective at the next game his adversary may go broke or start again with a capital of  $N-1$  units with, respective, probabilities  $1-\rho_N$  and  $\rho_N$ . Thus the probability of ruin partially is given by

$$P_0(j) \equiv P_0(j; p_j, q_j; 0, \rho_0, 0, \rho_N)$$

$$= 1 - \frac{(1-\rho_N) \left( \rho_0 / (1-\rho_0) + \sum_{m=1}^j v_{m-1} \right)}{(1-\rho_N) \rho_0 / (1-\rho_0) + \sum_{m=1}^N v_{m-1} - \rho_N \sum_{m=1}^{N-1} v_{m-1}}, \quad j \in S \quad (3.4)$$

where  $v_0 = 1$  and  $v_m = \prod_{i=1}^m (q_i / p_i)$ .

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; 0, \rho_0, 0, \rho_N) = 1 - P_0(j), \quad j \in S \quad (3.5)$$

where  $P_0(j)$  is given by (3.4).

a) If  $\rho_0 = 0$ ,  $\delta_0 = 0$ ,  $\delta_N = 0$  and  $0 < \rho_N < 1$ , the barriers at 0 is perfectly absorbing and at  $N$  is partially absorbing- partially reflecting, that is when the gambler is reduced to penury he may be ruined and the game ends. Hence, when the gambler reaches his objective at the next game his adversary may go broke or start again with a capital of  $N-1$  units with, respective, probabilities  $1-\rho_N$  and  $\rho_N$ . Thus the probability of ruin partially is given by

$$P_0(j) \equiv P_0(j; p_j, q_j; 0, 0, 0, \rho_N)$$

$$= 1 - \frac{(1-\rho_N) \sum_{m=1}^j v_{m-1}}{\sum_{m=1}^N v_{m-1} - \rho_N \sum_{m=1}^{N-1} v_{m-1}}, \quad j \in S \quad (3.6)$$

where  $v_0 = 1$  and  $v_m = \prod_{i=1}^m (q_i / p_i)$ .

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; 0, 0, 0, \rho_N) = 1 - P_0(j), \quad j \in S \quad (3.7)$$

where  $P_0(j)$  is given by (3.6).

If the gambler starts from the boundary  $N$  or 0, then

$$P_0(j) \equiv P_0(j; p_j, q_j; 0, 0, 0, \rho_N) = \begin{cases} \frac{\rho_N \nu_{N-1}}{\sum_{m=1}^N \nu_{m-1} - \rho_N \sum_{m=1}^{N-1} \nu_{m-1}}, & j = N \\ 1, & j = 0 \end{cases} \quad (3.8)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

b) If  $\delta_0 = 0$ ,  $\delta_N = 0$ ,  $\rho_N = 0$  and  $0 < \rho_0 < 1$ , the barriers at  $N$  is perfectly absorbing and at  $0$  is partially absorbing- partially reflecting, that is when the gambler reaches his objective his adversary may be ruined and the game ends. When the gambler is reduced to penury, at the next game the gambler goes broke or start again with one unit capital with, respective probabilities  $1 - \rho_0$  and  $\rho_0$ . Thus the probability of ruin partially is given by

$$\begin{aligned} P_0(j) &\equiv P_0(j; p_j, q_j; 0, \rho_0, 0, 0) \\ &= 1 - \frac{\rho_0 / (1 - \rho_0) + \sum_{m=1}^j \nu_{m-1}}{\rho_0 / (1 - \rho_0) + \sum_{m=1}^N \nu_{m-1}}, \quad j \in S \end{aligned} \quad (3.9)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; 0, \rho_0, 0, 0) = 1 - P_0(j), \quad j \in S \quad (3.10)$$

where  $P_0(j)$  is given by (3.9).

If the gambler starts from the boundary  $0$  or  $N$ , then

$$P_N(j) \equiv P_N(j; p_j, q_j; 0, \rho_0, 0, 0) = \begin{cases} \frac{\rho_0 / (1 - \rho_0)}{\rho_0 / (1 - \rho_0) + \sum_{m=1}^N \nu_{m-1}}, & j = 0 \\ 1, & j = N \end{cases} \quad (3.11)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

c) If  $\delta_0 = 0$ ,  $\delta_N = 0$ ,  $\rho_0 = \rho_N = \rho$  and  $0 < \rho < 1$ , the symmetric boundaries at  $0$  and  $N$  are partially-absorbing, partially-reflecting with respective probabilities  $1 - \rho$  and  $\rho$ . Then the probability of ruin partially is

$$P_0(j) = P_0(j; p_j, q_j; 0, \rho) \equiv P_0(j; p_j, q_j; 0, \rho, 0, \rho)$$

$$= 1 - \frac{\rho + (1 - \rho) \sum_{m=1}^j \nu_{m-1}}{\sum_{m=1}^N \nu_{m-1} + \rho \left( 1 - \sum_{m=1}^{N-1} \nu_{m-1} \right)}, \quad j \in S \quad (3.12)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; 0, \rho) = 1 - P_0(j), \quad j \in S \quad (3.13)$$

where  $P_0(j)$  is given by (3.12).

If the gambler starts from the boundary  $N$  or  $0$ , then

$$P_0(j) \equiv P_0(j; p_j, q_j; 0, \rho) = \begin{cases} \frac{\rho \nu_{N-1}}{\sum_{m=1}^N \nu_{m-1} + \rho \left(1 - \sum_{m=1}^{N-1} \nu_{m-1}\right)}, & j = N \\ 1 - \frac{\rho}{\sum_{m=1}^N \nu_{m-1} + \rho \left(1 - \sum_{m=1}^{N-1} \nu_{m-1}\right)}, & j = 0 \end{cases} \quad (3.14)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

We see that with the appropriate change of notation, in the special case,  $q_j = 1 - p_j$ ,  $1 \leq j \leq N - 1$ . i.e., ties are not allowed in the single games (staying with the same fortune when no unit is changed impossible) formulas (3.4) and (3.5), agree with that of El-Shehawey (2000). Formulas (3.6) and (3.7) with  $\rho_N = 0$  (also (3.9), (3.10) with  $\rho_0 = 0$ , and (3.12), (3.13) with  $\rho = 0$ ), agree with the well known results for a MC-GR problem between two perfectly absorbing barriers (see Parzen (1962), p. 233, and El-Shehawey (2000)).

ii) If  $\delta_0 = \delta_N = \delta$ ,  $\rho_N = \rho_0 = \rho$  and  $0 < \delta, \rho < 1$ , the boundary probabilities at the two boundaries  $0$  and  $N$  are symmetry, i.e., there are partially segregating-partially reflecting-partially absorbing barriers at  $0$  and  $N$  with respective probabilities  $\delta, \rho$  and  $1 - \delta - \rho$ . Thus the probability of ruin partially is given by

$$P_0(j) = P_0(j; p_j, q_j; \delta, \rho) \equiv P_0(j; p_j, q_j; \delta, \rho, \delta, \rho) \\ = 1 - \frac{1 + ((1 - \delta - \rho)/\rho) \sum_{m=1}^j \nu_{m-1}}{1 + ((1 - \delta - \rho)/\rho) \sum_{m=1}^{N-1} \nu_{m-1} + ((1 - \delta)/\rho) \nu_{N-1}}, \quad j \in S \quad (3.15)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; \delta, \rho) = 1 - P_0(j), \quad j \in S \quad (3.16)$$

where  $P_0(j)$  is given by (3.15).

a) If  $\delta_0 = \delta_N = \delta$ ,  $\rho_N = \rho_0 = 0$  and  $0 < \delta < 1$ , the symmetric boundaries at  $0$  and  $N$  are partially-absorbing, partially- segregating with respective probabilities

$1 - \delta$  and  $\delta$ , i.e., If the gambler fortune drops down to 0 unit (reaches his objective to  $N$ ) at the next game the gambler goes broke (adversary may go broke) with probability  $1 - \delta$ ,  $0 \leq \delta \leq 1$ , win the right to spin again with probability  $\delta$ . Then the probability of ruin partially is

$$P_0(j) = P_0(j; p_j, q_j; \delta, 0) \equiv P_0(j; p_j, q_j; \delta, 0, \delta, 0) \\ = \sum_{m=j}^{N-1} v_m / \sum_{m=0}^{N-1} v_m, \quad j \in S. \quad (3.17)$$

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; \delta, 0) = \sum_{m=0}^{j-1} v_m / \sum_{m=0}^{N-1} v_m, \quad j \in S \quad (3.18)$$

where  $v_0 = 1$  and  $v_m = \prod_{i=1}^m (q_i / p_i)$ .

Notice that formulas (3.17) and (3.18) are

- 1- independent on the segregating probability  $\delta$ ,
- 2- well known in the special case,  $\delta = 0$  (cf. Parzen (1962) p.233, and El-Shehawey (2000)).

b) If  $\delta_0 = \delta_N = \delta$ ,  $\rho_N = \rho_0 = \rho$  and  $\delta + \rho = 1$ , the probabilities at the two boundaries 0 and  $N$  are symmetry, i.e., there is a partially segregating-partially reflecting barrier at both 0 and  $N$  with respective probabilities  $\rho = 1 - \delta$  and  $\delta$ . Thus the probability of ruin partially is given by

$$P_0(j) = P_0(j; p_j, q_j; \delta, 1 - \delta) \equiv P_0(j; p_j, q_j; \delta, 1 - \delta, \delta, 1 - \delta) \\ = \frac{v_{N-1}}{1 + v_{N-1}}, \quad j \in S \quad (3.19)$$

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; \delta, 1 - \delta) = \frac{1}{1 + v_{N-1}}, \quad j \in S \quad (3.20)$$

where  $v_0 = 1$  and  $v_m = \prod_{i=1}^m (q_i / p_i)$ .

Notice that formulas (3.19) and (3.20) are

- 1- independent on both the segregating probability  $\delta$ , and the initial capital of the gambler  $j$ ,  $j \in S$
- 2- also obtained in the presence of two perfectly
  - i- reflecting barriers at both 0 and  $N$  (i.e.,  $\delta = 0$ )
  - ii- segregating barrier at both 0 and  $N$  (i.e.,  $\delta = 1$ ).

(iii) If the gambler starts from the initial capital  $j$ ,  $j \in S$ , the probability of winning partially the game in the presence of partially segregating-partially absorbing barrier at 0 with respective probabilities  $\delta_0$  and  $1 - \delta_0$ ,  $0 \leq \delta_0 \leq 1$ , and partially segregating- partially reflecting- partially absorbing barrier at  $N$  with respective probabilities  $\delta_N$ ,  $\rho_N$  and  $1 - \delta_N - \rho_N$ ,  $0 \leq \rho_N, \delta_N \leq 1$ , is

$$P_0(j) \equiv P_0(j; p_j, q_j; \delta_0, 0, \delta_N, \rho_N) \\ = \frac{\nu_{N-1} + ((1 - \delta_N - \rho_N)/(1 - \delta_N)) \sum_{m=j+1}^{N-1} \nu_{m-1}}{\nu_{N-1} + ((1 - \delta_N - \rho_N)/(1 - \delta_N)) \sum_{m=1}^{N-1} \nu_{m-1}}, \quad j \in S \quad (3.21)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i/p_i)$ .

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; \delta_0, 0, \delta_N, \rho_N) = 1 - P_0(j), \quad j \in S \quad (3.22)$$

where  $P_0(j)$  is given by (3.21).

Notice that formulas (3.21) and (3.22) are

1- independent on the segregating probability  $\delta_0$ .

2- also given in the presence of

i- totally absorbing barrier at 0, and partially-absorbing, partially-segregating, partially-reflecting at  $N$  ( $\delta_0 = 0$  and  $\rho_0 = 0$ ,  $0 \leq \delta_N, \rho_N \leq 1$ ).

ii- totally segregating at 0, and partially-absorbing, partially-segregating, partially-reflecting at  $N$  ( $\delta_0 = 0$  and  $\rho_0 = 0$ ,  $0 \leq \delta_N, \rho_N \leq 1$ ).

(iv) If the initial capital of the gambler is  $j$ ,  $j \in S$ , the probability of winning partially the game in the presence of partially segregating-partially absorbing barrier at  $N$  with respective probabilities  $\delta_N$  and  $1 - \delta_N$ ,  $0 \leq \delta_N \leq 1$ , and partially segregating- partially reflecting- partially absorbing barrier at 0 with respective probabilities  $\delta_0, \rho_0$  and  $1 - \delta_0 - \rho_0$ ,  $0 \leq \rho_0, \delta_0 \leq 1$ , is given by

$$P_0(j) \equiv P_0(j; p_j, q_j; \delta_0, \rho_0, \delta_N, 0) \\ = \frac{\sum_{m=j+1}^{N-1} \nu_{m-1}}{\rho_0/(1 - \delta_0 - \rho_0) + \sum_{m=1}^{N-1} \nu_{m-1}}, \quad j \in S \quad (3.23)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i/p_i)$ .

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; \delta_0, \rho_0, \delta_N, 0) = 1 - P_0(j), \quad j \in S \quad (3.24)$$

where  $P_0(j)$  is given by (3.23).

Notice that formulas (3.23) and (3.24) are

1- independent on the segregating probability  $\delta_N$ .

2- also given in the presence of

i- totally absorbing barrier at  $N$ , and partially-absorbing, partially-segregating, partially-reflecting at 0 ( $\delta_N = 0$  and  $\rho_N = 0$ ,  $0 \leq \delta_0, \rho_0 \leq 1$ ).

ii- totally segregating at  $N$ , and partially-absorbing, partially-segregating,

partially-reflecting at 0 ( $\delta_N = 0$  and  $\rho_N = 0$ ,  $0 \leq \delta_0, \rho_0 \leq 1$ ).

a) If  $\rho_N = \rho_0 = 0$  and  $0 \leq \delta_0, \delta_N \leq 1$ , the boundaries at 0 and  $N$  are partially-absorbing, partially-segregating with respective probabilities  $1 - \delta_0$  and  $\delta_0$  at 0,  $1 - \delta_N$  and  $\delta_N$  at  $N$  i.e., If the gambler fortune drops down to 0 unit (reaches his objective to  $N$ ) at the next game the gambler goes broke (adversary may go broke) with probability  $1 - \delta_0$  at 0,  $1 - \delta_N$  at  $N$ ,  $0 \leq \delta \leq 1$ , win the right to spin again with probability  $\delta_0$  at 0,  $\delta_N$  at  $N$ . Then the probability of ruin partially is

$$P_0(j) \equiv P_0(j; p_j, q_j; \delta_0, 0, \delta_N, 0) = \sum_{m=j}^{N-1} \nu_m / \sum_{m=0}^{N-1} \nu_m, \quad j \in S. \quad (3.25)$$

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, q_j; \delta_0, 0, \delta_N, 0) = \sum_{m=0}^{j-1} \nu_m / \sum_{m=0}^{N-1} \nu_m, \quad j \in S \quad (3.26)$$

where  $\nu_0 = 1$  and  $\nu_m = \prod_{i=1}^m (q_i / p_i)$ .

Notice that formulas (3.25) and (3.26) (see also (3.17) and (3.18)) are

- 1- independent of the segregating probabilities  $\delta_0$  and  $\delta_N$  ( $\delta_0 = \delta_N = \delta$ )
- 2- well known in the special case,  $\delta_0 = \delta_N = \delta = 0$  (see Parzen (1962), p.233, and El-Shehawey (2000)).

### 3.3. MC-GR with various winning and losing probabilities

In this section the initial capital of the gambler is  $j$ ,  $j \in S$ , and at the boundaries there are two partially segregating-partially reflecting-partially absorbing barriers with probabilities  $\delta_0, \rho_0$  and  $1 - \delta_0 - \rho_0$  at 0, and  $\delta_N, \rho_N$  and  $1 - \delta_N - \rho_N$  at  $N$ , respectively.

(i) If the probabilities of winning, losing, and neither winning nor losing one unit, respectively, are  $q_j = \lambda_j p_j$ , and  $r_j = 1 - (1 + \lambda_j) p_j$ ,  $0 < \lambda_j \leq 1$ ,  $1 \leq j \leq N - 1$ , the probability of ruin partially is given by

$$P_0(j) \equiv P_0(j; p_j, \lambda_j p_j; \delta_0, \rho_0, \delta_N, \rho_N) = \frac{\frac{1}{1 - \sigma_N} \prod_{i=1}^{N-1} \lambda_i + \sum_{m=1}^{N-1} \prod_{i=1}^{m-1} \lambda_i - \sum_{m=1}^j \prod_{i=1}^{m-1} \lambda_i}{\frac{\sigma_0}{1 - \sigma_0} + \frac{1}{1 - \sigma_N} \prod_{i=1}^{N-1} \lambda_i + \sum_{m=1}^{N-1} \prod_{i=1}^{m-1} \lambda_i}, \quad j \in S \quad (3.27)$$

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, \lambda_j p_j; \delta_0, \rho_0, \delta_N, \rho_N)$$

$$= \frac{\frac{\sigma_0}{1-\sigma_0} + \sum_{m=1}^j \prod_{i=1}^{m-1} \lambda_i}{\frac{\sigma_0}{1-\sigma_0} + \frac{1}{1-\sigma_N} \prod_{i=1}^{N-1} \lambda_i + \sum_{m=1}^{N-1} \prod_{i=1}^{m-1} \lambda_i}, \quad 0 < \lambda_j \leq 1, \quad j \in S \quad (3.28)$$

a) For the symmetric game ( $\lambda_j = 1, j \in S$ ), the probabilities of winning, losing, and neither winning nor losing one unit, respectively, are  $p_j, p_j$ , and  $r_j = 1 - 2p_j$ ,  $0 < j < N$ , the probability of ruin partially is given by

$$P_0(j) \equiv P_0(j; p_j, p_j; \delta_0, \rho_0, \delta_N, \rho_N) \\ = \frac{N - j + \sigma_N / (1 - \sigma_N)}{N + \sigma_0 / (1 - \sigma_0) + \sigma_N / (1 - \sigma_N)}, \quad j \in S \quad (3.29)$$

The complementary probability of ruin partially is

$$P_N(j) \equiv P_N(j; p_j, p_j; \delta_0, \rho_0, \delta_N, \rho_N) = 1 - P_0(j), \quad j \in S \quad (3.30)$$

where  $P_0(j)$  is given by (3.29).

Notice that

- 1- formulas (3.28) and (3.30) can be obtained from (3.27) and (3.29), respectively by interchanging  $\sigma_0$  and  $\sigma_N$ , and replacing  $j$  by  $N - j$ .
- 2- formulas (3.27) and (3.28) are dependent of  $\lambda_j = 1, j \in S$ , and independent of the probability of winning  $p_j, 1 \leq j \leq N - 1$ .
- 3- formulas (3.29) and (3.30) are independent of the probability of winning (or losing),  $p_j = q_j, r_j \neq 0, 1 \leq j \leq N - 1$ .
- 4- formulas (3.29) and (3.30) are also given for the following symmetric two cases,
  - i-  $p_j = q_j = 1/2$  (i.e.,  $r_j = 0$ ),  $1 \leq j \leq N - 1$ , and
  - ii-  $p_j = q_j = r_j = 1/3, 1 \leq j \leq N - 1$ .

(ii) In the case, the transition probabilities are given as (1.1), generalizes the spatial homogeneity  $q_j = q, p_j = p, r_j = r, j \in S \setminus \{0, N\}$  and  $p + q + r = 1$ . The probability of ruin partially is given by

$$P_0(j) \equiv P_0\left(j; \frac{p}{2\gamma j + 1}, \frac{q}{2\gamma j + 1}; \delta_0, \rho_0, \delta_N, \rho_N\right) \\ = \begin{cases} \left[ \left(\frac{q}{p}\right)^j - \frac{(q/p)(1 - \delta_N) - \rho_N}{1 - \delta_N - \rho_N} \left(\frac{q}{p}\right)^{N-1} \right] \left[ \frac{1 - \delta_0 - \rho_0 q/p}{1 - \delta_0 - \rho_0} \right] \\ - \frac{(q/p)(1 - \delta_N) - \rho_N}{1 - \delta_N - \rho_N} \left(\frac{q}{p}\right)^{N-1} \Bigg], & \text{if } p \neq q \quad (3.31) \\ \frac{N - j + \rho_N / (1 - \delta_N - \rho_N)}{N + \rho_0 / (1 - \delta_0 - \rho_0) + \rho_N / (1 - \delta_N - \rho_N)}, & \text{if } p = q \end{cases}$$



The complementary probability of ruin partially is

$$P_N(j) \equiv P_N\left(j; \frac{p}{2\gamma j + 1}, \frac{q}{2\gamma j + 1}; \delta_0, \rho_0, \delta_N, \rho_N\right) = 1 - P_0(j), \quad j \in S \quad (3.32)$$

where  $P_0(j)$  is given by (3.31).

The case  $p = q$  immediately follows from the case  $p \neq q$  after taking the limit as  $q/p \rightarrow 1$ .

a) The one-boundary case with the assumption that there is one partially segregating-partially reflecting-partially absorbing barriers with probabilities  $\delta_0, \rho_0$  and  $1 - \delta_0 - \rho_0$  at 0, and the other at  $\infty$ , can be obtained from (3.31) and (3.32) by taking the limit as  $N \rightarrow \infty$ :

$$\begin{aligned} P_0(j) &\equiv P_0\left(j; \frac{p}{2\gamma j + 1}, \frac{q}{2\gamma j + 1}; \delta_0, \rho_0, \delta_N, \rho_N\right) \\ &= \begin{cases} \frac{1 - \delta_0 - \rho_0}{1 - \delta_0 - \rho_0 q/p} \left(\frac{q}{p}\right)^j, & \text{if } q < p \\ 1, & \text{if } q \geq p \end{cases} \end{aligned} \quad (3.33)$$

and

$$\begin{aligned} P_N(j) &\equiv P_N\left(j; \frac{p}{2\gamma j + 1}, \frac{q}{2\gamma j + 1}; \delta_0, \rho_0, \delta_N, \rho_N\right) \\ &= \begin{cases} 1 - \frac{1 - \delta_0 - \rho_0}{1 - \delta_0 - \rho_0 q/p} \left(\frac{q}{p}\right)^j, & \text{if } q < p \\ 0, & \text{if } q \geq p \end{cases} \end{aligned} \quad (3.34)$$

Notice that, the solutions of MC-GR problem in the case given in (1.1), is the same as for the spatial homogeneity " $p_i = p, q_i = q, r_i = r = 1 - p - q, 1 \leq i \leq N - 1$ ". Therefore, the constant  $\gamma$  has no effect at all on the probabilities  $U_N(x)$ , as could have been expected.

It is straightforward to show that the present results are generalized some well known results, see for example, with the appropriate change of notation, formulas (3.33) and (3.34) in the special cases  $\delta_0 = \delta_N = 0$  agree with that of El-Shehawey (2002); and agree with that of Kac (1954, pp 295) in the special cases  $\delta_0 = \delta_N = 0$  and  $\rho_0 = 1$  for  $q \geq p$  ( see also Percus (1985)); expressions (3.31) and (3.32) in the special cases  $\delta_0 = \delta_N = 0$  and  $r_j = 0, 1 \leq j \leq N - 1$  agree with that of El-Shehawey ((2000), (2008)<sub>a</sub>, (2008)<sub>b</sub>) (see also Percus (1985) in the cases  $\delta_0 = \delta_N = 0$  and  $\rho_0 = \rho_N$ ), and in the special case  $\delta_0 = \delta_N = 0$  and  $\rho_N = 0$  agree with those of El-Shehawey and El-Shreef (2009), and in the special cases  $\delta_0 = \delta_N = 0, \rho_0 = \rho_N = 0$  and  $p = q = 1/2$  agree with those of Lefebvre (2008) (see also Feller (1968, pp 344-349), Cox and Miller (1965), Beyer and Waterman (1977), Vannucci (1990), El-Shehawey (1994), Palacios (1999) and Lengyel (2009) among others).

#### 4. The Expected Duration of the game

Let  $D_j(p_j, q_j; \delta_0, \rho_0; \delta_N, \rho_N) \equiv D_j$  be the expected duration of the game until the game ends. By conditioning on the outcome of the first bet, we have

$$D_j = p_j D_{j+1} + r_j D_j + q_j D_{j-1} + 1, \quad j \in S \setminus \{0, N\}, \quad (4.1)$$

and the quitting policy implies the following boundary conditions

$$D_0 = \sigma_0 D_1 + (\sigma_0 + \omega_0), \quad (4.2)$$

$$D_N = \sigma_N D_{N-1} + (\sigma_N + \omega_N) \quad (4.3)$$

**Theorem 4.1.** The expected number of bets  $D_j$ , until the gambler either reaches a fortune of  $N$  dollars or goes broke, is given by

$$D_j = D_1 \left( 1 + (1 - \sigma_0) \lambda_j \right) - (\sigma_0 + \omega_0) U_j - \sum_{s=1}^{j-1} \frac{1}{p_s} V_s, \quad (4.4)$$

where

$$D_1 = \left( (\sigma_0 + \omega_0) \left( (1 - \sigma_N) U_{N-1} + \prod_{k=1}^{N-1} \frac{q_k}{p_k} \right) + (1 - \sigma_N) \sum_{s=1}^{N-2} \frac{1}{p_s} V_s + \frac{1}{p_{N-1}} V_{N-1} + (\sigma_N + \omega_N) \right) / \left( (1 - \sigma_N) + (1 - \sigma_0) \left( (1 - \sigma_N) U_{N-1} + \prod_{k=1}^{N-1} \frac{q_k}{p_k} \right) \right), \quad (4.5)$$

and

$$U_m = \sum_{s=1}^{m-1} \prod_{k=1}^s \frac{q_k}{p_k}, \quad V_m = 1 + \sum_{s=2}^m \prod_{k=s}^m \frac{q_k}{p_{k-1}}, \quad \sigma_i = \frac{\rho_i}{1 - \delta_i}, \quad \omega_i = \frac{\delta_i}{1 - \delta_i}, \quad i = 0, N \quad (4.6)$$

**Corollary 4.1.** In the homogeneous case  $p_j = p, q_j = q, r_j = 0$  and partially absorbing partially reflecting symmetric boundaries  $\rho_0 = \rho_N = \rho$  and  $\delta_0 = \delta_N = 0 = \omega_0 = \omega_N$ , the expected duration of the game is given by

$$D_j = \frac{1}{p - q} \left( \frac{x^j - 1 - \rho(x^j - x)}{x^N - 1 - \rho(x^{N-1} - x)} \left( N + \frac{2\rho}{1 - \rho} \right) - j - \frac{2\rho q}{1 - \rho} \right), \quad (4.7)$$

where  $x = \frac{q}{p}$ .

This result agrees with Percus (1985).

#### 5. Conclusion.

In this paper, we have considered the so-called Markov chain- gambler's ruin problem with variable win/loss probabilities and ties allowed in the present of two partially absorbing- partially segregating- partially reflecting boundaries that generalizes the classical MC-GR problem. The probabilities of winning, losing, and neither winning nor losing depend on the amount of the current fortune, from probability boundary conditions viewpoint are taking into account. We derived exact and explicit expressions for the solutions of the introduced general MC-GR problem and the expected duration of the game, these formulas enable us to find many interesting particular cases ( some results are known and the others are "new" apparently not covered by the literature) through as appropriate choice of the boundary probabilities or the transition probabilities.

## References

- Bell G.I. 1976. Models of carcinogenesis as an escape from mitotic inhibitors, *Science* 192, 569-572.
- Beyer, W. A., and Waterman, M. S. 1977. Symmetries for conditional ruin problem. *Math. Mag.* 50, 42-45.
- Cox, D. R., and Miller, H. D. 1965. *The Theory of Stochastic Processes*. Methuen, London.
- Dua S, Khadilkar S., and Sen K. 1976. A modified random walk in the presence of partially reflecting barriers, *J. Appl. Prob.*, 13, 169-175.
- El-Shehawey, M. A. 1994. The frequency count for a random walk with absorbing boundaries: a carcinogenesis example. I, *J. Phys. A: Math. Gen.* Vol. 27, 7035-7046.
- El-Shehawey, M. A. 2000. Absorption probabilities for a random walk between two partially absorbing boundaries: I. *J Phys. A: Math. Gen.* 33, 9005- 9013..
- El-Shehawey, M. A. 2002. A semi-infinite random walk associated with the game of roulette, *J. Phys. A: Math. Gen.* Vol. 35, 1813-1820.
- El-Shehawey, M. A. 2008a. On inverses of tridiagonal matrices arising from Markov chain-random walk I. *Siam J. Matrix Anal. Appl.*, 30, 497- 508.
- El-Shehawey, M. A. 2008b. Tridiagonal random walk with one or two imperfect absorbing barriers. *Math. Slovaca.* 58, 353- 376.
- El-Shehawey, M. A., and Al-Shreef, Gh. A. 2009. On Markov chain roulette type game. *J Phys. A: Math. Theor.* 42, 195005 (11pp).
- Feller, W. 1968. *An Introduction to Probability Theory and Its Applications*, 3rd. vol. I. Wiley, New York.
- Kac, M. 1954. Random walk and the theory of Brownian motion . In *Selected Papers on Noise and Stochastic Processes* (N. Wax ed.), Dover, New York.
- Lefebvre, M. 2008. The gambler's ruin problem for a Markov chain related to the Bessel process. *Statistics and Probability Letters.* 78, 2314-2320.
- Lehner, G. 1963. One-dimensional random walk with a partially reflecting barrier, *Ann. Math. Stat.*, 34, 405-412.
- Lengyel, T. 2009. The conditional gambler's ruin problem with ties allowed. *Applied Mathematics Letters.* 22, 351- 355.
- Palacios, J.L. 1999. The ruin problem via electric networks. *American. Stat. Assoc.* 53, 67-70.
- Parzen, E. 1962. *Stochastic Processes*. Holden- Day, Inc. London.
- Percus, O. E. 1985. Phase transition in one- dimensional random walk with partially reflecting boundaries. *Adv. Appl. Prob.* 17, 594- 606.
- Samuels, S.M. 1975. The classical ruin problem with equal initial fortunes, *Math. Mag.* 48 (5) 286-288.
- Siegrist K. 1989. n-point, win-by-k games, *J. Appl. Probab.* 26, 807-814.
- Srinivasan, S. K., and Mehata, K., M. 1978. *Stochastic Processes*, Mc Graw Hill, New Delhi.
- Uem, van, T. 2013. Maximum and minimum of modified gambler's ruin problem. [arXiv: 1301.2702](https://arxiv.org/abs/1301.2702).
- Vannucci, L. 1990. Gambler's ruin problem with Markovian turns. *Riv. Mat. Sci. Econom. Social.* 13, No.1-2, 73-85.
- Weesakul, B. 1961. The random walk between a reflecting and an absorbing barrier, *Ann. Math. Statist.* 32, 765-769.
- Yamamoto, K. 2012. Exact solution to a gambler's ruin problem with a nonzero halting probability. . [arXiv: 1211.4314](https://arxiv.org/abs/1211.4314).

the first of these is the fact that the  
the second is the fact that the  
the third is the fact that the

the fourth is the fact that the  
the fifth is the fact that the  
the sixth is the fact that the

the seventh is the fact that the  
the eighth is the fact that the  
the ninth is the fact that the  
the tenth is the fact that the  
the eleventh is the fact that the  
the twelfth is the fact that the  
the thirteenth is the fact that the  
the fourteenth is the fact that the  
the fifteenth is the fact that the  
the sixteenth is the fact that the  
the seventeenth is the fact that the  
the eighteenth is the fact that the  
the nineteenth is the fact that the  
the twentieth is the fact that the  
the twenty-first is the fact that the  
the twenty-second is the fact that the  
the twenty-third is the fact that the  
the twenty-fourth is the fact that the  
the twenty-fifth is the fact that the  
the twenty-sixth is the fact that the  
the twenty-seventh is the fact that the  
the twenty-eighth is the fact that the  
the twenty-ninth is the fact that the  
the thirtieth is the fact that the  
the thirty-first is the fact that the  
the thirty-second is the fact that the  
the thirty-third is the fact that the  
the thirty-fourth is the fact that the  
the thirty-fifth is the fact that the  
the thirty-sixth is the fact that the  
the thirty-seventh is the fact that the  
the thirty-eighth is the fact that the  
the thirty-ninth is the fact that the  
the fortieth is the fact that the  
the forty-first is the fact that the  
the forty-second is the fact that the  
the forty-third is the fact that the  
the forty-fourth is the fact that the  
the forty-fifth is the fact that the  
the forty-sixth is the fact that the  
the forty-seventh is the fact that the  
the forty-eighth is the fact that the  
the forty-ninth is the fact that the  
the fiftieth is the fact that the  
the fifty-first is the fact that the  
the fifty-second is the fact that the  
the fifty-third is the fact that the  
the fifty-fourth is the fact that the  
the fifty-fifth is the fact that the  
the fifty-sixth is the fact that the  
the fifty-seventh is the fact that the  
the fifty-eighth is the fact that the  
the fifty-ninth is the fact that the  
the sixtieth is the fact that the  
the sixty-first is the fact that the  
the sixty-second is the fact that the  
the sixty-third is the fact that the  
the sixty-fourth is the fact that the  
the sixty-fifth is the fact that the  
the sixty-sixth is the fact that the  
the sixty-seventh is the fact that the  
the sixty-eighth is the fact that the  
the sixty-ninth is the fact that the  
the seventieth is the fact that the  
the seventy-first is the fact that the  
the seventy-second is the fact that the  
the seventy-third is the fact that the  
the seventy-fourth is the fact that the  
the seventy-fifth is the fact that the  
the seventy-sixth is the fact that the  
the seventy-seventh is the fact that the  
the seventy-eighth is the fact that the  
the seventy-ninth is the fact that the  
the eightieth is the fact that the  
the eighty-first is the fact that the  
the eighty-second is the fact that the  
the eighty-third is the fact that the  
the eighty-fourth is the fact that the  
the eighty-fifth is the fact that the  
the eighty-sixth is the fact that the  
the eighty-seventh is the fact that the  
the eighty-eighth is the fact that the  
the eighty-ninth is the fact that the  
the ninetieth is the fact that the  
the ninety-first is the fact that the  
the ninety-second is the fact that the  
the ninety-third is the fact that the  
the ninety-fourth is the fact that the  
the ninety-fifth is the fact that the  
the ninety-sixth is the fact that the  
the ninety-seventh is the fact that the  
the ninety-eighth is the fact that the  
the ninety-ninth is the fact that the  
the hundredth is the fact that the

the hundred and first is the fact that the  
the hundred and second is the fact that the  
the hundred and third is the fact that the

the hundred and fourth is the fact that the  
the hundred and fifth is the fact that the  
the hundred and sixth is the fact that the

the hundred and seventh is the fact that the  
the hundred and eighth is the fact that the  
the hundred and ninth is the fact that the

the hundred and tenth is the fact that the  
the hundred and eleventh is the fact that the  
the hundred and twelfth is the fact that the

the hundred and thirteenth is the fact that the  
the hundred and fourteenth is the fact that the  
the hundred and fifteenth is the fact that the

the hundred and sixteenth is the fact that the  
the hundred and seventeenth is the fact that the  
the hundred and eighteenth is the fact that the

the hundred and nineteenth is the fact that the  
the hundred and twentieth is the fact that the  
the hundred and twenty-first is the fact that the

the hundred and twenty-second is the fact that the  
the hundred and twenty-third is the fact that the  
the hundred and twenty-fourth is the fact that the

the hundred and twenty-fifth is the fact that the  
the hundred and twenty-sixth is the fact that the  
the hundred and twenty-seventh is the fact that the

the hundred and twenty-eighth is the fact that the  
the hundred and twenty-ninth is the fact that the  
the hundred and thirtieth is the fact that the

## Parameters Estimation of the Bivariate Weibull Distribution by EM Algorithm Based on Censored Samples

Sahar, A. M. Ibrahim<sup>1</sup> and Radwan, E. Bedar<sup>2</sup>

### Abstract

In this paper, the parameters of Marshall-Olkin bivariate Weibull distribution are estimated. The maximum likelihood estimators under type I censoring are estimated. Also, the expectation-maximization algorithm is considered. Numerical studies are considered to compare the estimates. Hanagal and Ahmadi [8] results may be considered as special case from our results.

**Key words:** Bivariate Weibull, censored sample, EM algorithm, maximum likelihood estimation, missing data.

### 1. Introduction

Hanagal and Ahmadi [8] estimated the parameters Marshall-Olkin bivariate exponential [11] under censored samples using maximum likelihood estimators (MLEs) and compared them by that expectation-maximization (EM) algorithm. We follow the same arguments to estimate the parameters of Marshall-Olkin bivariate Weibull (MOBW) distribution, in which the marginals are Weibull distributions

The censoring time ( $T$ ) is assumed to be independent of the life times  $(X, Y)$  of the two components. The bivariate density function of  $(X, Y)$  is denoted by  $f_{X,Y}(x, y)$ . The considered situation occurs for example in medical studies of paired organs like kidneys, eyes, lungs, or any other paired organs of an individual as a two components system which works under interdependency circumstances. Failure of an individual may censor failure of either one of the paired organ or both. This scheme of censoring is right censoring.

There are similar situation in engineering science whenever sub-systems are considered having two components with life times  $(X, Y)$  being independent of the life time ( $T$ ) of the entire system. However, failure of the main system may censor failure of

---

<sup>1,2</sup>Department of Mathematical Statistics, Institute of Statistical Studies & Research, Cairo University, Cairo City, Egypt.

either one component or both. Hanagal [6, 7] derived maximum likelihood estimators of the parameters of the bivariate exponential distribution under right censoring samples.

Kundu and Gupta [9] and Kundu and Dey [10] studied an EM algorithm for computing the MLEs of the parameters of the MOBW and bivariate generalized exponential distribution (MOBGE) in case of complete data, that is, data on all pairs of variables are available. Dewan and Nandi [3, 4] introduced an EM algorithm for computing the MLEs of the parameters of the MOBW and MOBGE in case of censored samples. They also provided the observed and the expected Fisher information matrices using 'pseudo observations' to form pseudo log-likelihood.

Censoring may also occur in other ways. Patients may be lost to follow up during the study. They may decide to move elsewhere or may become non-cooperative therefore the experimenter cannot follow them or her again. Such cases are called withdrawal from the study. A patient with censored data contributes valuable information and should therefore not be omitted from the analysis.

Because of lack of data of real processes, the data in this study are generated from the MOBW of Marshall-Olkin [11] using Mathcad software. Subsequently the EM algorithm is used to estimate the parameters.

## 2. Marshall-Olkin Bivariate Weibull Distribution

Marshall-Olkin [11] suggested a more flexible bivariate MOBW distribution with many interesting properties like joint probability density function, marginal probability density functions and joint survival function of the bivariate Weibull distribution given for  $x, y > 0$ ,  $\lambda_1, \lambda_2, \lambda_3 > 0$  and  $\alpha > 0$  as follows:

$$\bar{F}_{X,Y}(x,y) = e^{-\lambda_1 x^\alpha - \lambda_2 y^\alpha - \lambda_3 \max(x,y)} \quad (1)$$

The distribution (1) is not absolutely continuous with respect to the Lebesgue measure in  $R^2$ . It has singularities on the diagonal  $X = Y$ . The parameter  $\lambda_3$  reflects the dependence between the two lifetimes  $X$  and  $Y$  and it is also related to the simultaneous failures of the two components. When  $\lambda_3 = 0$ , the lifetimes  $X$  and  $Y$  are independent. Simultaneous failure occurs of  $X$  and  $Y$  have the same lifetime. The probability of simultaneous failure is given by

$$P_{X,Y}(\{(x,y)|x=y\}) = \frac{\lambda_3}{(\lambda_1 + \lambda_2 + \lambda_3)} \quad (2)$$

which is also equal to the correlation coefficient between  $X$  and  $Y$ .

The density functions of  $X|\{(x,y) | x > y\}$ ,  $Y|\{(x,y) | y > x\}$  and  $Z = \min(X,Y)$  are given as follows:

$$\begin{aligned} f_{X|\{(x,y) | x > y\}}(x) &= (\lambda_1 + \lambda_3)\alpha x^{\alpha-1}e^{-(\lambda_1+\lambda_3)(x-y)^\alpha} \\ f_{Y|\{(x,y) | y > x\}}(y) &= (\lambda_2 + \lambda_3)\alpha y^{\alpha-1}e^{-(\lambda_2+\lambda_3)(y-x)^\alpha} \\ f_Z(z) &= (\lambda_1 + \lambda_2 + \lambda_3)\alpha z^{\alpha-1}e^{-(\lambda_1+\lambda_2+\lambda_3)z^\alpha} \end{aligned} \quad (3)$$

This paper aims at deriving an estimation method for the parameters of a bivariate Weibull distribution of Marshall-Olkin by the EM algorithm. In Section 3, the parameters are estimated by applying the EM algorithm. Finally in Section 4, we present the results of a simulation study.

### 3. The EM Algorithm and Parameters Estimation

The EM algorithm was introduced by Dempster et al. [2], we follow the same steps of Hanagal and Ahmadi [8]. The algorithm is an iterative procedure for finding the maximum likelihood estimates for incomplete, missing, unobserved or censored data. Because it is easy to implement, the impact of the EM algorithm has been far reaching, not only as computational tool but also as a way of solving difficult statistical problems.

The density function of  $(X,Y)$  is given by

$$f_{X,Y}(x,y) = \begin{cases} \lambda_1 \alpha x^{\alpha-1} e^{-\lambda_1 x^\alpha} (\lambda_2 + \lambda_3) \alpha y^{\alpha-1} e^{-(\lambda_2+\lambda_3)y^\alpha} & \text{for } 0 < x < y \\ (\lambda_1 + \lambda_3) \alpha x^{\alpha-1} e^{-(\lambda_1+\lambda_3)x^\alpha} \lambda_2 \alpha y^{\alpha-1} e^{-\lambda_2 y^\alpha} & \text{for } 0 < y < x \\ \lambda_3 \alpha x^{\alpha-1} e^{-(\lambda_1+\lambda_2+\lambda_3)x^\alpha} & \text{for } 0 < x = y \end{cases} \quad (4)$$

For the bivariate life time distribution, we use the univariate censoring scheme given by Hanagal [6, 7] because the individuals do not enter at the same time the study and withdrawal or death of an individual or termination of the study will censor both life times of the components. Here the censoring time is independent of the life times of both components. This is the standard univariate right censoring for both failure times  $X$  and  $Y$ .

Suppose that there are  $n$  independent pairs of components, for example, paired kidneys, lungs, eyes, ears in an individual under study and  $i$ -th pair of the components have life times  $(X_i, Y_i)$  and a censoring time  $(T_i)$ . The life times associated with  $i$ -th pair of the components are given by

$$(X_i, Y_i) = \begin{cases} (X_i, Y_i) & \text{for } \max(X_i, Y_i) < T_i \\ (X_i, T_{iy}) & \text{for } X_i < T_i < T_{iy} = Y_i \\ (T_{ix}, Y_i) & \text{for } Y_i < T_i < T_{ix} = X_i \\ (T_{ix}, T_{ixy}) & \text{for } T_i < T_{ixy} = \min(X_i, Y_i) \end{cases} \quad (5)$$

where  $T_{ix}, T_{iy}$  and  $T_{ixy}$  represent the unobserved random variables.

The likelihood of the sample of size  $n$  after discarding factors which do not contain any of the parameters of interest is given as follows:

$$L_c(\underline{\lambda}, \alpha, t_{ix}, t_{iy}, t_{ixy} | x_i, y_i, t_i) = \prod_{j=1}^6 \prod_{i=1}^{n_j} f_{X,Y}^{(j)}(x_i, y_i) \quad (6)$$

where

$$\begin{aligned} f_{X,Y}^{(1)}(x, y) &= \lambda_1 \alpha x^{\alpha-1} e^{-\lambda_1 x^\alpha} (\lambda_2 + \lambda_3) \alpha y^{\alpha-1} e^{-(\lambda_2 + \lambda_3) y^\alpha} & \text{if } 0 < x < y < \infty \\ f_{X,Y}^{(2)}(x, y) &= (\lambda_1 + \lambda_3) \alpha x^{\alpha-1} e^{-(\lambda_1 + \lambda_3) x^\alpha} \lambda_2 \alpha y^{\alpha-1} e^{-\lambda_2 y^\alpha} & \text{if } 0 < y < x < \infty \\ f_{X,Y}^{(3)}(x, y) &= \lambda_3 \alpha x^{\alpha-1} e^{-(\lambda_1 + \lambda_2 + \lambda_3) x^\alpha} & \text{if } 0 < x = y < \infty \\ f_{X,Y}^{(4)}(x, y) &= \lambda_1 \alpha x^{\alpha-1} e^{-\lambda_1 x^\alpha} (\lambda_2 + \lambda_3) \alpha (t_{iy})^{\alpha-1} e^{-(\lambda_2 + \lambda_3) (t_{iy})^\alpha} & \text{if } 0 < x < t < t_y = y \\ f_{X,Y}^{(5)}(x, y) &= (\lambda_1 + \lambda_3) \alpha (t_{ix})^{\alpha-1} e^{-(\lambda_1 + \lambda_3) (t_{ix})^\alpha} \lambda_2 \alpha y^{\alpha-1} e^{-\lambda_2 y^\alpha} & \text{if } 0 < y < t < t_x = x \\ f_{X,Y}^{(6)}(x, y) &= f_{X,Y}(t_{xy}) = (\lambda_1 + \lambda_2 + \lambda_3) \alpha (t_{ixy})^{\alpha-1} & \text{if } t < t_{xy} = \min(x, y) \\ &\times e^{-(\lambda_1 + \lambda_2 + \lambda_3) (t_{ixy})^\alpha} \end{aligned} \quad (7)$$

Let  $n_1, n_2, n_3, n_4, n_5$  and  $n_6$  the number of realizations falling in the range corresponding to  $f_{X,Y}^{(1)}(x, y), f_{X,Y}^{(2)}(x, y), f_{X,Y}^{(3)}(x, y), f_{X,Y}^{(4)}(x, y), f_{X,Y}^{(5)}(x, y)$  and  $f_{X,Y}^{(6)}(x, y)$  respectively.  $f_{X,Y}^{(1)}(x, y), f_{X,Y}^{(2)}(x, y), f_{X,Y}^{(4)}(x, y)$  and  $f_{X,Y}^{(5)}(x, y)$  are density functions with respect to the Lebesgue measure on  $R^2$ , while  $f_{X,Y}^{(3)}(x, y)$  and  $f_{X,Y}^{(6)}(x, y)$  are density functions with respect to the Lebesgue measure on  $R$ .

The log-likelihood function  $\log L_c(\underline{\lambda}, \alpha, t_{ix}, t_{iy}, t_{ixy} | x_i, y_i, t_i)$  can be written as follows:

$$\begin{aligned} \ln L_c(\underline{\lambda}, \alpha, t_{ix}, t_{iy}, t_{ixy} | x_i, y_i, t_i) &= (n_1 + n_4) \ln(\lambda_1) + (n_1 + n_4) \ln(\lambda_2 + \lambda_3) + (n_2 + n_5) \ln(\lambda_2) \\ &+ (n_2 + n_5) \ln(\lambda_1 + \lambda_3) + (n_3) \ln(\lambda_3) + n_6 \ln(\lambda_1 + \lambda_2 + \lambda_3) \\ &+ (2n_1 + 2n_2 + n_3 + 2n_4 + 2n_5 + n_6) \ln(\alpha) \\ &+ (\alpha) \\ &- 1) \left\{ \sum_{i=1}^{n_1} \ln(x_i) + \sum_{i=1}^{n_1} \ln(y_i) + \sum_{i=1}^{n_2} \ln(x_i) + \sum_{i=1}^{n_2} \ln(y_i) + \sum_{i=1}^{n_3} \ln(x_i) + \sum_{i=1}^{n_4} \ln(x_i) \right. \\ &\quad \left. + \sum_{i=1}^{n_4} \ln(t_{iy}) + \sum_{i=1}^{n_5} \ln(y_i) + \sum_{i=1}^{n_5} \ln(t_{ix}) + \sum_{i=1}^{n_6} \ln(t_{ixy}) \right\} \end{aligned}$$



$$\begin{aligned}
 & -\lambda_1 \left\{ \sum_{i=1}^{n_1} (x_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (x_i)^\alpha + \sum_{i=1}^{n_4} (t_{ix})^\alpha + \sum_{i=1}^{n_4} (t_{ixy})^\alpha \right\} \\
 & -\lambda_2 \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (y_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t_{iy})^\alpha + \sum_{i=1}^{n_4} (y_i)^\alpha + \sum_{i=1}^{n_4} (t_{ixy})^\alpha \right\} \\
 & -\lambda_3 \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t_{iy})^\alpha + \sum_{i=1}^{n_4} (t_{ix})^\alpha + \sum_{i=1}^{n_4} (t_{ixy})^\alpha \right\}
 \end{aligned}$$

Thus, the following E-step and M-step are obtained:

• **E-step:**

The unobserved random variables,  $T_x, T_y$  and  $T_{xy}$  follow Weibull distributions respectively as stated in (3). Because of the loss of memory property of the Weibull distribution, the conditional distributions of  $T_x | \{(t_x, t, y) | t_x > t > y\}$ ,  $T_y | \{(t_y, t, x) | t_y > t > x\}$  and  $T_{xy} | \{(t_{xy}, t) | t_{xy} > t\}$  still follow the Weibull distributions and the corresponding density functions are as follows:

$$f_{T_x | \{(t_x, t, y) | t_x > t > y\}}(t_x) = (\lambda_1 + \lambda_3) \alpha (t_x)^{\alpha-1} e^{-(\lambda_1 + \lambda_3)(t_x)^\alpha - t^\alpha}$$

$$f_{T_y | \{(t_y, t, x) | t_y > t > x\}}(t_y) = (\lambda_2 + \lambda_3) \alpha (t_y)^{\alpha-1} e^{-(\lambda_2 + \lambda_3)(t_y)^\alpha - t^\alpha}$$

$$f_{T_{xy} | \{(t_{xy}, t) | t_{xy} > t\}}(t_{xy}) = (\lambda_1 + \lambda_2 + \lambda_3) \alpha (t_{xy})^{\alpha-1} e^{-(\lambda_1 + \lambda_2 + \lambda_3)(t_{xy})^\alpha - t^\alpha}$$

The values of the first moments of the conditional unobserved random variables,  $T_x | \{(t_x, t, y) | t_x > t > y\}$ ,  $T_y | \{(t_y, t, x) | t_y > t > x\}$  and  $T_{xy} | \{(t_{xy}, t) | t_{xy} > t\}$  are as follows.

$$E[T_x^\alpha | \{(t_x, t, y) | t_x > t > y\}] = t^\alpha + \frac{1}{(\lambda_1 + \lambda_3)} = t^\alpha + a$$

$$E[T_y^\alpha | \{(t_y, t, x) | t_y > t > x\}] = t^\alpha + \frac{1}{(\lambda_2 + \lambda_3)} = t^\alpha + b$$

$$E[T_{xy}^\alpha | \{(t_{xy}, t) | t_{xy} > t\}] = t^\alpha + \frac{1}{(\lambda_1 + \lambda_2 + \lambda_3)} = t^\alpha + c$$

The conditional expectation  $Q(\underline{\lambda} | \underline{\lambda}^{(k)})$  is obtained as follows:

$$\begin{aligned}
 Q(\underline{\lambda} | \underline{\lambda}^{(k)}) &= (n_1 + n_4) \ln(\lambda_1) + (n_1 + n_4) \ln(\lambda_2 + \lambda_3) + (n_2 + n_5) \ln(\lambda_2) \\
 &+ (n_2 + n_5) \ln(\lambda_1 + \lambda_3) + (n_3) \ln(\lambda_3) + n_6 \ln(\lambda_1 + \lambda_2 + \lambda_3) \\
 &+ (2n_1 + 2n_2 + n_3 + 2n_4 + 2n_5 + n_6) \ln(\alpha)
 \end{aligned}$$

$$\begin{aligned}
 & +(\alpha - 1) \left\{ \sum_{i=1}^{n_1} \ln(x_i) + \sum_{i=1}^{n_1} \ln(y_i) + \sum_{i=1}^{n_2} \ln(x_i) + \sum_{i=1}^{n_2} \ln(y_i) + \sum_{i=1}^{n_3} \ln(x_i) + \sum_{i=1}^{n_4} \ln(x_i) \right. \\
 & \quad \left. + \sum_{i=1}^{n_4} \ln(t_{iy}) + \sum_{i=1}^{n_5} \ln(y_i) + \sum_{i=1}^{n_5} \ln(t_{ix}) + \sum_{i=1}^{n_6} \ln(t_{ixy}) \right\} \\
 & -\lambda_1 \left\{ \sum_{i=1}^{n_1} (x_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (x_i)^\alpha + \sum_{i=1}^{n_5} (t^\alpha + a^{(k)}) + \sum_{i=1}^{n_6} (t^\alpha + c^{(k)}) \right\} \\
 & -\lambda_2 \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (y_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t^\alpha + b^{(k)}) + \sum_{i=1}^{n_5} (y_i)^\alpha + \sum_{i=1}^{n_6} (t^\alpha + c^{(k)}) \right\} \\
 & -\lambda_3 \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t^\alpha + b^{(k)}) + \sum_{i=1}^{n_5} (t^\alpha + a^{(k)}) \right. \\
 & \quad \left. + \sum_{i=1}^{n_6} (t^\alpha + c^{(k)}) \right\}
 \end{aligned}$$

where  $a_1, a_2$  and  $a_3$  equal to  $\lambda_1^{(k)}, \lambda_2^{(k)}$  and  $\lambda_3^{(k)}$  at  $k^{th}$  iteration.

• **M-Step:**

The following likelihood equations are obtained by equating the partial derivatives of  $Q(\underline{\lambda}|\underline{\lambda}^{(k)})$  with respect to  $\lambda_1, \lambda_2$  and  $\lambda_3$  to zero:

$$\begin{aligned}
 \frac{\partial Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_1} &= \frac{(n_1 + n_4)}{\lambda_1} + \frac{(n_2 + n_5)}{(\lambda_1 + \lambda_3)} + \frac{n_6}{(\lambda_1 + \lambda_2 + \lambda_3)} \\
 &- \left\{ \sum_{i=1}^{n_1} (x_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (x_i)^\alpha + \sum_{i=1}^{n_5} (t^\alpha + a^{(k)}) + \sum_{i=1}^{n_6} (t^\alpha + c^{(k)}) \right\} \\
 \frac{\partial Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_2} &= \frac{(n_1 + n_4)}{(\lambda_2 + \lambda_3)} + \frac{(n_2 + n_5)}{\lambda_2} + \frac{n_6}{(\lambda_1 + \lambda_2 + \lambda_3)} \\
 &- \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (y_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t^\alpha + b^{(k)}) + \sum_{i=1}^{n_5} (y_i)^\alpha + \sum_{i=1}^{n_6} (t^\alpha + c^{(k)}) \right\} \\
 \frac{\partial Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_3} &= \frac{(n_1 + n_4)}{(\lambda_2 + \lambda_3)} + \frac{(n_2 + n_5)}{(\lambda_1 + \lambda_3)} + \frac{n_3}{\lambda_3} + \frac{n_6}{(\lambda_1 + \lambda_2 + \lambda_3)} \\
 &- \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t^\alpha + b^{(k)}) + \sum_{i=1}^{n_5} (t^\alpha + a^{(k)}) + \sum_{i=1}^{n_6} (t^\alpha + c^{(k)}) \right\}
 \end{aligned}$$

The above likelihood equations are solved for the maximum likelihood estimates  $(\lambda_1^{(k)}, \lambda_2^{(k)}, \lambda_3^{(k)})$  using the Newton-Raphson procedure. The observed symmetric information matrix are given below as follows

$$\begin{aligned}
 -\frac{\partial^2 Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_1^2} &= \frac{(n_1 + n_4)}{(\lambda_1)^2} + \frac{(n_2 + n_5)}{(\lambda_1 + \lambda_3)^2} + \frac{n_6}{(\lambda_1 + \lambda_2 + \lambda_3)^2} \\
 -\frac{\partial^2 Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_1 \partial \lambda_2} &= 0 \\
 -\frac{\partial^2 Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_1 \partial \lambda_3} &= \frac{(n_2 + n_5)}{(\lambda_1 + \lambda_3)^2} + \frac{n_6}{(\lambda_1 + \lambda_2 + \lambda_3)^2} \\
 -\frac{\partial^2 Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_2^2} &= \frac{(n_1 + n_4)}{(\lambda_2 + \lambda_3)^2} + \frac{(n_2 + n_5)}{(\lambda_2)^2} + \frac{n_6}{(\lambda_1 + \lambda_2 + \lambda_3)^2} \\
 -\frac{\partial^2 Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_2 \partial \lambda_3} &= \frac{(n_1 + n_4)}{(\lambda_2 + \lambda_3)^2} + \frac{n_6}{(\lambda_1 + \lambda_2 + \lambda_3)^2} \\
 -\frac{\partial^2 Q(\underline{\lambda}|\underline{\lambda}^{(k)})}{\partial \lambda_3^2} &= \frac{(n_1 + n_4)}{(\lambda_2 + \lambda_3)^2} + \frac{(n_2 + n_5)}{(\lambda_1 + \lambda_3)^2} + \frac{n_3}{(\lambda_3)^2} + \frac{n_6}{(\lambda_1 + \lambda_2 + \lambda_3)^2}
 \end{aligned}$$

#### 4. Simulation Study and Conclusions

The sample data in the first case are generated based on the following algorithms:

- Step 1: generate  $z_i$  using the Weibull distribution with  $\lambda_1 = 0.08$ ,  $\lambda_2 = 0.1$ ,  $\lambda_3 = 0.0$  for  $i = 1, 2, 3$  and  $\alpha = 3$ .
- Step 2: take  $x = \min(z_1, z_3)$  and  $y = \min(z_2, z_3)$  and, therefore,  $(X, Y)$  follows the bivariate Weibull distribution of Marshall-Olkin type.
- Step 3: generate  $t_i$  using the Weibull distribution with  $\theta = 0.03$ , where the  $t_i$ s are the censoring time.

For two cases with respect to  $\lambda_i$ s, we generate 1000 set samples. Each set consisted of three samples  $n = 50, 75, 100$ . The corresponding maximum likelihood estimates are displayed in Table 1 together with the estimates and standard error (SE). The estimates denoted by  $MLE_{em}$  and  $SE_{em}$  are obtained by using the EM algorithm, while the estimates denoted by  $MLE$  and  $SE$  are obtained without EM algorithm.

The estimates  $MLE_{em}$  are close to the true parameter values and the  $SE_{em}$  decreases as the sample size increases. The estimates for both methods are obtained by taking the mean of the 1000 maximum likelihood estimates and the biased is the difference between the true parameter and the mean of the estimated parameter from the 1000 samples  $n = 50, 75, 100$ .

One can also extend the estimation procedure based on the EM algorithm for other types of bivariate Weibull distribution like those of Freund [5], Block and Basu [1], or Proschan and Sullo [12].

|                   | $\theta = 0.03 \alpha = 3$ |             |             | $\theta = 0.03 \alpha = 2$ |             |             |
|-------------------|----------------------------|-------------|-------------|----------------------------|-------------|-------------|
|                   | Case 1                     |             |             | Case 1                     |             |             |
| Parameters        | $\lambda_1$                | $\lambda_2$ | $\lambda_3$ | $\lambda_1$                | $\lambda_2$ | $\lambda_3$ |
|                   | 0.08                       | 0.1         | 0.03        | 0.1                        | 0.12        | 0.04        |
|                   | n=50                       |             |             |                            |             |             |
| MLE <sub>em</sub> | 0.081796                   | 0.101429    | 0.029931    | 0.102119                   | 0.122057    | 0.040519    |
| SE <sub>em</sub>  | 0.0003393                  | 0.000363    | 0.00022     | 0.0004232                  | 0.0004413   | 0.0002811   |
| MLE               | 0.081639                   | 0.102724    | 0.030519    | 0.102027                   | 0.123128    | 0.040752    |
| SE                | 0.0003427                  | 0.0003841   | 0.00022     | 0.000425                   | 0.0004623   | 0.0002815   |
|                   | n=75                       |             |             |                            |             |             |
| MLE <sub>em</sub> | 0.081                      | 0.099046    | 0.030233    | 0.101213                   | 0.120058    | 0.040519    |
| SE <sub>em</sub>  | 0.0001794                  | 0.0001925   | 0.0001188   | 0.0002189                  | 0.0002431   | 0.000151    |
| MLE               | 0.080878                   | 0.100963    | 0.030514    | 0.101029                   | 0.122003    | 0.040399    |
| SE                | 0.0001797                  | 0.0002141   | 0.0001219   | 0.0002199                  | 0.0002564   | 0.0001512   |
|                   | n=100                      |             |             |                            |             |             |
| MLE <sub>em</sub> | 0.080548                   | 0.099368    | 0.03021     | 0.100885                   | 0.120494    | 0.040198    |
| SE <sub>em</sub>  | 0.0001144                  | 0.0001257   | 0.000079    | 0.0001407                  | 0.0001586   | 0.0000987   |
| MLE               | 0.080532                   | 0.101133    | 0.030445    | 0.100789                   | 0.121521    | 0.04046     |
| SE                | 0.000115                   | 0.0001383   | 0.000079    | 0.0001414                  | 0.0001666   | 0.0000989   |

Table 1: Comparison of MLEs obtained using the EM algorithm and without EM algorithm.

## Appendix: Computations Without EM Algorithm

As stated above the observable life times associated with the  $i$ -th pair of components are given as follows:

$$(X_i, Y_i) = \begin{cases} (X_i, Y_i) & \text{for } \max(X_i, Y_i) < T_i \\ (X_i, T_{iy}) & \text{for } X_i < T_i < T_{iy} = Y_i \\ (T_{ix}, Y_i) & \text{for } Y_i < T_i < T_{ix} = X_i \\ (T_{ixy}, T_{ixy}) & \text{for } T_i < T_{ixy} = \min(X_i, Y_i) \end{cases} \quad (8)$$

The likelihood function for a given sample of size  $n$  is:

$$L_c(\lambda, \alpha | x_i, y_i, t_i) = \prod_{j=1}^6 \prod_{i=1}^{n_j} f_{X,Y}^{(j)}(x_i, y_i) \quad (9)$$

where

$$\begin{aligned} f_{X,Y}^{(1)}(x, y) &= \lambda_1 \alpha x^{\alpha-1} e^{-\lambda_1 x^\alpha} (\lambda_2 + \lambda_3) \alpha y^{\alpha-1} e^{-(\lambda_2 + \lambda_3) y^\alpha} & \text{if } 0 < x < y < t \\ f_{X,Y}^{(2)}(x, y) &= (\lambda_1 + \lambda_3) \alpha x^{\alpha-1} e^{-(\lambda_1 + \lambda_3) x^\alpha} \lambda_2 \alpha y^{\alpha-1} e^{-\lambda_2 y^\alpha} & \text{if } 0 < y < x < t \\ f_{X,Y}^{(3)}(x, y) &= \lambda_3 \alpha x^{\alpha-1} e^{-(\lambda_1 + \lambda_2 + \lambda_3) x^\alpha} & \text{if } 0 < x = y < t \\ f_{X,Y}^{(4)}(x, y) &= \lambda_1 \alpha x^{\alpha-1} e^{-\lambda_1 x^\alpha} e^{-(\lambda_2 + \lambda_3) t^\alpha} & \text{if } 0 < x < t < y \\ f_{X,Y}^{(5)}(x, y) &= e^{-(\lambda_1 + \lambda_3) t^\alpha} \lambda_2 \alpha y^{\alpha-1} e^{-\lambda_2 y^\alpha} & \text{if } 0 < y < t < x \\ f_{X,Y}^{(6)}(x, y) &= e^{-(\lambda_1 + \lambda_2 + \lambda_3) t^\alpha} & \text{if } t < t_i = \min(x, y) \end{aligned}$$

where  $f_{X,Y}^{(1)}, f_{X,Y}^{(2)}$  are density functions with respect to the Lebesgue measure on  $R^2$ , while  $f_{X,Y}^{(3)}, f_{X,Y}^{(4)}$  and  $f_{X,Y}^{(6)}$  are density functions with respect to The Lebesgue measure on  $R$ .

The log-likelihood is given by:

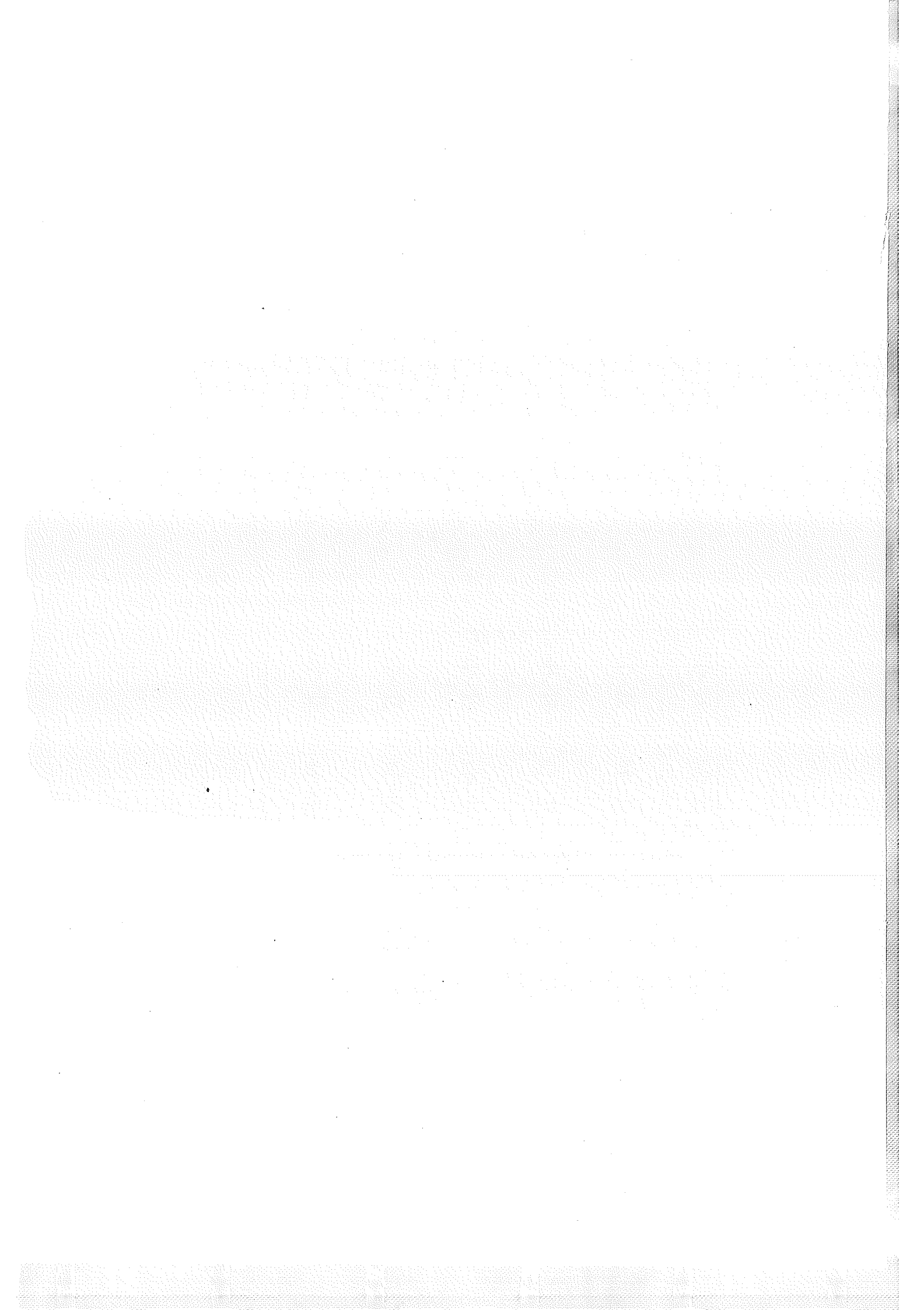
$$\begin{aligned} \ln L_c(\underline{\lambda}, \alpha | x_i, y_i, t_i) &= (n_1 + n_4) \ln(\lambda_1) + (n_1) \ln(\lambda_2 + \lambda_3) + (n_2 + n_5) \ln(\lambda_2) + (n_2) \ln(\lambda_1 + \lambda_3) \\ &+ n_3 \ln(\lambda_3) + (2n_1 + 2n_2 + n_3 + n_4 + n_5) \ln(\alpha) \\ &+ (\alpha - 1) \left\{ \sum_{i=1}^{n_1} \ln(x_i) + \sum_{i=1}^{n_1} \ln(y_i) + \sum_{i=1}^{n_2} \ln(x_i) + \sum_{i=1}^{n_2} \ln(y_i) + \sum_{i=1}^{n_3} \ln(x_i) + \sum_{i=1}^{n_4} \ln(x_i) + \sum_{i=1}^{n_5} \ln(y_i) \right\} \\ &- \lambda_1 \left\{ \sum_{i=1}^{n_1} (x_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (x_i)^\alpha + \sum_{i=1}^{n_5} (t_i)^\alpha + \sum_{i=1}^{n_6} (t_i)^\alpha \right\} \\ &- \lambda_2 \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (y_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t_i)^\alpha + \sum_{i=1}^{n_5} (y_i)^\alpha + \sum_{i=1}^{n_6} (t_i)^\alpha \right\} \\ &- \lambda_3 \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t_i)^\alpha + \sum_{i=1}^{n_5} (t_i)^\alpha + \sum_{i=1}^{n_6} (t_i)^\alpha \right\} \end{aligned}$$

The first order partial derivatives of the log-likelihood can be obtained as follows:

$$\begin{aligned} \frac{\partial \ln L_c(\underline{\lambda}, \alpha | x_i, y_i, t_i)}{\partial \lambda_1} &= \frac{(n_1 + n_4)}{\lambda_1} + \frac{n_2}{(\lambda_1 + \lambda_3)} \\ &- \left\{ \sum_{i=1}^{n_1} (x_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (x_i)^\alpha + \sum_{i=1}^{n_5} (t_i)^\alpha + \sum_{i=1}^{n_6} (t_i)^\alpha \right\} \\ \frac{\partial \ln L_c(\underline{\lambda}, \alpha | x_i, y_i, t_i)}{\partial \lambda_2} &= \frac{n_1}{(\lambda_2 + \lambda_3)} + \frac{(n_2 + n_5)}{(\lambda_2)} \\ &- \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (y_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t_i)^\alpha + \sum_{i=1}^{n_5} (y_i)^\alpha + \sum_{i=1}^{n_6} (t_i)^\alpha \right\} \\ \frac{\partial \ln L_c(\underline{\lambda}, \alpha | x_i, y_i, t_i)}{\partial \lambda_3} &= \frac{n_1}{(\lambda_2 + \lambda_3)} + \frac{n_2}{(\lambda_1 + \lambda_3)} + \frac{n_3}{\lambda_3} \\ &- \left\{ \sum_{i=1}^{n_1} (y_i)^\alpha + \sum_{i=1}^{n_2} (x_i)^\alpha + \sum_{i=1}^{n_3} (x_i)^\alpha + \sum_{i=1}^{n_4} (t_i)^\alpha + \sum_{i=1}^{n_5} (t_i)^\alpha + \sum_{i=1}^{n_6} (t_i)^\alpha \right\} \end{aligned}$$

The second order partial derivatives can be obtained as follows:

$$\begin{aligned} - \frac{\partial^2 \ln L_c(\underline{\lambda}, \alpha | x_i, y_i, t_i)}{\partial \lambda_1^2} &= \frac{(n_1 + n_4)}{(\lambda_1)^2} + \frac{n_2}{(\lambda_1 + \lambda_3)^2} \\ - \frac{\partial^2 \ln L_c(\underline{\lambda}, \alpha | x_i, y_i, t_i)}{\partial \lambda_1 \partial \lambda_2} &= 0 \\ - \frac{\partial^2 \ln L_c(\underline{\lambda}, \alpha | x_i, y_i, t_i)}{\partial \lambda_1 \partial \lambda_3} &= \frac{n_2}{(\lambda_1 + \lambda_3)^2} \end{aligned}$$



1870

1871

1872

1873

1874

1875



جامعة القاهرة  
معهد الدراسات والبحوث الإحصائية



المؤتمر السنوى الثامن والأربعون للإحصاء  
وعلوم الحاسب وبحوث العمليات

مجلد  
إحصاء رياضى

٢٣-٢٦ ديسمبر ٢٠١٣