

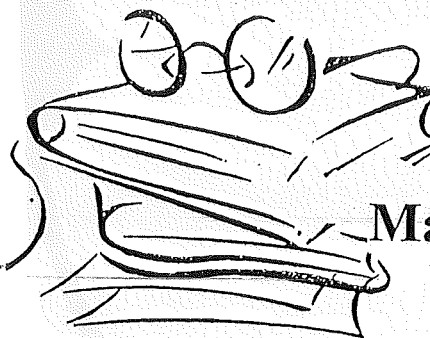


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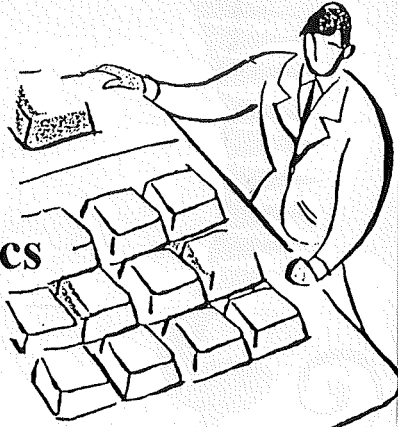
Institute of Statistical Studies and Research



THE ANNUAL CONFERENCE
ON STATISTICS, COMPUTER SCIENCE
AND OPERATIONS RESEARCH



Mathematical Statistics



26-29, Dec ,2011

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Index

MATHEMTICAL STATISTICS

1	On Distributions of Generalized Order Statistics from Kumaraswamy Distribution in Closed Forms Samir K. Safi , Rehab H. Al Sheikh	1-11
2	Estimation of the Bivariate Generalized Linear Failure Rate Distribution Parameters Under Random Censoring Hanan M. Aly , Hiba Z.Muhammed	12-28
3	Estimating and Planning Constant Stress Accelerated Life Test for Generalized Logistic Distribution Under Type-II Censoring Attia,A.F , H.M.Aly , S.O.Bleed	29-44
4	On the Correlated Ruin Problem With Boundaries El-Shehawey, M. A , Al-Shreef,Gh.A	45-59
5	Cumulative Exposure Model Based on Type II Hybrid Censored Data for Burr Type XII Distribution Abd-Elfattah.A.M. , Essawy.A. , Mubarak,Sh.	60-78
6	Parameters Estimation of the Two-Parameter Gamma Distribution Using Different Methods Elsayed A.Elsherpieny , Ahlam M.Saad	79-86
7	Inference for Partially Step Stress Modified Weibull Model Based on Censoring Schemes Elsayed A.Elsherpieny , Marwa Ahmed	87-100
8	The Empirical Probability Generating Method for Intervened Poisson Distribution Based on Rapid Estimation Elsayed A.Elsherpieny , Ola F.Abd El-Mabood	101-113
9	On the Kumaraswamy-Gumbel Minimum Distribution Elsayed A.Elsherpieny , MohamedA.Ahmed	114-132

1871. 1872. 1873. 1874. 1875. 1876. 1877. 1878. 1879. 1880.

1881. 1882. 1883. 1884. 1885. 1886. 1887. 1888. 1889. 1890.

1891. 1892. 1893. 1894. 1895. 1896. 1897. 1898. 1899. 1900.

1901. 1902. 1903. 1904. 1905. 1906. 1907. 1908. 1909. 1910.

1911. 1912. 1913. 1914. 1915. 1916. 1917. 1918. 1919. 1920.

1921. 1922. 1923. 1924. 1925. 1926. 1927. 1928. 1929. 1930.

1931. 1932. 1933. 1934. 1935. 1936. 1937. 1938. 1939. 1940.

1941. 1942. 1943. 1944. 1945. 1946. 1947. 1948. 1949. 1950.

1951. 1952. 1953. 1954. 1955. 1956. 1957. 1958. 1959. 1960.

1961. 1962. 1963. 1964. 1965. 1966. 1967. 1968. 1969. 1970.

1971. 1972. 1973. 1974. 1975. 1976. 1977. 1978. 1979. 1980.

1981. 1982. 1983. 1984. 1985. 1986. 1987. 1988. 1989. 1990.

1991. 1992. 1993. 1994. 1995. 1996. 1997. 1998. 1999. 2000.

2001. 2002. 2003. 2004. 2005. 2006. 2007. 2008. 2009. 2010.

2011. 2012. 2013. 2014. 2015. 2016. 2017. 2018. 2019. 2020.

2021. 2022. 2023. 2024. 2025. 2026. 2027. 2028. 2029. 2030.

2031. 2032. 2033. 2034. 2035. 2036. 2037. 2038. 2039. 2040.

2041. 2042. 2043. 2044. 2045. 2046. 2047. 2048. 2049. 2050.

2051. 2052. 2053. 2054. 2055. 2056. 2057. 2058. 2059. 2060.

2061. 2062. 2063. 2064. 2065. 2066. 2067. 2068. 2069. 2070.

2071. 2072. 2073. 2074. 2075. 2076. 2077. 2078. 2079. 2080.

2081. 2082. 2083. 2084. 2085. 2086. 2087. 2088. 2089. 2090.

2091. 2092. 2093. 2094. 2095. 2096. 2097. 2098. 2099. 2100.

2101. 2102. 2103. 2104. 2105. 2106. 2107. 2108. 2109. 2110.

2111. 2112. 2113. 2114. 2115. 2116. 2117. 2118. 2119. 2120.

2121. 2122. 2123. 2124. 2125. 2126. 2127. 2128. 2129. 2130.

2131. 2132. 2133. 2134. 2135. 2136. 2137. 2138. 2139. 2140.

2141. 2142. 2143. 2144. 2145. 2146. 2147. 2148. 2149. 2150.

On Distributions of Generalized Order Statistics from Kumaraswamy Distribution in Closed Forms

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Abstract

The Kumaraswamy distribution which introduced by Poondi Kumaraswamy (1980) is similar to the Beta distribution but has the key advantage of a closed-form cumulative distribution function. With its two non-negative shape parameters p and q , it was originally conceived to model hydrological phenomena, (See for example Mitnik, 2008). In this paper, we study some distributions of generalized order statistics on Kumaraswamy distribution. We have derived the joint probability density functions for generalized order statistics from Kumaraswamy distribution in closed-forms. In addition, the *pdf* of the conditional distribution for generalized order statistics from Kumaraswamy distribution is obtained. Furthermore, some special cases have been discussed.

Keywords: Kumaraswamy Distribution, Generalized Order Statistics, Random Variables, Conditional Distribution.

1. Introduction

Kumaraswamy (1980) was a leading Indian engineer and hydrologist, he introduced the distribution for variables that are lower and upper bounded. The Kumaraswamy distribution is a continuous probability distribution with double-bounded support, defined on the interval $[0,1]$ differing in the values of their two non-negative shape parameters p and q . It is similar to the Beta distribution but has the key advantage of a closed form cumulative distribution function, Carrasco (2010). The probability density function of the Kumaraswamy distribution is

$$f_z(z) = \frac{1}{(b-c)^p} p q \left(\frac{z-c}{b-c} \right)^{p-1} \left[1 - \left(\frac{z-c}{b-c} \right)^p \right]^{q-1}, c < z < b, \quad (1.1)$$

with shape parameters $p > 0$ and $q > 0$, and boundary parameters c and b . The standard form of the Kumaraswamy density function ($c=0, b=1$), $kum(p, q)$ is given by

$$f_x(x; p, q) = p q x^{p-1} (1-x^p)^{q-1} \quad (1.2)$$

The closed form of the cumulative distribution function of the Kumaraswamy distribution is given by

$$F(x; p, q) = 1 - (1-x^p)^q \quad (1.3)$$

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The median of the Kumaraswamy distribution can be written as $md(x) = \left(1 - 0.5^{\frac{1}{q}}\right)^{\frac{1}{p}}$.

The expectation of the $kum(p, q)$ is

$$E(x) = \frac{q\Gamma\left(1+\frac{1}{p}\right)\Gamma(q)}{\Gamma\left(1+q+\frac{1}{p}\right)} = qB\left(1+\frac{1}{p}, q\right)$$

Where, $B\left(1+\frac{1}{p}, q\right)$ is Beta distribution with parameters $1+\frac{1}{p}$ and q

The variance of the $kum(p, q)$ is

$$\begin{aligned} \text{Var}(x) &= \frac{q\Gamma\left(1+\frac{2}{p}\right)\Gamma(q)}{\Gamma\left(1+q+\frac{2}{p}\right)} - \left(\frac{q\Gamma\left(1+\frac{1}{p}\right)\Gamma(q)}{\Gamma\left(1+q+\frac{1}{p}\right)}\right)^2 \\ &= qB\left(1+\frac{2}{p}, q\right) - \left(qB\left(1+\frac{1}{p}, q\right)\right)^2. \end{aligned}$$

Relations between the Kumaraswamy Distribution and other Distributions
We derived some relations between the $kum(p, q)$ and other distributions.

For any non-negative shape parameters p and q ($p > 0$ and $q > 0$), we have derived the following propositions

1. $X \sqsubset Beta(1, q) \Leftrightarrow X \sqsubset kum(1, q)$.
2. $X \sqsubset Beta(p, 1) \Leftrightarrow X \sqsubset kum(p, 1)$.
3. If $Y = x^p$ then $Y \sqsubset Beta(1, q) \Leftrightarrow X \sqsubset kum(p, q)$.
4. $X \sqsubset Beta(1, q)$ then $X^{\frac{1}{p}} \sqsubset kum(p, q)$.
5. $X \sqsubset U(0, 1) \Leftrightarrow X \sqsubset kum(1, 1)$.

2. Joint Distribution of all Generalized Order Statistics

Definition 2.1 If the random variables $U(1, n, \tilde{m}, k), U(2, n, \tilde{m}, k), \dots, U(n, n, \tilde{m}, k)$ possess a joint density function of the form

$$f^{U(1, n, \tilde{m}, k), \dots, U(n, n, \tilde{m}, k)}(u_1, u_2, \dots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left[\prod_{i=1}^{n-1} (1-u_i)^{m_i} \right] (1-u_n)^{k-1}, \quad \text{on the cone}$$

$0 \leq u_1 \leq \dots \leq u_n \leq 1$ of $\square^n$, then they are called Uniform Generalized Order Statistics.

Note that $m_1, m_2, \dots, m_{n-1} \in R, K \geq 1, n \in N, 0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq 1, M_r = \sum_{j=r}^{n-1} m_j$ are parameters which satisfy $\gamma_r = k + n - r + M_r \geq 1$ for all values of $r \in \{1, 2, \dots, n-1\}$, $\tilde{m} = (m_1, m_2, \dots, m_{n-1})$. (Kamps, U., 1995)

Definition 2.2 The random variables $X(r, n, \tilde{m}, k) = F^{-1}(U(r, n, \tilde{m}, k))$ are called generalized order statistics (based on F), which have a joint density function of the form

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n) = k \left[\prod_{j=1}^{n-1} \gamma_j \right] \left[\prod_{i=1}^{n-1} (1-F(x_i))^{m_i} f(x_i) \right] (1-F(x_n))^{k-1} f(x_n),$$

$$F^{-1}(0) < X_1 \leq X_2 \leq \dots \leq X_n < F^{-1}(1). \text{ (Kamps, U., 1995)}$$

Special Case (The joint pdf of all the ordinary order statistics)

In Definition 2.2, if $k=1$ and $m_1=m_2=\dots=m_{n-1}=\text{zero}$, then

$$f^{X(1,n,0,1),\dots,X(n,n,0,1)}(x_1,\dots,x_n) \left[\prod_{j=1}^{n-1} \gamma_j \right] \left[\prod_{i=1}^{n-1} (1-F(x_i))^0 f(x_i) \right] (1-F(x_n))^{1-1} f(x_n).$$

$$\left[\prod_{j=1}^{n-1} \gamma_j \right] = \prod_{j=1}^{n-1} (k+n-j+M_j), \quad M_j = \sum_{r=j}^{n-1} m_r = \text{zero}.$$

$$\text{Then } \left[\prod_{j=1}^{n-1} \gamma_j \right] = \prod_{j=1}^{n-1} (1+n-j) = (1+n-1)(1+n-2)(1+n-3)\dots(1+n-n+1).$$

$$= n(n-1)(n-2)\dots 4.3.2.1 = n!.$$

Therefore, $f^{X(1,n,0,1),\dots,X(n,n,0,1)}(x_1,\dots,x_n) = n! \prod_{i=1}^n f(x_i)$, which is the well known pdf of all ordinary order statistics.

Theorem 2.3 The joint pdf of $X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)$ for Kumaraswamy distribution is

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n) =$$

$$k \prod_{j=1}^{n-1} \gamma_j \prod_{i=1}^{n-1} \left[\left((1-x_i^p)^q \right)^{m_i} p q x_i^{p-1} (1-x_i^p)^{q-1} \right] \left[\left((1-x_n^p)^q \right)^{k-1} p q x_n^{p-1} (1-x_n^p)^{q-1} \right]$$

We discuss some special cases in Corollary 2.4.

Corollary 2.4

1. Let $p=q=1$, then

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n) = k \prod_{j=1}^{n-1} \gamma_j \left[\prod_{i=1}^{n-1} (1-x_i)^{m_i} \right] (1-x_n)^{k-1}.$$

2. Let $p=1$, then

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n) =$$

$$k \prod_{j=1}^{n-1} \gamma_j \left[\prod_{i=1}^{n-1} \left((1-x_i)^q \right)^{m_i} q (1-x_i)^{q-1} \right] \left[\left((1-x_n)^q \right)^{k-1} q (1-x_n)^{q-1} \right].$$

3. Let $q=1$, then

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n) = k \prod_{j=1}^{n-1} \gamma_j \left[\prod_{i=1}^{n-1} (1-x_i^p)^{m_i} p x_i^{p-1} \right] (1-x_n^p)^{k-1} p x_n^{p-1}.$$

3. Joint Distribution of Two Generalized Order Statistics

Definition 3.1 The joint pdf of i^{th} and j^{th} generalized order statistics, $X(i,n,\tilde{m},k)$ and

$X(j,n,\tilde{m},k)$, is given by

$$f_{i,j,n,\tilde{m},k}(x_i, x_j) = \frac{c_j}{(i-1)!(j-i-1)!} [1-F(x_i)]^m [1-F(x_j)]^{\gamma_j-1} [g_m F(x_i)]^{i-1} \times \\ [g_m F(x_j) - g_m F(x_i)]^{j-i-1} f(x_i) f(x_j), \text{ for } 0 < x_i < x_j < \infty, 1 \leq i < j \leq n,$$

where, $c_r = \prod_{j=1}^r \gamma_j$, $\gamma_j = k + (n-j)(m+1)$ and

$$g_m(x) = \begin{cases} \frac{1}{m+1} [1 - (1-x)^{m+1}], & m \neq -1. \\ -\ln(1-x) & , m = -1, x \in (0,1) \end{cases}.$$

Since $\lim_{m \rightarrow -1} \frac{1}{m+1} [1 - (1-x)^{m+1}] = -\ln(1-x)$, we shall write $g_m(x) = \frac{1}{m+1} [1 - (1-x)^{m+1}]$ for all $x \in (0,1)$ and for all m with $g_{-1}(x) = \lim_{m \rightarrow -1} g_m(x)$. (Garg, M., 2009)

Special Case (The joint pdf of two ordinary order statistics)

In Definition (3.1), if $k = 1$ and $m = 0$, then

$$f_{i,j,n,0,1}(x_i, x_j) = \frac{c_j}{(i-1)!(j-i-1)!} [1-F(x_j)]^{\gamma_j-1} [g_m(F(x_i))]^{i-1} \times \\ [g_m(F(x_j)) - g_m(F(x_i))]^{j-i-1} f(x_i) f(x_j). \\ g_m(F(x_i)) = \frac{1}{0+1} [1 - (1-F(x_i))^{0+1}] = F(x_i), \quad g_m(F(x_j)) = F(x_j). \\ \gamma_j = n-j+1, \quad c_j = \prod_{i=1}^j \gamma_i = [1+(n-1)][1+(n-2)] \dots [1+(n-j)] = \frac{n!}{(n-j)!}.$$

Then

$$f_{i,j,n,0,1}(x_i, x_j) = \frac{c_j}{(i-1)!(j-i-1)!} [1-F(x_j)]^{n-j} [F(x_i)]^{i-1} \times \\ [F(x_j) - F(x_i)]^{j-i-1} f(x_i) f(x_j),$$

which is the well known joint pdf of two ordinary order statistics x_i and x_j .

Theorem 3.2 The joint pdf of $X(1, n, \tilde{m}, k)$ and $X(n, n, \tilde{m}, k)$ for Kumaraswamy distribution is

$$f_{1,n,n,\tilde{m},k}(x_1, x_n) = \frac{c_n}{(n-2)!} [1-x_1^p]^{mq} [1-x_n^p]^{qk-q} \times \\ \left[\frac{1}{m+1} \left((1-x_1^p)^{qm+q} - (1-x_n^p)^{qm+q} \right) \right]^{n-2} (pq)^2 (x_1 x_n)^{p-1} [(1-x_1^p)(1-x_n^p)]^{q-1}$$

Some special cases are obtained in Corollary 3.3.

Corollary 3.3

1. Let $p = q = 1$, then

$$f_{1,n,n,\tilde{m},k}(x_1, x_n) = \frac{k^n}{(n-2)!} [1-x_1]^m [1-x_n]^{k-1} \left[\frac{1}{m+1} \left((1-x_1)^{m+1} - (1-x_n)^{m+1} \right) \right]^{n-2}.$$

2. Let $p = 1$, then

$$f_{1,n,n,\bar{m},k}(x_1, x_n) = \frac{k^n}{(n-2)!} [1-x_1]^{mq} [1-x_n]^{qk-q} \left[\frac{1}{m+1} \left((1-x_1)^{q^{m+q}} - (1-x_n)^{q^{m+q}} \right) \right]^{n-2} \times q^2 [(1-x_1)(1-x_n)]^{q-1}.$$

3. Let $q = 1$, then

$$f_{1,n,n,\bar{m},k}(x_1, x_n) = \frac{k^n}{(n-2)!} [1-x_1^p]^m [1-x_n^p]^{k-1} \left[\frac{1}{m+1} \left((1-x_1^p)^{m+1} - (1-x_n^p)^{m+1} \right) \right]^{n-2} p^2 (x_1 x_n)^{p-1}$$

4. Distribution of Single Generalized Order Statistic

Definition 4.1 Further integrating out $x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n$ from Definition 2.2, we get the pdf $f_{r,n,\bar{m},k}$ of $X(r, n, m, k)$ as

$$f_{r,n,\bar{m},k}(x) = \frac{c_r}{(r-1)!} [1-F(x_r)]^{\gamma_r-1} g_m^{r-1}(F(x_r)) f(x_r)$$

Where, $c_r = \prod_{j=1}^r \gamma_j$, $\gamma_r = k + (n-r)(m+1)$ and $g_m(x) = \frac{1}{m+1} [1-(1-x)^{m+1}]$

for all $x \in (0,1)$ and for all m with $g_{-1}(x) = \lim_{x \rightarrow 1} g_m(x)$. (Garg, M., 2009)

Special Case (The pdf of the single ordinary order statistic x_r)

Using Definition 4.1, if $k=1$ and $m=0$, then $\gamma_j = n-j+1 \Rightarrow c_r = \prod_{j=1}^r n-j+1 = \frac{n!}{(n-r)!}$

$$g_0(F(x_r)) = [1-(1-F(x_r))] = F(x_r) \Rightarrow g_0^{-1}(F(x_r)) = [F(x_r)]^{r-1}.$$

$f_{r,n,0,1}(x) = \frac{n!}{(n-r)!(r-1)!} [1-F(x_r)]^{n-r} [F(x_r)]^{r-1} f(x_r)$, which is the well known pdf of the single ordinary order statistic x_r .

Lemma 4.2 The pdf of the minimum generalized order statistic is

$$f_{1,n,\bar{m},k}(x) = [k + (n-1)(m+1)] [1-F(x_1)]^{k+(n-1)(m+1)-1} f(x_1) \quad (4.1)$$

Proof:

Using Definition 4.1, let $r = 1$, then $c_1 = \gamma_1 = k + (n-1)(m+1)$, we get (4.1) and that completes the proof.

Corollary 4.3 (The pdf of the minimum ordinary order statistic)

Using (4.1), let $k = 1$ and $m = 0$ then, $f_{1,n,0,1}(x) = n [1-F(x_1)]^{n-1} f(x_1)$, which is the well known pdf of the minimum ordinary order statistic x_1 .

Lemma 4.4 The pdf of the maximum generalized order statistic is

$$f_{n,n,\tilde{m},k}(x) = \frac{[k+(n-1)(m+1)][k+(n-2)(m+1)]\dots[k]}{(n-1)!} [1-F(x_n)]^{k-1} \times \left(\frac{1}{m+1} [1-(1-F(x_n))^{m+1}] \right)^{n-1} f(x_n) \quad (4.2)$$

Proof:

Using Definition (4.1), let $r = n$, then $g_m(F(x_n)) = \frac{1}{m+1} [1-(1-F(x_n))^{m+1}]$, $\gamma_n = k$

$c_n = \prod_{j=1}^n k+(n-j)(m+1) = [k+(n-1)(m+1)][k+(n-2)(m+1)]\dots[k]$, we get (4.2) and that completes the proof.

Corollary 4.5 (The pdf of the maximum ordinary order statistic)

Using (4.2), let $k = 1$ and $m = 0$, then $f_{n,n,0,1}(x) = n[F(x_n)]^{n-1}f(x_n)$, which is the well known pdf of the maximum ordinary order statistic x_n .

Lemma 4.6 The pdf of the median generalized order statistic is

$$f_{t+1,n,\tilde{m},k}(\bar{x}) = \frac{[k+(n-1)(m+1)][k+(n-2)(m+1)]\dots[k+t(m+1)]}{t!} [1-F(\bar{x})]^{k+t(m+1)-1} \left(\frac{1}{m+1} [1-(1-F(\bar{x}))^{m+1}] \right)^t f(\bar{x}) \quad (4.3)$$

Proof:

Using Definition (4.1), based on median property, let $r-1=t$, $n-r=t$, and $n=2t+1$,

then $\gamma_r = k+t(m+1)$, $g'_m(F(\bar{x})) = \left(\frac{1}{m+1} [1-(1-F(\bar{x}))^{m+1}] \right)^t$,

$c_r = [k+(n-1)(m+1)][k+(n-2)(m+1)]\dots[k+t(m+1)]$, we get (4.3) and that completes the proof.

Corollary 4.7 (The pdf of the median ordinary order statistic)

Using (4.3), $k = 1$ and $m = 0$, then

$$f_{t+1,n,0,1}(\bar{x}) = \frac{[1+(n-1)][1+(n-2)]\dots[1+t]}{t!} [1-F(\bar{x})]^t (F(\bar{x}))^t f(\bar{x}) = \frac{(2t+1)!}{(t!)^2} [1-F(\bar{x})]^t (F(\bar{x}))^t f(\bar{x}), \text{ which is the pdf of the median ordinary order statistic.}$$

Lemma 4.8 The pdf of the r th generalized order statistic for Kumaraswamy distribution is

$$f_{r,n,\tilde{m},k}(x) = \frac{c_r}{(m+1)(r-1)!} \left[(1-x_r^p)^q \right]^{r-1} \left[\frac{1}{m+1} (1-(1-x_r^p)^{q(m+1)}) \right]^{r-1} p q (x_r)^{p-1} (1-x_r^p)^{q-1} \quad (4.4)$$

Some special cases are obtained in Corollary 4.9.

Corollary 4.9

1. Let $p = q = 1$, then

$$f_{r,n,\bar{m},k}(x) = \frac{c_r}{(r-1)!} \left[(1-x_r) \right]^{r-1} \left(\frac{1}{m+1} \left[1 - (1-x_r)^{m+1} \right] \right)^{r-1}.$$

2. Let $p = 1$, then

$$f_{r,n,\bar{m},k}(x) = \frac{c_r}{(r-1)!} \left[(1-x_r)^q \right]^{r-1} \left(\frac{1}{m+1} \left[1 - \left((1-x_r)^q \right)^{m+1} \right] \right)^{r-1} q (1-x_r)^{q-1}.$$

3. Let $q = 1$, then

$$f_{r,n,\bar{m},k}(x) = \frac{c_r}{(r-1)!} \left[(1-x_r^p) \right]^{r-1} \left(\frac{1}{m+1} \left[1 - (1-x_r^p)^{m+1} \right] \right)^{r-1} p (x_r)^{p-1}.$$

Lemma 4.10 The pdf of the minimum generalized order statistic for Kumaraswamy distribution is

$$f_{1,n,\bar{m},k}(x) = k + (n-1)(m+1) (1-x_1^p)^{q[k+(n-1)(m+1)-1]} p q x_1^{p-1} (1-x_1^p)^{q-1} \quad (4.5)$$

Proof:

Using Definition (4.1), let $r = 1$, then $c_1 = \prod_{j=1}^1 \gamma_j$, $\gamma_1 = k + (n-1)(m+1)$, we get (4.5) and that completes the proof.

We discuss some special cases in Corollary 4.11.

Corollary 4.11

1. Let $p = q = 1$ then $f_{1,n,\bar{m},k}(x) = k + (n-1)(m+1) \left[(1-x_1) \right]^{k+(n-1)(m+1)-1}$.
2. Let $p = 1$ then $f_{1,n,\bar{m},k}(x) = k + (n-1)(m+1) \left[(1-x_1)^q \right]^{k+(n-1)(m+1)-1} q (1-x_1)^{q-1}$.
3. Let $q = 1$ then $f_{1,n,\bar{m},k}(x) = k + (n-1)(m+1) \left[(1-x_1^p) \right]^{k+(n-1)(m+1)-1} p (x_1)^{p-1}$.

Lemma 4.12 The pdf of the maximum generalized order statistic for Kumaraswamy distribution is

$$f_{n,n,\bar{m},k}(x) = \frac{[k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k]}{(n-1)!} \left[(1-x_n^p)^q \right]^{k-1} \left(\frac{1}{m+1} \left[1 - \left((1-x_n^p)^q \right)^{m+1} \right] \right)^{n-1} p q (x_n)^{p-1} (1-x_n^p)^{q-1} \quad (4.6)$$

Proof:

Using Definition (4.1), let $r = n$, then $c_n = \prod_{j=1}^n \gamma_j$, $\gamma_j = k + (n-j)(m+1)$, $\gamma_n = k$,

$c_n = [k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k]$, and

$g_m(F(x_n)) = \frac{1}{m+1} [1 - (1 - F(x_n))^{m+1}]$, we get (4.6) and that completes the proof.

Some special cases are obtained in Corollary 4.13.

Corollary 4.13

1. Let $p = q = 1$, then

$$f_{n,n,\bar{m},k}(x) = \frac{[k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k]}{(n-1)!} \times \\ \left[(1-x_n)^{k-1} \left(\frac{1}{m+1} [1 - (1-x_n)^{m+1}] \right)^{n-1} \right].$$

2. Let $p = 1$, then

$$f_{n,n,\bar{m},k}(x) = \frac{[k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k]}{(n-1)!} \left[(1-x_n)^q \right]^{k-1} \times \\ \left(\frac{1}{m+1} [1 - ((1-x_n)^q)^{m+1}] \right)^{n-1} q (1-x_n)^{q-1}.$$

3. Let $q = 1$, then

$$f_{n,n,\bar{m},k}(x) = \frac{[k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k]}{(n-1)!} \left[(1-x_n^p)^{k-1} \right] \times \\ \left(\frac{1}{m+1} [1 - (1-x_n^p)^{m+1}] \right)^{n-1} p (x_n)^{p-1}.$$

Lemma 4.14 The pdf of the median generalized order statistic for Kumaraswamy distribution is

$$f_{t+1,n,\bar{m},k}(\bar{x}) = \frac{[k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k + t(m+1)]}{t!} \times \\ \left[(1-\bar{x}^p)^q \right]^{k+t(m+1)-1} \left(\frac{1}{m+1} [1 - ((1-\bar{x}^p)^q)^{m+1}] \right)^t p q (\bar{x})^{p-1} (1-\bar{x}^p)^{q-1} \quad (4.7)$$

Proof:

Using Definition (4.1), based on median property, let $r-1=t$, $n-r=t$, and $n=2t+1$, then

$$c_r = [k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k + t(m+1)]$$

$\gamma_r = k + (n-r)(m+1) = k + t(m+1)$, and $g_m'[F(\bar{x})] = \left(\frac{1}{m+1} [1 - (1-F(x))^{m+1}] \right)'$, we get (4.7) and that completes the proof.

Some special cases are obtained in Corollary 4.15.

Corollary 4.15

1. Let $p = q = 1$, then

$$f_{t+1,n,\tilde{m},k}(\bar{x}) = \frac{[k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k + t(m+1)]}{t!} \times \\ \left[(1-\bar{x}) \right]^{k+t(m+1)-1} \left(\frac{1}{m+1} \left[1 - ((1-\bar{x}))^{m+1} \right] \right)^t.$$

2. Let $p = 1$, then

$$f_{t+1,n,\tilde{m},k}(\bar{x}) = \frac{[k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k + t(m+1)]}{t!} \times \\ \left[(1-\bar{x})^q \right]^{k+t(m+1)-1} \left(\frac{1}{m+1} \left[1 - ((1-\bar{x})^q)^{m+1} \right] \right)^t q (1-\bar{x})^{q-1}.$$

3. Let $q = 1$, then

$$f_{t+1,n,\tilde{m},k}(\bar{x}) = \frac{[k + (n-1)(m+1)][k + (n-2)(m+1)] \dots [k + t(m+1)]}{t!}$$

5. Conditional Distribution of Generalized Order Statistics

Theorem 5.1 (Samuel, 2005)

Let $X_1, \dots, X_n$ be a random sample from a continuous population with cdf $F(x)$ and pdf $f(x)$. Let $X(1, n, \tilde{m}, k), X(2, n, \tilde{m}, k), \dots, X(n, n, \tilde{m}, k)$ denote the generalized order statistics obtained from this sample. Then the conditional distribution of $X(s, n, \tilde{m}, k) | X(r, n, \tilde{m}, k) = x$ for $r < s$ is

$$h(X(s, n, \tilde{m}, k) | X(r, n, \tilde{m}, k) = x) = \\ \frac{c_s [1 - F(x)]^m [1 - F(y)]^{\gamma_s - 1} [g_m(F(y)) - g_m(F(x))]^{\gamma_s - r - 1} f(y)}{c_r (s - r - 1)! [1 - F(x)]^{\gamma_r - 1}},$$

$$F^{-1}(0) < x \leq y < F^{-1}(1)$$

Notes

$$c_r = \prod_{j=1}^r \gamma_j, \quad \gamma_j = k + (n-j)(m+1) \text{ and } g_m(x) = \frac{1}{m+1} [1 - (1-x)^{m+1}] \text{ for all } x \in (0,1)$$

and for all m with $g_{-1}(x) = \lim_{m \rightarrow -1} g_m(x)$.

Special Case (Conditional distribution of an ordinary order statistics)

Using Theorem (5.1), if $k = 1$ and $m = 0$, then $h(X(s, n, 0, 1) | X(r, n, 0, 1) = x)$

$$\frac{[1-F(x)]^0 [1-F(y)]^{\gamma_s-1} [g_0(F(y)) - g_0(F(x))]^{s-r-1} f(y) c_s}{[1-F(x)]^{\gamma_r-1} c_r (s-r-1)!}, x(r, n, 0, 1) \leq y(s, n, 0, 1) < \infty$$

$$\gamma_r = 1 + (n-r), \quad \gamma_s = 1 + (n-s), \quad \gamma_{r+1} = 1 + (n-r-1) = n-r, g_0(F(y)) = F(y),$$

$$g_0(F(x)) = F(x), c_s = n(n-1)(n-2)\dots(1+n-s) = \frac{n!}{(n-s)!}, \text{ and}$$

$$c_r = n(n-1)(n-2)\dots(1+n-r) = \frac{n!}{(n-r)!}. \text{ Then}$$

$$h(X(s, n, 0, 1) | X(r, n, 0, 1)) = \frac{[1-F(y)]^{n-s} [(F(y)) - (F(x))]^{s-r-1} f(y)(n-r)!}{[1-F(x)]^{n-r} (n-s)!(s-r-1)!}.$$

Which is the well known conditional distribution of an ordinary order statistics $X(s, n, 0, 1) | X(r, n, 0, 1)$.

Theorem 5.2 The conditional pdf of $X(s, n, \tilde{m}, k) | X(r, n, \tilde{m}, k) = x$ for Kumaraswamy distribution is

$$h(X(s, n, \tilde{m}, k) | X(r, n, \tilde{m}, k)) = \frac{c_s}{c_r (s-r-1)!} [1-x^p]^{q(m-\gamma_r+1)} \left[\frac{1}{m+1} [(1-x^p)^{q(m+1)} - (1-y^p)^{q(m+1)}] \right]^{s-r-1} \times [1-y^p]^{q(\gamma_s-1)} p q(y)^{p-1} (1-y^p)^{q-1}, F^{-1}(0) < x \leq y < F^{-1}(1)$$

Some special cases are derived in Corollary 5.3.

Corollary 5.3

1. Let $p = q = 1$, then

$$h(X(s, n, \tilde{m}, k) | X(r, n, \tilde{m}, k)) =$$

$$\frac{c_s}{c_r (s-r-1)!} [1-x]^{(m-\gamma_r+1)} \left[\frac{1}{m+1} [(1-x)^{(m+1)} - (1-y)^{(m+1)}] \right]^{s-r-1} [1-y]^{(\gamma_s-1)}$$

2. Let $p = 1$, then

$$h(X(s, n, \tilde{m}, k) | X(r, n, \tilde{m}, k)) =$$

$$\frac{c_s}{c_r (s-r-1)!} [1-x]^{q(m-\gamma_r+1)} \left[\frac{1}{m+1} [(1-x)^{q(m+1)} - (1-y)^{q(m+1)}] \right]^{s-r-1} [1-y]^{q(\gamma_s-1)} q (1-y^p)^{q-1}$$

3. Let $q = 1$, then

$$h(X(s, n, \tilde{m}, k) | X(r, n, \tilde{m}, k)) =$$

$$\frac{c_s}{c_r (s-r-1)!} [1-x^p]^{(m-\gamma_r+1)} \left[\frac{1}{m+1} [(1-x^p)^{(m+1)} - (1-y^p)^{(m+1)}] \right]^{s-r-1} [1-y^p]^{(\gamma_s-1)} p (y)^{p-1}$$

6. Conclusion and Future Research

In this paper, we have derived the joint pdfs of generalized order statistics for different cases from Kumaraswamy distribution in closed-forms. In addition, the pdf of the conditional distribution of generalized order statistics from Kumaraswamy distribution is given. Furthermore, some special cases have been discussed.

Many opportunities of future research are available. The plan for the future research on generalized order Statistics from Kumaraswamy distribution can be split into two main areas. Estimation and hypothesis testing of Kumaraswamy parameters based on generalized order statistics.

References

1. Carrasco, J., et al. "A New Generalized Kumaraswamy Distribution", (2010).
2. Garg M. "On Generalized Order Statistics From Kumaraswamy Distribution", Tamsui Oxford Journal of Mathematical Sciences 25(2), (2009).
3. Kamps, U. "A concept of generalized order statistics", ELSEVIER Journal of Statistical planning and inference 48, (1995).
4. Kumaraswamy, P. "A generalized probability density function for double bounded random processes". Journal of Hydrology 46, 79-88, (1980).
5. Mitnik, P. "New Facts About Kumaraswamy Distribution". Department of Sociology, University of Wisconsin-Madison, USA, (2008). Electronic copy available at <http://ssrn.com/abstract=1231467>.
6. Samuel, P. "Characterization of Distributions by Conditional Expectation of Generalized Order Statistics" Statistical Papers 49, 101-108 (2008).

Estimation of the Bivariate Generalized Linear Failure Rate Distribution parameters under Random Censoring

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Abstract

Recently a new distribution, named a bivariate generalized linear failure rate distribution has been introduced by Sarhan et al. (2011). In this paper, we obtained the moment generating function for the bivariate generalized linear failure rate distribution and the maximum likelihood estimates for the unknown parameters of this distribution and their approximate variance- covariance matrix in case of left random censoring. A numerical example is carried out to discuss the properties of the estimators.

Keywords: Maximum Likelihood estimators, random censoring, generalized linear failure rate distribution, bivariate generalized linear failure rate distribution, moment generating function.

1. Introduction

In analyzing lifetime data, one often uses the exponential, Rayleigh, linear failure rate or generalized exponential distributions. It is well known that the exponential distribution can have constant hazard function whereas Rayleigh, linear failure rate and the generalized exponential distribution can have monotone (increasing in the case of Rayleigh or linear failure rate and increasing / decreasing in case of the generalized exponential distribution) hazard functions. Unfortunately, in practice often one needs to consider non-monotonic functions such as bathtub shaped hazard function. Recently Sarhan and Kundu (2009) presented a new distribution; called the generalized linear failure rate distribution

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(GLFR) which may has bathtub shaped hazard function. The GLFR has a cdf in the form

$$F_{GLFR}(x; \alpha, \beta, \gamma) = [1 - e^{-(\beta x + \frac{\gamma}{2} x^2)}]^\alpha, \quad x > 0, \alpha > 0, \beta > 0, \gamma > 0.$$

Its corresponding pdf has the form

$$f_{GLFR}(x; \alpha, \beta, \gamma) = \alpha(\beta + \gamma x) e^{-(\beta x + \frac{\gamma}{2} x^2)} [1 - e^{-(\beta x + \frac{\gamma}{2} x^2)}]^{\alpha-1}, \quad x > 0.$$

Several properties of this distribution were established by the authors. They observed that several known distributions like the exponential, the Rayleigh and the linear failure rate distribution distributions can be obtained as special cases of the GLFR distribution.

Recently, Sarhan et al. (2011) introduced a new bivariate generalized linear failure rate (BGLFR) distribution, whose marginals were GLFR distribution. This new five parameters BGLFR distribution was obtained using a method similar to that used to obtain the Marshall – Olkin bivariate exponential model [Marshall and Olkin (1967)]. The proposed BGLFR was constructed from three independent GLFR distributions using a maximization process. This new distribution is a singular distribution, and it can be used quit conveniently if there are ties in the data. They provided the following two interpretations for the BGLFR model

Competing risks model: Assume a system has two components, labeled 1 and 2, and the survival time of component i is denoted by $X_i, i = 1, 2$. It is considered that there are three independent causes of failures, which may affect the system. Only component 1 can fail due to cause 1, and similarly only component 2 can fail due to cause 2, while both the components fail at the same time due to cause 3. Let U_i be the lifetime of cause $i, i = 1, 2, 3$. If U_1, U_2 and U_3 follow a GLFR distribution, then (X_1, X_2) follow the BGLFR model.

Shock model: Suppose there are three independent sources of shocks; say 1, 2 and 3. Suppose these shocks are affecting a system with two components, say 1 and 2. It is assumed that the shock from source 1 reaches the system and destroys component 1 immediately, the shock from source 2 reaches the system and destroys component 1 immediately, while if the shock from source 3 hits the system it destroys both components immediately. Let U_i denote the inter-interval times, between the shocks in source $i, i = 1, 2, 3$, which follow the distribution GLFR. If X_1, X_2 denote the survival times of the components, then (X_1, X_2) follows the BGLFR model. Sarhan et al. (2011) discussed several

properties of this distribution, and used the EM algorithm to compute the maximum likelihood estimators of the unknown parameters in the case of complete sampling.

Random censored data arises often in clinical trails Patients may enter the study in a more or less random fashion, according to their time of diagnosis. If the study is terminated some prearranged date, then censoring times, that is the lengths of time from an individual's entry into the study until the termination of the study, are random. This type of censoring will be the main censoring mechanism. It occurs when the censoring time varies from individual to individual and is unknown in advance.

The data are left censored when some job durations are incomplete at the beginning of the job spells and are observed. For example, in the study of epidemics, such as AIDS, the time of onset of infection is typically unknown. What is known is the time at which the patient reports to the doctor. Hence the time from infection to the development of the disease is left censored.

This Paper is concerned with the moment generating function for the BGLFR distribution that provided in section 3. The maximum likelihood estimation, estimated variance-covariance matrix and asymptotic confidence intervals for BGLFR distribution under random left censoring are provided in Section 4. In Section 5, we introduce a numerical example to illustrate the results. Finally we conclude the paper in Section 6.

2. The Bivariate Generalized Linear Failure Rate Distribution

Sarhan et al. (2011) defined the BGLFR distribution as follows: Suppose U_1, U_2 and U_3 are three independent random variables such that $U_i \sim \text{GLFR}(\alpha_i, \beta, \gamma)$ for $i = 1, 2$ and 3 . Define $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$, then it is said that the bivariate vector (X_1, X_2) has BGLFR distribution with parameters $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$, denoted by $\text{BGLFR}(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$. Then, the joint cdf of (X_1, X_2) is given as follows

$$F_{\text{BGLFR}}(x_1, x_2) = \prod_{i=1}^3 F_{\text{GLFR}}(x_i; \alpha_i, \beta, \gamma)$$

where $x_3 = \min(x_1, x_2)$, $\alpha_{13} = \alpha_1 + \alpha_3$, $\alpha_{23} = \alpha_2 + \alpha_3$ and $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$.

The joint cdf of (X_1, X_2) can be written as

$$F_{BGLFR}(x_1, x_2) = \begin{cases} F_1(x_1, x_2) & \text{if } 0 < x_1 < x_2 < \infty \\ F_2(x_1, x_2) & \text{if } 0 < x_2 < x_1 < \infty \\ F_3(x) & \text{if } 0 < x_1 = x_2 = x < \infty \end{cases} \quad (2.1)$$

where

$$\begin{aligned} F_1(x_1, x_2) &= F_{GLFR}(x_1; \alpha_{13}, \beta, \gamma) F_{GLFR}(x_2; \alpha_2, \beta, \gamma) \\ &= [1 - e^{-(\beta x_1 + \frac{\gamma}{2} x_1^2)}]^{\alpha_{13}} [1 - e^{-(\beta x_2 + \frac{\gamma}{2} x_2^2)}]^{\alpha_2}, \\ F_2(x_1, x_2) &= F_{GLFR}(x_1; \alpha_1, \beta, \gamma) F_{GLFR}(x_2; \alpha_{23}, \beta, \gamma) \\ &= [1 - e^{-(\beta x_1 + \frac{\gamma}{2} x_1^2)}]^{\alpha_1} [1 - e^{-(\beta x_2 + \frac{\gamma}{2} x_2^2)}]^{\alpha_{23}}, \\ F_3(x) &= F_{GLFR}(x; \alpha_{123}, \beta, \gamma) \\ &= [1 - e^{-(\beta x + \frac{\gamma}{2} x^2)}]^{\alpha_{123}}. \end{aligned}$$

Then, joint pdf of (X_1, X_2) is given as

$$f_{BGLFR}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } 0 < x_1 < x_2 < \infty \\ f_2(x_1, x_2) & \text{if } 0 < x_2 < x_1 < \infty \\ f_3(x) & \text{if } 0 < x_1 = x_2 = x < \infty. \end{cases} \quad (2.2)$$

where

$$\begin{aligned} f_1(x_1, x_2) &= f_{GLFR}(x_1; \alpha_{13}, \beta, \gamma) f_{GLFR}(x_2; \alpha_2, \beta, \gamma) \\ &= \alpha_{13} \alpha_2 (\beta + \gamma x_1) (\beta + \gamma x_2) e^{-(\beta x_1 + \frac{\gamma}{2} x_1^2)} e^{-(\beta x_2 + \frac{\gamma}{2} x_2^2)} \\ &\quad [1 - e^{-(\beta x_1 + \frac{\gamma}{2} x_1^2)}]^{\alpha_{13}-1} [1 - e^{-(\beta x_2 + \frac{\gamma}{2} x_2^2)}]^{\alpha_2-1}, \\ f_2(x_1, x_2) &= f_{GLFR}(x_1; \alpha_1, \beta, \gamma) f_{GLFR}(x_2; \alpha_{23}, \beta, \gamma) \\ &= \alpha_1 \alpha_{23} (\beta + \gamma x_1) (\beta + \gamma x_2) e^{-(\beta x_1 + \frac{\gamma}{2} x_1^2)} e^{-(\beta x_2 + \frac{\gamma}{2} x_2^2)} \\ &\quad [1 - e^{-(\beta x_1 + \frac{\gamma}{2} x_1^2)}]^{\alpha_1-1} [1 - e^{-(\beta x_2 + \frac{\gamma}{2} x_2^2)}]^{\alpha_{23}-1}, \end{aligned}$$

$$\text{and } f_3(x) = \alpha_3 (\beta + \gamma x) e^{-(\beta x + \frac{\gamma}{2} x^2)} [1 - e^{-(\beta x + \frac{\gamma}{2} x^2)}]^{\alpha_{123}-1}.$$

The marginals of both X_1 and X_2 have GLFR $(\alpha_{13}, \beta, \gamma)$ and GLFR $(\alpha_{23}, \beta, \gamma)$, respectively.

3. The Moment Generating Function for BGLFR Distribution

In this section, we derive the joint moment generating function for the BGLFR distribution. We then obtain the joint moment of (X_1, X_2) and the marginal moments of X_1 and X_2 respectively. The moment generating function for the BGLFR distribution is given by:

$$\begin{aligned}
 M(t_1, t_2) = & \alpha_2 \alpha_{13} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{ij} \left(\left[\frac{1}{(i+1)(j+1)} \left\{ -1 - \frac{t_1}{2\gamma} \left[\frac{\gamma(i+1)}{2} \right]^{-\frac{1}{2}} \Gamma_{s_1^2(i)}^* \left(\frac{1}{2} \right) \right. \right. \right. \\
 & \left. \left. \left. e^{\frac{\gamma}{2}[(i+1)s_1^2(i) + (j+1)s_2^2(j)]} \right\} \cdot \left\{ e^{-\frac{\gamma}{2}(j+1)s_2^2(j)} + \frac{t_2}{2} \left[\frac{\gamma(j+1)}{2} \right]^{-\frac{1}{2}} \Gamma_{s_2^2(j)} \left(\frac{1}{2} \right) \right\} \right] \right. \\
 & + \left[\frac{e^{\frac{\gamma}{2}[2(i+1)s_1^2(i) + (j-i)s_3^2(i,j)]}}{(i+1)(j-i)} \cdot \left\{ e^{-\frac{\gamma}{2}(j-i)s_3^2(i,j)} + \frac{t_2 - t_1}{2} \left[\frac{\gamma(j-i)}{2} \right]^{-\frac{1}{2}} \Gamma_{s_3^2(i,j)} \left(\frac{1}{2} \right) \right\} \right] \\
 & + \left[\frac{t_1 e^{\frac{\gamma}{2}[(i+1)s_1^2(i) + (j+1)s_2^2(j)]}}{2\gamma(i+1)} \left[\frac{\gamma(i+1)}{2} \right]^{-\frac{1}{2}} \tau(m, k, r; t_2; s_1(i), s_2(j)) \right] \\
 & + \alpha_2 \alpha_{13} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k'_{ij} \left(\left[\frac{1}{(i+1)(j+1)} \left\{ -1 - \frac{t_2}{2\gamma} \left[\frac{\gamma(i+1)}{2} \right]^{-\frac{1}{2}} \Gamma_{s_2^2(j)}^* \left(\frac{1}{2} \right) \right. \right. \right. \\
 & \left. \left. \left. \cdot e^{\frac{\gamma}{2}[(i+1)s_1^2(i) + (j+1)s_2^2(j)]} \right\} \cdot \left\{ e^{-\frac{\gamma}{2}(j+1)s_1^2(i)} + \frac{t_1}{2} \left[\frac{\gamma(j+1)}{2} \right]^{-\frac{1}{2}} \Gamma_{s_1^2(i)} \left(\frac{1}{2} \right) \right\} \right] \right. \\
 & + \left[\frac{e^{\frac{\gamma}{2}[2(i+1)s_1^2(j) + (j-i)s_3^2(i,j)]}}{(i+1)(j-i)} \cdot \left\{ e^{-\frac{\gamma}{2}(j-i)s_3^2(i,j)} + \frac{t_1 - t_2}{2} \left[\frac{\gamma(j-i)}{2} \right]^{-\frac{1}{2}} \Gamma_{s_3^2(i,j)} \left(\frac{1}{2} \right) \right\} \right] \\
 & + \left[\frac{t_2 e^{\frac{\gamma}{2}[(i+1)s_2^2(j) + (j+1)s_1^2(i)]}}{2(i+1)\gamma} \left[\frac{\gamma(i+1)}{2} \right]^{-\frac{1}{2}} \tau(m, k, r; t_1; s_2(j), s_1(i)) \right] \\
 & + \alpha_3 \sum_{i=0}^{\infty} k_i \frac{e^{\frac{\gamma}{2}(i+1)s_4^2(j)}}{i+1} \left[e^{-\frac{\gamma}{2}(i+1)s_4^2(j)} + \frac{t_1 + t_2}{2} \left[\frac{\gamma(i+1)}{2} \right]^{-\frac{1}{2}} \Gamma_{s_4^2(j)} \left(\frac{1}{2} \right) \right] \quad ; i < j
 \end{aligned}
 \tag{3.1}$$

where $k_{ij} = (-1)^{i+j} \alpha_{13}(i) \alpha_2(j)$, $k'_{ij} = (-1)^{i+j} \alpha_{23}(i) \alpha_1(j)$, $k_i = (-1)^i \alpha_{123}(i)$,

$$\alpha(i) = \frac{(\alpha-1)(\alpha-2)\dots(\alpha-i)}{i!}, \quad s_1(i) = \frac{\beta}{\gamma} - \frac{t_1}{\gamma(i+1)}, \quad s_2(j) = \frac{\beta}{\gamma} - \frac{t_2}{\gamma(j+1)},$$

$$s_3(i, j) = \frac{\beta}{\gamma} - \frac{t_2 - t_1}{\gamma(j-i)}, \quad \tilde{s}_3(i, j) = \frac{\beta}{\gamma} - \frac{t_1 - t_2}{\gamma(j-i)}, \quad s_4(j) = \frac{\beta}{\gamma} - \frac{t_1 + t_2}{\gamma(j+1)},$$

$$\Gamma_x^*(a) = \int_0^x t^{a-1} e^{-t} dt, \quad \text{and} \quad .$$

$$\tau(m, k, r; t_2; s_1(i), s_2(j)) = \sum_{m=0}^{\infty} \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{r=0}^k \frac{(-1)^{m+k-r} (s_1(i))^{2(m-k)+1} (s_2(j))^{k-r}}{2 \left(m + \frac{1}{2}\right) m!}$$

$$\left(m + \frac{1}{2} \right) \binom{k}{r} \cdot \left\{ \gamma \left[\frac{\gamma(j+1)}{2} \right]^{-\left(\frac{r+2}{2}\right)} \Gamma_{s_2^2(j)} \left(\frac{r+2}{2} \right) \right.$$

$$\left. + \frac{t_2}{j+1} \left[\frac{\gamma(j+1)}{2} \right]^{-\left(\frac{r+1}{2}\right)} \Gamma_{s_2^2(j)} \left(\frac{r+1}{2} \right) \right\}.$$

It easily to obtain different moments of X_1 and X_2 as follows:

- The joint moment of X_1 and X_2 is given as:

$$E(X_1, X_2) = \left(\frac{\beta}{\gamma}\right)^2 \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}(1) \left[\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{\alpha_{13} \alpha_2 c_{ij} + \alpha_{23} \alpha_1 c'_{ij}}{(i+1)(j+1)} \right] \right.$$

$$\left. \cdot \left[\left[\frac{\gamma(i+1)}{2} \right]^{-\frac{1}{2}} \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^* \left(\frac{3}{2} \right) - \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^* (1) \right] + \alpha_3 \sum_{i=0}^{\infty} \frac{c_i}{(i+1)} \right]$$

$$+ \left(\frac{\beta}{\gamma}\right)^2 \left(\frac{\gamma}{2}\right)^{-\frac{1}{2}} \Gamma_{\left(\frac{\beta}{\gamma}\right)^2} \left(\frac{3}{2} \right) \left[\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{\alpha_{13} \alpha_2 c_{ij} + \alpha_{23} \alpha_1 c'_{ij}}{(i+1)(j+1)^{\frac{3}{2}}} \right] \right.$$

$$\left. \cdot \left[\Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^* (1) - (i+1)^{-\frac{1}{2}} \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^* \left(\frac{3}{2} \right) \right] + 2\alpha_3 \sum_{i=0}^{\infty} \frac{c_i}{(i+1)^{\frac{3}{2}}} \right]$$

$$+ \alpha_3 \left(\frac{\gamma}{2}\right)^{-\frac{1}{2}} \Gamma_{\left(\frac{\beta}{\gamma}\right)^2} (2) \sum_{i=0}^{\infty} \frac{c_i}{(i+1)^2}$$

$$+ \left(\frac{\beta}{\gamma}\right) \left(\frac{\gamma}{2}\right) \left[\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [\alpha_{13} \alpha_2 c_{ij} + \alpha_{23} \alpha_1 c'_{ij}] \cdot \right.$$

$$\left. \cdot \left[\left(\frac{\gamma}{2}\right)^{-\frac{1}{2}} \frac{\eta(u, l, 1, j, 1)}{(i+1)^{\frac{3}{2}}} - \frac{\eta(u, l, 1, j, 0)}{(i+1)} \right] \right].$$

where $c_{ij} = (-1)^{i+j} \alpha_{13}(i) \alpha_2(j) e^{\frac{\gamma(\beta)^2}{2} (i+j+1)}$, $c'_{ij} = (-1)^{i+j} \alpha_{23}(i) \alpha_1(j) e^{\frac{\gamma(\beta)^2}{2} (i+j+1)}$,

$c_i = (-1)^i \alpha_{123}(i) e^{\frac{\gamma(\beta)^2}{2} (i+1)}$ and

$$\eta(u, l, s, j, k) = \sum_{u=0}^{\infty} \sum_{l=0}^s \frac{(-1)^{s+u-l} \left(\frac{\beta}{\gamma}\right)^{s-l} \binom{s}{l}}{\left(\frac{k}{2} + u + 1\right) u!} \left[\frac{\gamma}{2}(j+1)\right]^{-\frac{k+u+l+4}{2}}$$

$$\Gamma_{\left(\frac{\beta}{\gamma}\right)^2} \left(\frac{k+u+l+4}{2}\right), \quad k, s \in \mathbb{N}^*.$$

- The first moment of X_1 is given as:

$$\begin{aligned} E(X_1) = & \left(\frac{\beta}{\gamma}\right) \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}(1) \left\{ \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^*(1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{\alpha_{13} \alpha_2 c_{ij} + \alpha_{23} \alpha_1 c'_{ij}}{(i+1)(j+1)} \right] - \alpha_3 \sum_{i=0}^{\infty} \frac{c_i}{(i+1)} \right. \\ & \left. - \alpha_{13} \alpha_2 \left(\frac{\gamma}{2}\right)^{-\frac{1}{2}} \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^* \left(\frac{3}{2}\right) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c_{ij}}{(i+1)^{\frac{3}{2}}(j+1)} \right\} \\ & + \left(\frac{\beta}{\gamma}\right) \left(\frac{\gamma}{2}\right)^{-\frac{1}{2}} \Gamma_{\left(\frac{\beta}{\gamma}\right)^2} \left(\frac{3}{2}\right) \left\{ \alpha_3 \sum_{i=0}^{\infty} \frac{c_i}{(i+1)^{\frac{3}{2}}} - \alpha_{23} \alpha_1 \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^*(1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c'_{ij}}{(i+1)(j+1)^{\frac{3}{2}}} \right\} \\ & + \left(\frac{\beta}{\gamma}\right) \left(\frac{\gamma}{2}\right)^{-\frac{1}{2}} \alpha_{13} \alpha_2 \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c_{ij}}{(i+1)} \left[\frac{\eta(u, 0, 0, j, 1)}{(i+1)^{\frac{1}{2}}} - \left(\frac{\gamma}{2}\right)^{-\frac{1}{2}} \eta(u, 0, 0, j, 0) \right] \right\} \\ & + \left(\frac{\beta}{2}\right) \alpha_{23} \alpha_1 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c'_{ij}}{(i+1)} \eta(u, l, 1, j, 0). \end{aligned}$$

- The second moment of X_1 is given as:

$$\begin{aligned} E(X_1^2) = & \left(\frac{\beta}{\gamma}\right) \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}(1) \left\{ \alpha_{13} \alpha_2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c_{ij}}{(i+1)(j+1)} \left[\Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^*(1) \right. \right. \\ & \left. \left. + 2 \frac{\Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^* \left(\frac{3}{2}\right)}{\left[\frac{\gamma(i+1)}{2}\right]^{\frac{1}{2}}} + \frac{\Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^*(2)}{\left[\frac{\gamma(i+1)}{2}\right]} \right] + \alpha_{23} \alpha_1 \left(\frac{\beta}{\gamma}\right) \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^*(1) \right. \\ & \left. \left. + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c'_{ij}}{(i+1)(j+1)} + \alpha_3 \left(\frac{\beta}{\gamma}\right) \sum_{i=0}^{\infty} \frac{c_i}{(i+1)} \right\} \end{aligned}$$

$$\begin{aligned}
& + \beta \left(\frac{\gamma}{2}\right)^{-\frac{3}{2}} \Gamma_{\left(\frac{\beta}{\gamma}\right)^2} \left(\frac{3}{2}\right) \left\{ \alpha_3 \sum_{i=0}^{\infty} \frac{c_i}{(i+1)^{\frac{3}{2}}} - \alpha_{23} \alpha_1 \left(\frac{\beta}{2}\right) \left(\frac{\gamma}{2}\right)^{-1} \right. \\
& \quad \left. \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^* (1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c'_{ij}}{(i+1)(j+1)^{\frac{3}{2}}} \right\} \\
& + \left(\frac{\gamma}{2}\right)^{-1} \Gamma_{\left(\frac{\beta}{\gamma}\right)^2} (2) \left\{ \alpha_3 \sum_{i=0}^{\infty} \frac{c_i}{(i+1)^2} - \alpha_{23} \alpha_1 \left(\frac{\beta}{\gamma}\right) \left(\frac{\gamma}{2}\right)^{-1} \right. \\
& \quad \left. \Gamma_{\left(\frac{\beta}{\gamma}\right)^2}^* (1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c'_{ij}}{(i+1)(j+1)^2} \right\} \\
& + \left(\frac{\beta}{\gamma}\right) \left(\frac{\gamma}{2}\right) \alpha_{13} \alpha_2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \left\{ \frac{\eta(u,0,0,j,0)}{(i+1)} - 2 \frac{\eta(u,0,0,j,1)}{(i+1)^{\frac{1}{2}}} + \frac{\eta(u,0,0,j,2)}{(i+1)} \right\} \\
& + \left(\frac{\beta}{\gamma}\right) \left(\frac{\gamma}{2}\right) \alpha_{23} \alpha_1 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c'_{ij}}{(i+1)} \eta(u,l,2,j,0).
\end{aligned}$$

Similarly, we can obtain $E(X_2)$ and $E(X_2^2)$ respectively as follows.

Hence the covariance between X_1 and X_2 will be obtained by substituting in the following equation

$$\text{Cov}(X_1, X_2) = E(X_1, X_2) - E(X_1) E(X_2).$$

4. Maximum Likelihood Estimation under Left Random Censoring

Suppose $\{(x_{1i}, x_{2i}), \dots, (x_{1n}, x_{2n})\}$ is a random sample from BGLFR distribution is subject to random left censoring by an independent vector of any random variables $\{(y_{1i}, y_{2i}), \dots, (y_{1n}, y_{2n})\}$. We observe $(t_{1i}, \delta_{1i}, t_{2i}, \delta_{2i})$, $i = 1 \dots n$ Where

$$\begin{aligned}
t_{1i} &= \max(x_{1i}, y_{1i}), \quad \delta_{1i} = \begin{cases} 1 & ; \quad x_{1i} > y_{1i} \\ 0 & ; \quad \text{otherwise} \end{cases} \\
t_{2i} &= \max(x_{2i}, y_{2i}), \quad \delta_{2i} = \begin{cases} 1 & ; \quad x_{2i} > y_{2i} \\ 0 & ; \quad \text{otherwise} \end{cases}
\end{aligned}$$

Therefore, if $x_{1i} < y_{1i}$, x_{1i} is left censored and if $x_{1i} > y_{1i}$, x_{1i} is observed.

In order to write down the likelihood, we note the following four cases:

- i. When $\delta_{1i} = \delta_{2i} = 1$, both failure times are observed and the contribution to the likelihood is $f(t_{1i}, t_{2i})$.
- ii. When $\delta_{1i} = 1$ and $\delta_{2i} = 0$, the first component fails at t_{1i} and the second component is censored (fails before t_{2i}). The contribution to the likelihood is $\int_0^{t_{2i}} f(t_{1i}, y) dy$.
- iii. When $\delta_{1i} = 0$ and $\delta_{2i} = 1$, the first component is censored (fails before t_{1i}) and the second one fails at t_{2i} . The contribution to the likelihood is $\int_0^{t_{1i}} f(x, t_{2i}) dx$.
- iv. When $\delta_{1i} = \delta_{2i} = 0$, both x_{1i} and x_{2i} are censored and the contribution to the likelihood is $F(t_{1i}, t_{2i})$.

According to Dewan and Nandi (2010) the likelihood function of the sample of size n based on $(t_{1i}, \delta_{1i}, t_{2i}, \delta_{2i})$, $i = 1, 2, \dots, n$ is given by:

$$L(\underline{\alpha}) = \prod_{i=1}^n L(t_{1i}, \delta_{1i}; t_{2i}, \delta_{2i})$$

$$= \prod_{i=1}^n [f(t_{1i}, t_{2i})]^{\delta_{1i}\delta_{2i}} [\varphi(t_{1i}, t_{2i})]^{\delta_{1i}(1-\delta_{2i})} [\psi(t_{1i}, t_{2i})]^{\delta_{2i}(1-\delta_{1i})} [F(t_{1i}, t_{2i})]^{(1-\delta_{1i})(1-\delta_{2i})}.$$

Where

$$f(t_{1i}, t_{2i}) \text{ is the joint pdf of } (x_{1i}, x_{2i}), \quad \varphi(t_{1i}, t_{2i}) = \int_0^{t_{2i}} f(t_{1i}, y) dy,$$

$$\psi(t_{1i}, t_{2i}) = \int_0^{t_{1i}} f(x, t_{2i}) dx, \text{ and } F(t_{1i}, t_{2i}) \text{ is the joint cdf of } (x_{1i}, x_{2i}).$$

Let $\underline{\alpha}$ be the vector of the parameters, and let I_1, I_2, I_3 denote the following sets

$I_1 = \{i | t_{1i} < t_{2i}\}$, $I_2 = \{i | t_{1i} > t_{2i}\}$ and $I_3 = \{i | t_{1i} = t_{2i} = t_i\}$, then the likelihood function can be written as

$$L(\underline{\alpha}) = \prod_{j=1}^3 \prod_{i \in I_j} L(t_{1i}, \delta_{1i}; t_{2i}, \delta_{2i}) \quad (4.1)$$

Let n_1, n_2 and n_3 denote the number of elements in the sets I_1, I_2, I_3 respectively,

In the case of BGLFR distribution we can evaluate $\varphi(t_{1i}, t_{2i}) = \int_0^{t_{2i}} f(t_{1i}, y) dy$ and $\psi(t_{1i}, t_{2i}) = \int_0^{t_{1i}} f(x, t_{2i}) dx$ as follows:

$$\varphi(t_{1i}, t_{2i}) = \int_0^{t_{2i}} f(t_{1i}, y) dy = \begin{cases} \varphi_1(t_{1i}, t_{2i}) & \text{if } t_{1i} < t_{2i} \\ \varphi_2(t_{1i}, t_{2i}) & \text{if } t_{1i} > t_{2i} \\ \varphi_3(t_i) & \text{if } t_{1i} = t_{2i} = t_i \end{cases} \quad (4.2)$$

where

$$\varphi_1(t_{1i}, t_{2i}) = f_{GLFR}(t_{1i}; \alpha_{13}) [F_{GLFR}(t_{2i}; \alpha_2) - F_{GLFR}(t_{1i}; \alpha_2)],$$

$$\varphi_2(t_{1i}, t_{2i}) = f_{GLFR}(t_{1i}; \alpha_1) F_{GLFR}(t_{2i}; \alpha_{23}),$$

$$\varphi_3(t_i) = f_{GLFR}(t_i; \alpha_1) F_{GLFR}(t_i; \alpha_{23}).$$

and

$$\psi(t_{1i}, t_{2i}) = \int_0^{t_{1i}} f(x, t_{2i}) dx = \begin{cases} \psi_1(t_{1i}, t_{2i}) & \text{if } t_{1i} < t_{2i} \\ \psi_2(t_{1i}, t_{2i}) & \text{if } t_{1i} > t_{2i} \\ \psi_3(t_i) & \text{if } t_{1i} = t_{2i} = t_i \end{cases} \quad (4.3)$$

where

$$\psi_1(t_{1i}, t_{2i}) = f_{GLFR}(t_{2i}; \alpha_2) F_{GLFR}(t_{1i}; \alpha_{13}),$$

$$\psi_2(t_{1i}, t_{2i}) = f_{GLFR}(t_{2i}; \alpha_{23}) [F_{GLFR}(t_{1i}; \alpha_1) - F_{GLFR}(t_{2i}; \alpha_1)],$$

$$\psi_3(t_i) = f_{GLFR}(t_i; \alpha_2) F_{GLFR}(t_i; \alpha_{13}).$$

The Likelihood on the set I_1 is:

$$L_1(\underline{\alpha}) = \prod_{i=1}^n ([f_1(t_{1i}, t_{2i})]^{\delta_{1i}\delta_{2i}} \cdot [\varphi_1(t_{1i}, t_{2i})]^{\delta_{1i}(1-\delta_{2i})} \cdot [\psi_1(t_{1i}, t_{2i})]^{\delta_{2i}(1-\delta_{1i})} \cdot [F_1(t_{1i}, t_{2i})]^{(1-\delta_{1i})(1-\delta_{2i})}). \quad (4.4)$$

After substituting from (2.1), (2.2), (4.2) and (4.3) in (4.4) and taking the logarithm we get:

$$\begin{aligned} \ln L_1(\underline{\alpha}) = \sum_{i \in I_1} & [\delta_{1i} \ln \alpha_{13} + \delta_{2i} \ln \alpha_2 + \alpha_{13}(1-\delta_{1i}) \ln u_{1i}(\beta, \gamma) + (\alpha_{13}-1)\delta_{1i} \ln u_{1i}(\beta, \gamma) \\ & + \alpha_2(1-\delta_{1i})(1-\delta_{2i}) \ln u_{2i}(\beta, \gamma) + (\alpha_2-1)\delta_{2i} \ln u_{2i}(\beta, \gamma) \\ & + \delta_{1i}(1-\delta_{2i}) \ln [(u_{1i}(\beta, \gamma))^{\alpha_2} - (u_{1i}(\beta, \gamma))^{\alpha_1}] + \delta_{1i} \ln(\beta + \gamma t_{1i}) \\ & + \delta_{2i} \ln(\beta + \gamma t_{2i}) - \beta(\delta_{1i}t_{1i} + \delta_{2i}t_{2i}) - \frac{\gamma}{2}(\delta_{1i}t_{1i}^2 + \delta_{2i}t_{2i}^2)]. \end{aligned} \quad (4.5)$$

where $u_{ki}(\beta, \gamma) = 1 - e^{-(\beta t_{ki} + \frac{\gamma}{2}t_{ki}^2)}$ $k = 1, 2$, and $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$.

The Likelihood on the set I_2 is:

$$L_2(\underline{\alpha}) = \prod_{i=1}^n ([f_2(t_{1i}, t_{2i})]^{\delta_{1i}\delta_{2i}} \cdot [\varphi_2(t_{1i}, t_{2i})]^{\delta_{1i}(1-\delta_{2i})} \cdot [\psi_2(t_{1i}, t_{2i})]^{\delta_{2i}(1-\delta_{1i})} \cdot [F_2(t_{1i}, t_{2i})]^{(1-\delta_{1i})(1-\delta_{2i})}). \quad (4.6)$$

After substituting from (2.1), (2.2), (4.2) and (4.3) in (4.6) and taking the logarithm we get:

$$\begin{aligned} \ln L_2(\underline{\alpha}) = \sum_{i \in I_2} & [\delta_{1i} \ln \alpha_1 + \delta_{2i} \ln \alpha_{23} + \alpha_{23}(1-\delta_{2i}) \ln u_{2i}(\beta, \gamma) + (\alpha_1-1)\delta_{1i} \ln u_{1i}(\beta, \gamma) \\ & + \alpha_1(1-\delta_{1i})(1-\delta_{2i}) \ln u_{1i}(\beta, \gamma) + (\alpha_{23}-1)\delta_{2i} \ln u_{2i}(\beta, \gamma) \\ & + \delta_{2i}(1-\delta_{1i}) \ln [(u_{1i}(\beta, \gamma))^{\alpha_1} - (u_{2i}(\beta, \gamma))^{\alpha_1}] + \delta_{1i} \ln(\beta + \gamma t_{1i}) \\ & + \delta_{2i} \ln(\beta + \gamma t_{2i}) - \beta(\delta_{1i}t_{1i} + \delta_{2i}t_{2i}) - \frac{\gamma}{2}(\delta_{1i}t_{1i}^2 + \delta_{2i}t_{2i}^2)]. \end{aligned} \quad (4.7)$$

and the Likelihood on the set I_3 is :

$$L_3(\underline{\alpha}) = \prod_{i=1}^n \left([f_3(t_i)]^{\delta_{1i}\delta_{2i}} \cdot [\varphi_3(t_i)]^{\delta_{1i}(1-\delta_{2i})} \cdot [\psi_3(t_i)]^{\delta_{2i}(1-\delta_{1i})} \cdot [F_3(t_i)]^{(1-\delta_{1i})(1-\delta_{2i})} \right) \quad (4.8)$$

after substituting from (2.1), (2.2), (4.2) and (4.3) in (4.8) and taking the logarithm we get :

$$\begin{aligned} \ln L_3(\underline{\alpha}) = & \sum_{i \in I_3} (\delta_{1i}\delta_{2i} \ln \alpha_3 + \delta_{1i}(1-\delta_{2i}) \ln \alpha_1 + \delta_{2i}(1-\delta_{1i}) \ln \alpha_2 \\ & + [\delta_{1i} + \delta_{2i}(1-\delta_{1i})](\alpha_{123} - 1) \ln u_i(\beta, \gamma) \\ & + \alpha_{123}(1-\delta_{1i})(1-\delta_{2i}) \ln u_i(\beta, \gamma) \\ & + [\delta_{1i} + \delta_{2i}(1-\delta_{1i})] \ln(\beta + \gamma t_{1i}) \\ & - \beta t_{1i} [\delta_{1i} + \delta_{2i}(1-\delta_{1i})] - \frac{\gamma}{2} t_{1i}^2 [\delta_{1i} + \delta_{2i}(1-\delta_{1i})]). \end{aligned} \quad (4.9)$$

Now taking the logarithm of (4.1) and substitute from (4.5), (4.7) and (4.9) we get

$$\begin{aligned} \ln L(\underline{\alpha}) \propto & \left[\sum_{i \in I_2} \delta_{1i} + \sum_{i \in I_3} \delta_{1i}(1-\delta_{2i}) \right] \ln \alpha_1 + \left[\sum_{i \in I_1} \delta_{2i} + \sum_{i \in I_3} \delta_{2i}(1-\delta_{1i}) \right] \ln \alpha_2 \\ & + \left[\sum_{i \in I_3} \delta_{1i}\delta_{2i} \right] \ln \alpha_3 + \left[\sum_{i \in I_1} \delta_{1i} \right] \ln(\alpha_1 + \alpha_3) + \left[\sum_{i \in I_2} \delta_{2i} \right] \ln(\alpha_2 + \alpha_3) \\ & + (\alpha_1 + \alpha_3) \left[\sum_{i \in I_1} (1-\delta_{1i}) \ln u_{1i}(\beta, \gamma) \right] + (\alpha_2 + \alpha_3) \left[\sum_{i \in I_2} (1-\delta_{2i}) \ln u_{2i}(\beta, \gamma) \right] \\ & + \alpha_1 \left[\sum_{i \in I_2} (1-\delta_{1i})(1-\delta_{2i}) \ln u_{1i}(\beta, \gamma) \right] + \alpha_2 \left[\sum_{i \in I_1} (1-\delta_{1i})(1-\delta_{2i}) \ln u_{2i}(\beta, \gamma) \right] \\ & + (\alpha_1 + \alpha_2 + \alpha_3) \left[\sum_{i \in I_3} (1-\delta_{1i})(1-\delta_{2i}) \ln u_i(\beta, \gamma) \right] \\ & + (\alpha_1 + \alpha_2 + \alpha_3) \left(\sum_{i \in I_3} [\delta_{1i} + \delta_{2i}(1-\delta_{1i})] \ln u_i(\beta, \gamma) \right) \\ & + (\alpha_2 - 1) \left[\sum_{i \in I_1} \delta_{2i} \ln u_{2i}(\beta, \gamma) \right] + (\alpha_1 + \alpha_3 - 1) \left[\sum_{i \in I_1} \delta_{1i} \ln u_{1i}(\beta, \gamma) \right] \\ & + (\alpha_2 + \alpha_3 - 1) \left[\sum_{i \in I_2} \delta_{2i} \ln u_{2i}(\beta, \gamma) \right] + (\alpha_1 - 1) \left[\sum_{i \in I_2} \delta_{1i} \ln u_{1i}(\beta, \gamma) \right] \\ & + \sum_{i \in I_1} \delta_{1i}(1-\delta_{2i}) \ln [(u_{2i}(\beta, \gamma))^{\alpha_2} - (u_{1i}(\beta, \gamma))^{\alpha_2}] \\ & + \sum_{i \in I_2} \delta_{2i}(1-\delta_{1i}) \ln [(u_{1i}(\beta, \gamma))^{\alpha_1} - (u_{2i}(\beta, \gamma))^{\alpha_1}]. \end{aligned} \quad (4.10)$$

On differentiating (4.10) with respect to $\alpha_1, \alpha_2, \alpha_3$ in turn and equating to zero, we obtain the following likelihood equations:

$$\frac{1}{\hat{\alpha}_1} \left[\sum_{i \in I_2} \delta_{1i} + \sum_{i \in I_3} \delta_{1i} (1 - \delta_{2i}) \right] + \frac{1}{\hat{\alpha}_1 + \hat{\alpha}_3} \sum_{i \in I_1} \delta_{1i} + k_1 + w_1(\hat{\alpha}_1, u_1, u_2, \delta_1, \delta_2) = 0,$$

$$\frac{1}{\hat{\alpha}_2} \left[\sum_{i \in I_1} \delta_{2i} + \sum_{i \in I_3} \delta_{2i} (1 - \delta_{1i}) \right] + \frac{1}{\hat{\alpha}_2 + \hat{\alpha}_3} \sum_{i \in I_2} \delta_{2i} + k_2 + w_2(\hat{\alpha}_2, u_1, u_2, \delta_1, \delta_2) = 0,$$

and

$$\frac{1}{\hat{\alpha}_3} \sum_{i \in I_3} \delta_{1i} \delta_{2i} + \frac{1}{\hat{\alpha}_1 + \hat{\alpha}_3} \sum_{i \in I_1} \delta_{1i} + \frac{1}{\hat{\alpha}_2 + \hat{\alpha}_3} \sum_{i \in I_2} \delta_{2i} + k_3 = 0.$$

(4.11)

where

$$w_1(\hat{\alpha}_1, u_1, u_2, \delta_1, \delta_2) = \sum_{i \in I_2} \delta_{1i} (1 - \delta_{2i}) \frac{[u_{1i}(\beta, \gamma)]^{\hat{\alpha}_1} \ln u_{1i}(\beta, \gamma) - [u_{2i}(\beta, \gamma)]^{\hat{\alpha}_1} \ln u_{2i}(\beta, \gamma)}{[u_{1i}(\beta, \gamma)]^{\hat{\alpha}_1} - [u_{2i}(\beta, \gamma)]^{\hat{\alpha}_1}},$$

$$w_2(\hat{\alpha}_2, u_1, u_2, \delta_1, \delta_2) = \sum_{i \in I_1} \delta_{2i} (1 - \delta_{1i}) \frac{[u_{2i}(\beta, \gamma)]^{\hat{\alpha}_2} \ln u_{2i}(\beta, \gamma) - [u_{1i}(\beta, \gamma)]^{\hat{\alpha}_2} \ln u_{1i}(\beta, \gamma)}{[u_{2i}(\beta, \gamma)]^{\hat{\alpha}_2} - [u_{1i}(\beta, \gamma)]^{\hat{\alpha}_2}},$$

$$k_1 = \sum_{i \in I_1} \ln u_{1i}(\beta, \gamma) + \sum_{i \in I_3} \ln u_{1i}(\beta, \gamma) + \sum_{i \in I_2} \delta_{1i} \delta_{2i} (1 - \delta_{2i}) \ln u_{1i}(\beta, \gamma),$$

$$k_2 = \sum_{i \in I_2} \ln u_{2i}(\beta, \gamma) + \sum_{i \in I_3} \ln u_{2i}(\beta, \gamma) + \sum_{i \in I_1} \delta_{1i} \delta_{2i} (1 - \delta_{1i}) \ln u_{2i}(\beta, \gamma),$$

$$k_3 = \sum_{i \in I_1} \ln u_{1i}(\beta, \gamma) + \sum_{i \in I_2} \ln u_{2i}(\beta, \gamma) + \sum_{i \in I_3} \ln u_{1i}(\beta, \gamma),$$

$$\text{and } u_k = u_{ki}(\beta, \gamma) = 1 - e^{-(\beta'x_i + \frac{\gamma}{2}x_i^2)} \quad k = 1, 2.$$

These three equations have not explicit form; therefore, their solutions are numerically obtained using Newton-Raphson method as will be seen in Section 5. They are solved simultaneously to obtain $\hat{\alpha}_1, \hat{\alpha}_2$ and $\hat{\alpha}_3$. For simplification we assumed that β and γ are fixed known.

The asymptotic variance-covariance matrix of $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ is obtained by inverting the Fisher information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood function. In the present situation, it seems appropriate to approximate the expected values by their maximum likelihood estimates [see Cohen (1965)]. Accordingly; we have the following approximate variance-covariance matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}$$

where

$$a_{11} = -\frac{\partial^2 \ln L}{\partial \alpha_1^2} \Big|_{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3} = \frac{1}{\hat{\alpha}_1^2} \left[\sum_{i \in I_2} \delta_{1i} + \sum_{i \in I_3} \delta_{1i} (1 - \delta_{2i}) \right] + \frac{1}{(\hat{\alpha}_1 + \hat{\alpha}_3)^2} \sum_{i \in I_1} \delta_{1i} + v_1(\hat{\alpha}_1, u_1, u_2, \delta_1, \delta_2),$$

$$a_{22} = -\frac{\partial^2 \ln L}{\partial \alpha_2^2} \Big|_{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3} = \frac{1}{\hat{\alpha}_2^2} \left[\sum_{i \in I_1} \delta_{2i} + \sum_{i \in I_3} \delta_{2i} (1 - \delta_{1i}) \right] + \frac{1}{(\hat{\alpha}_2 + \hat{\alpha}_3)^2} \sum_{i \in I_2} \delta_{2i} + v_2(\hat{\alpha}_2, u_1, u_2, \delta_1, \delta_2),$$

$$a_{33} = -\frac{\partial^2 \ln L}{\partial \alpha_3^2} \Big|_{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3} = \frac{1}{\hat{\alpha}_3^2} \sum_{i \in I_3} \delta_{1i} \delta_{2i} + \frac{1}{(\hat{\alpha}_1 + \hat{\alpha}_3)^2} \sum_{i \in I_1} \delta_{1i} + \frac{1}{(\hat{\alpha}_2 + \hat{\alpha}_3)^2} \sum_{i \in I_2} \delta_{2i},$$

$$a_{12} = -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \alpha_2} \Big|_{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3} = 0,$$

$$a_{13} = -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \alpha_3} \Big|_{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3} = \frac{1}{(\hat{\alpha}_1 + \hat{\alpha}_3)^2} \sum_{i \in I_1} \delta_{1i},$$

$$a_{23} = -\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \alpha_3} \Big|_{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3} = \frac{1}{(\hat{\alpha}_2 + \hat{\alpha}_3)^2} \sum_{i \in I_2} \delta_{2i},$$

$$v_1(\hat{\alpha}_1, u_1, u_2, \delta_1, \delta_2) = \sum_{i \in I_2} \delta_{2i} (1 - \delta_{1i}) \frac{[u_{1i}(\beta, \gamma)]^{\hat{\alpha}_1} [u_{2i}(\beta, \gamma)]^{\hat{\alpha}_1} [\ln u_{1i}(\beta, \gamma) - \ln u_{2i}(\beta, \gamma)]^2}{([u_{1i}(\beta, \gamma)]^{\hat{\alpha}_1} - [u_{2i}(\beta, \gamma)]^{\hat{\alpha}_1})^2},$$

and

$$v_2(\hat{\alpha}_2, u_1, u_2, \delta_1, \delta_2) = \sum_{i \in I_1} \delta_{1i} (1 - \delta_{2i}) \frac{[u_{1i}(\beta, \gamma)]^{\hat{\alpha}_2} [u_{2i}(\beta, \gamma)]^{\hat{\alpha}_2} [\ln u_{1i}(\beta, \gamma) - \ln u_{2i}(\beta, \gamma)]^2}{([u_{1i}(\beta, \gamma)]^{\hat{\alpha}_2} - [u_{2i}(\beta, \gamma)]^{\hat{\alpha}_2})^2}.$$

Now we state the asymptotic normality results to obtain the asymptotic confidence intervals of α_1 , α_2 and α_3 . It can be stated as follows:

$$[\sqrt{n}(\hat{\underline{\alpha}} - \underline{\alpha})] \rightarrow N_3(\underline{0}, I^{-1}(\underline{\alpha})) \text{ as } n \rightarrow \infty \quad (4.12)$$

Where $I^{-1}(\underline{\alpha})$ is the variance-covariance matrix, $\hat{\underline{\alpha}} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$, and $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$.

Since $\underline{\alpha}$ is unknown in (4.12), $I^{-1}(\underline{\alpha})$ is estimated by $I^{-1}(\hat{\underline{\alpha}})$; the asymptotic variance-covariance matrix that defined above and this can be used to obtain the asymptotic confidence intervals of α_1 , α_2 and α_3 .

5. Numerical Results

To illustrate the results, a numerical example is given where the MLEs, their approximate variance- covariance matrix, relative bias, mean square error and asymptotic confidence intervals are determined using Mathcad (2001) software. Thirty seven observations from BGLFR distribution (X_1, X_2) have been obtained from Sarhan et al. (2011) and are listed in table (1). Two samples of thirty seven observations are selected at random from, exponential distribution with different parameters $\theta_1 = 2.5$ and $\theta_2 = 3.5$ where $Y_i \sim \exp(\theta_i)$, $i = 1, 2$ are also listed in table(1). After censoring the sample observations with the generated times $Y_i, i = 1, 2$, we observe $(t_{1i}, \delta_{1i}, t_{2i}, \delta_{2i})$ that defined in Section 4 as in table (2). From The likelihood equations (4.11), the Maximum likelihood estimators (MLEs) are obtained by iterative technique of Newton-Raphson using Mathcad (2001) software, their approximate variance- covariance matrix, relative bias, Mean Square Error (MSE) and a 95% asymptotic confidence intervals are listed in table (3).

Table (3): The estimates, Relative Bias, MSE, Confidence interval and variance- covariance matrix

Parameter	MLE	Relative Bias	MSE	Confidence Interval	Variance-Covariance matrix		
α_1	0.846	0.046	0.002152	(0.783-0.91)	0.039	0.00076	- 0.005
α_2	0.534	0.034	0.001169	(0.491-0.577)	0.00076	0.018	- 0.003
α_3	0.307	0.00741	0.000055	(0.274-0.341)	- 0.005	- 0.003	0.011

Table (1): Samples observations from BGLFR distribution and Exponential distribution with $\theta_1 = 2.5$ and $\theta_2 = 3.5$

No.	x_1	x_2	y_1	y_2	No.	x_1	x_2	y_1	y_2
1	0.26	0.20	0.6	0.08	20	0.34	0.34	0.26	0.06
2	0.63	0.18	0.66	0.22	21	0.53	0.39	0.05	0.47
3	0.19	0.19	0.21	0.08	22	0.54	0.07	0.02	0.06
4	0.66	0.85	0.42	0.15	23	0.51	0.28	0.25	0.53
5	0.4	0.4	0.08	0.09	24	0.76	0.64	0.31	0.09
6	0.49	0.49	0.7	0.16	25	0.64	0.15	0.06	0.36
7	0.08	0.08	0.14	0.54	26	0.26	0.48	0.1	0.11
8	0.69	0.71	0.48	0.24	27	0.16	0.16	0.00128	0.09
9	0.39	0.39	0.96	0.19	28	0.44	0.6	0.2	0.6
10	0.82	0.48	0.77	0.08	29	0.25	0.14	0.53	0.05
11	0.72	0.72	0.00462	0.51	30	0.55	0.11	0.07	0.19
12	0.66	0.62	0.85	0.2	31	0.49	0.49	0.39	0.24
13	0.25	0.09	1.89	0.1	32	0.24	0.24	0.16	0.01
14	0.41	0.03	0.25	0.55	33	0.44	0.30	1.89	0.17
15	0.16	0.75	0.2	0.56	34	0.42	0.03	0.52	0.21
16	0.18	0.18	0.72	0.1	35	0.27	0.47	0.21	0.05
17	0.22	0.14	0.32	0.24	36	0.28	0.28	0.07	0.22
18	0.42	0.42	1.15	0.0097	37	0.02	0.02	0.29	0.0049
19	0.36	0.52	0.1	0.54					

Table (2): values of $(t_{1i}, \delta_{1i}, t_{2i}, \delta_{2i})$ after left random censoring

No.	T_1	T_2	δ_1	δ_2	No.	T_1	T_2	δ_1	δ_2
1	0.60	0.20	0	1	20	0.34	0.34	1	1
2	0.66	0.22	0	0	21	0.53	0.47	1	0
3	0.21	0.19	0	1	22	0.54	0.07	1	1
4	0.66	0.85	1	1	23	0.51	0.53	1	0
5	0.4	0.4	1	1	24	0.76	0.64	1	1
6	0.7	0.49	0	1	25	0.64	0.36	1	0
7	0.14	0.54	0	0	26	0.26	0.48	1	1
8	0.69	0.71	1	1	27	0.16	0.16	1	1
9	0.96	0.39	0	1	28	0.44	0.6	1	0
10	0.82	0.48	1	1	29	0.53	0.14	0	1
11	0.72	0.72	1	1	30	0.55	0.19	1	0
12	0.85	0.62	0	1	31	0.49	0.49	1	1
13	1.89	0.1	0	0	32	0.24	0.24	1	1
14	0.41	0.55	1	0	33	1.89	0.30	0	1
15	0.2	0.75	0	1	34	0.52	0.21	0	0
16	0.72	0.18	0	1	35	0.27	0.47	1	1
17	0.32	0.24	0	0	36	0.28	0.28	1	1
18	1.15	0.42	0	1	37	0.29	0.02	0	1
19	0.36	0.54	1	0					

6. Conclusion

In this paper we have provide a review of the new bivariate generalized linear failure rate distribution whose marginals are generalized linear failure rate distributions. This new distribution is a singular distribution, Sarhan et al. (2011) concluded that it can be used quit effectively instead of the Marshall-Olkin bivariate exponential model or the bivariate generalized exponential model when there are ties in the data.

We have obtained the moment generating function for the BGLFR distribution and then obtained different joint and marginal moments for this distribution. And we have considered the maximum likelihood estimators of the unknown parameters of the BGLFR distribution when both components of the bivariate variable are subject to left random censoring. The associated asymptotic variance-covariance matrix for the unknown parameters was derived. To illustrate the results, a numerical example was given where the MLEs, their approximate variance- covariance matrix, relative bias, mean square error and asymptotic confidence intervals were determined using computer facilities and Mathcad (2001) software.

References

- 1) C. Cohen, Maximum Likelihood Estimation in The Weibull Distribution Based On Complete and Censored Samples, *Technometrics*, 7(1965), 579-588.
- 2) I. Dewan and S. Nandi, an EM Algorithm for Estimation of Parameters of Bivariate Generalized Exponential Distribution under Random Left Censoring. Technical report, Indian Statistical Institute, Delhi Center (2010).
- 3) W. Marshall and I. Olkin, a Multivariate Exponential Distribution. *Journal of the American Statistical Association*, 62 (1967), 30 -44.
- 4) M. Sarhan and D. Kundu Generalized Linear Failure Rate Distribution. *Communications in Statistics – Theory and Methods*, 38(5) (2009), 642-660.
- 5) M. Sarhan, D. C. Hamilton, B. Smith and D. Kundu, the Bivariate Generalized Linear Failure Rate Distribution and Its Multivariate Extension *Computational Statistics and Data Analysis*, 55 (2011), 644 – 654.

Estimating and Planning Constant Stress Accelerated Life Test for Generalized Logistic Distribution under Type-II Censoring

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Abstract

This paper presents estimation and optimal design of Constant Stress Accelerated Life Test (CSALT) under type-II censoring. The maximum likelihood (ML) method is applied to obtain the ML estimators of the parameters of generalized logistic distribution. The scale parameter of the lifetime distribution is assumed to be an inverse power law function of the stress level. The Fisher information matrix, the asymptomatic variance-covariance matrix, the confidence bounds, the predictive value of the scale parameter and the reliability function under the usual conditions are obtained. Moreover, the optimal values of the fraction of failure units are specified according to the D-optimality criterion. Finally, numerical studies are introduced to illustrate the results.

Key Words and Phrases: *Accelerated life test, Constant Stress, Type-II censoring, Maximum likelihood Estimation, Fisher Information Matrix, Optimum Test Plan, Generalized Logistic Distribution.*

1 Introduction

Life data analysis involves analyzing lifetime data of a device, system, or component obtained under normal operating conditions in order to quantify their life characteristics. In many situations, and for many reasons, such data is very difficult, if not impossible, to obtain. The reasons for this difficulty can include, the long life

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times of today's devices and the developments of new technologies need more accurate estimation of reliability of a device or system in a short time. A common way of tackling this problem is to expose the device to sufficient over stress (e.g., temperature, voltage, humidity, and so on), or forcing them to fail more quickly than they would under normal use conditions to accelerate their failures. Therefore, the failure data are analyzed in terms of a suitable physical statistical model to obtain desired information on a device or performance under normal use conditions. This approach is called Accelerated Life Testing (ALT).

The most common ALT loading is constant stress, step stress, and progressive stress (for more details, see Nelson (1990)). In constant stress accelerated life test (CSALT), the stress is kept at a constant level of stress throughout the life of the test, i.e., each unit is run at a constant high stress level until the occurrence of failure or the observation is censored. Practically, most devices such as lamps, semiconductors and microelectronics are run at a constant stress.

Many authors have studied statistical inference of CSALT, for example, see McCool (1980), Watkins (1991), Abdel Ghaly, et. al. (1998), El-Dessouky (2001), and Attia, et. al. (2011). Additional to the statistical inference studies, optimum CSALT plans were studied for different lifetime distributions based on different censoring scheme; for example, Nelson and Kielpinski (1976) studied optimum ALT plans for normal and log-normal life distributions. Nelson (1990) reviewed statistically optimal and compromise plans for the single stress ALT planning problem. Yang (1994) proposed an optimal design of 4-level constant-stress ALT plans considering different censoring times. Chang, et. al. (2010) dealt with Weibull distribution. Attia, et. al. (2011) studied optimum ALT plans considering type-I censoring data. However, no studies have been made on estimating or planning (optimal design) CSALT using the Generalized Logistic distribution (except Attia, et. al. (2011)).

Generalized Logistic (GL) distribution is used as the name for several different families of probability distributions (see, Johnson, et. al. (1995)). The main feature of this distribution is expected to be useful in many more practical situations, for example, in extreme value event evaluation, in hydrological risk analysis, in a quantal response data, to model the data with a unimodal density, and in analysis of survival data, (for more details, see Tolikas (2008), Tolikas and Gettinby (2009), Shabri, et. al. (2011), and Tolikas (2011)).

This paper is organized as follows: The underlying distribution and the test method are described in section 2. Section 3 introduces the ML estimators of the model parameters with their properties. Optimum failure test plan is developed in Section 4. The numerical results are presented in Section 5.

2 The Model

2.1 The Generalized Logistic Distribution as Lifetime Model

The probability density function (pdf) of a three-parameter generalized logistic distribution (Molenberghs and Verbeke (2011)) is given by

$$f(x) = \alpha \gamma e^{\alpha x} \left(1 + \frac{\gamma}{\theta} e^{\alpha x}\right)^{-(\theta+1)}, \quad -\infty < x < \infty, \quad \alpha, \gamma, \theta > 0. \quad (2.1)$$

The reliability function takes the form

$$R(x) = \left(1 + \frac{\gamma}{\theta} e^{\alpha x}\right)^{-\theta}, \quad -\infty < x < \infty, \quad \alpha, \gamma, \theta > 0. \quad (2.2)$$

and the corresponding failure rate is given by

$$h(x) = \alpha \gamma e^{\alpha x} \left(1 + \frac{\gamma}{\theta} e^{\alpha x}\right)^{-1}, \quad -\infty < x < \infty, \quad \alpha, \gamma, \theta > 0. \quad (2.3)$$

2.2 Assumptions

We assume the following assumptions for the CSALT procedure

- A total of N units are divided into $n_1, n_2, \dots, n_k$ units where $\sum_{j=1}^k n_j = N$.
- There are k levels of high stress V_j , $j = 1, \dots, k$ in the experiment, and V_u is the stress under usual conditions, where $V_u < V_1 < \dots < V_k$.
- Each n_j , $j = 1, \dots, k$ units in the experiment are run at a pre-specified constant stress V_j , $j = 1, \dots, k$.
- It is assumed that the stress affected only on the scale parameter of the underlying distribution.
- The failure times x_{ij} , $i = 1, \dots, r_j$ and $j = 1, \dots, k$ at stress levels V_j , $j = 1, \dots, k$ are the 3-parameter generalized logistic distribution with probability density function

$$f(x_{ij}, \alpha_j, \gamma, \theta) = \alpha_j \gamma e^{\alpha_j x_{ij}} \left(1 + \frac{\gamma}{\theta} e^{\alpha_j x_{ij}}\right)^{-(\theta+1)}, \quad -\infty < x_{ij} < \infty, \\ \alpha_j, \gamma, \theta > 0, \quad i = 1, \dots, r_j, \quad j = 1, \dots, k. \quad (2.4)$$

- The scale parameter α_j , $j = 1, \dots, k$, of the underlying lifetime distribution (2.4) is assumed to have an inverse power law function on stress levels, i.e.,

$$\alpha_j = CS_j^P, \quad C, P > 0, \quad \text{where } S_j = \frac{V^*}{V_j},$$

$$V^* = \prod_{j=1}^k V_j^{b_j}, \quad \text{and } b_j = \frac{r_j}{\sum_{j=1}^k r_j} \quad (2.5)$$

where C is the constant of proportionality, and P is the power of the applied stress.

3 Maximum Likelihood (ML) Estimation

Considering the common assumptions (2.2), and assuming that the experiment is terminated at a specified number of failure units r_j ($r_j < n_j$), $j = 1, \dots, k$, the likelihood function will be as the following form

$$L = \prod_{j=1}^k \left\{ \frac{n_j!}{(n_j - r_j)!} \left[\prod_{i=1}^{r_j} CS_j^P \gamma e^{CS_j^P x_{ij}} \left(1 + \frac{\gamma}{\theta} e^{CS_j^P x_{ij}} \right)^{-(\theta+1)} \right] \right. \\ \left. \left(1 + \frac{\gamma}{\theta} e^{CS_j^P x_{r_j j}} \right)^{-\theta(n_j - r_j)} \right\}. \quad (3.1)$$

The log-likelihood function

$$\ln L = A + \ln C \sum_{j=1}^k r_j + \ln \gamma \sum_{j=1}^k r_j + C \sum_{j=1}^k \sum_{i=1}^{r_j} S_j^P x_{ij} - (\theta + 1) \\ \sum_{j=1}^k \sum_{i=1}^{r_j} \ln \left(1 + \frac{\gamma}{\theta} e^{CS_j^P x_{ij}} \right) - \theta \sum_{j=1}^k (n_j - r_j) \ln \left(1 + \frac{\gamma}{\theta} e^{CS_j^P x_{r_j j}} \right), \quad (3.2)$$

where, $\sum_{j=1}^k r_j \ln S_j = 0$, $A = \sum_{j=1}^k \ln \left[\frac{n_j!}{(n_j - r_j)!} \right]$.

3.1 ML Estimation of the Parameters

The first derivatives of the log-likelihood function (3.2) with respect to the unknown parameters C , P , γ and θ are

$$\frac{\partial \ln L}{\partial C} = \frac{\sum_{j=1}^k r_j}{C} + \sum_{j=1}^k \sum_{i=1}^{r_j} S_j^P x_{ij} - (\theta + 1) \sum_{j=1}^k \sum_{i=1}^{r_j} \xi_{ij} - \theta \sum_{j=1}^k (n_j - r_j) \xi_{r_j j}. \quad (3.3)$$

$$\frac{\partial \ln L}{\partial P} = C \left\{ \sum_{j=1}^k \sum_{i=1}^{r_j} \sigma_{ij} - (\theta + 1) \sum_{j=1}^k \sum_{i=1}^{r_j} \ln S_j \xi_{ij} - \theta \sum_{j=1}^k (n_j - r_j) \ln S_j \xi_{r_{jj}} \right\}. \quad (3.4)$$

$$\frac{\partial \ln L}{\partial \gamma} = \frac{1}{\gamma} \left\{ \sum_{j=1}^k r_j - (\theta + 1) \sum_{j=1}^k \sum_{i=1}^{r_j} \nu_{ij} - \theta \sum_{j=1}^k (n_j - r_j) \nu_{r_{jj}} \right\}. \quad (3.5)$$

$$\frac{\partial \ln L}{\partial \theta} = \sum_{j=1}^k \sum_{i=1}^{r_j} \delta_{ij} \left(\frac{\theta + 1}{\theta} \nu_{ij} - \pi_{ij} \right) + \sum_{j=1}^k (n_j - r_j) (\nu_{r_{jj}} - \pi_{r_{jj}}), \quad (3.6)$$

where, $\xi_{ij} = S_j^P x_{ij} \nu_{ij}$, $\nu_{ij} = (1 + \frac{\theta}{\gamma} e^{-CS_j^P x_{ij}})^{-1}$, $\xi_{r_{jj}} = S_j^P x_{r_{jj}} \nu_{r_{jj}}$, $\nu_{r_{jj}} = (1 + \frac{\theta}{\gamma} e^{-CS_j^P x_{r_{jj}}})^{-1}$, $\sigma_{ij} = S_j^P x_{ij} \ln S_j$, $\pi_{ij} = -\ln(1 - \nu_{ij})$, and $\pi_{r_{jj}} = -\ln(1 - \nu_{r_{jj}})$.

The numerical solution of the equations (3.3) to (3.6) are placed in section (5.1). The second partial derivatives of the log-likelihood function (3.2) with respect to the model parameters C , P , γ and θ are as follows

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial C^2} = & - \left\{ C^{-2} \sum_{j=1}^k r_j + \frac{\theta(\theta + 1)}{\gamma} \sum_{j=1}^k \sum_{i=1}^{r_j} \xi_{ij}^2 e^{-CS_j^P x_{ij}} + \right. \\ & \left. \frac{\theta^2}{\gamma} \sum_{j=1}^k (n_j - r_j) \xi_{r_{jj}}^2 e^{-CS_j^P x_{r_{jj}}} \right\}. \end{aligned} \quad (3.7)$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial P^2} = & -C \left\{ \sum_{j=1}^k \sum_{i=1}^{r_j} \ln S_j [(\theta + 1) \xi_{ij} (C \sigma_{ij} (1 - \nu_{ij}) + \ln S_j) - \sigma_{ij}] + \right. \\ & \left. \theta \sum_{j=1}^k (n_j - r_j) \xi_{r_{jj}} \ln S_j (C \sigma_{r_{jj}} (1 - \nu_{r_{jj}}) + \ln S_j) \right\}. \end{aligned} \quad (3.8)$$

$$\frac{\partial^2 \ln L}{\partial \gamma^2} = \frac{-1}{\gamma^2} \left\{ \sum_{j=1}^k r_j - (\theta + 1) \sum_{j=1}^k \sum_{i=1}^{r_j} \nu_{ij}^2 - \theta \sum_{j=1}^k (n_j - r_j) \nu_{r_{jj}}^2 \right\}. \quad (3.9)$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-1}{\theta} \left\{ \frac{1}{\theta} \sum_{j=1}^k \sum_{i=1}^{r_j} \nu_{ij} [(1 - \theta) + (1 + \theta)(1 - \nu_{ij})] - \sum_{j=1}^k (n_j - r_j) \nu_{r_{jj}}^2 \right\}. \quad (3.10)$$

$$\frac{\partial^2 \ln L}{\partial C \partial P} = -\left\{ \sum_{j=1}^k \sum_{i=1}^{r_j} [(\theta + 1) \xi_{ij} (C \sigma_{ij} (1 - \nu_{ij}) + \ln S_j) - \sigma_{ij}] + \theta \sum_{j=1}^k (n_j - r_j) \xi_{r_j j} (C \sigma_{r_j j} (1 - \nu_{r_j j}) + \ln S_j) \right\}. \quad (3.11)$$

$$\frac{\partial^2 \ln L}{\partial C \partial \gamma} = \frac{-1}{\gamma} \left\{ (\theta + 1) \sum_{j=1}^k \sum_{i=1}^{r_j} \xi_{ij} (1 - \nu_{ij}) + \theta \sum_{j=1}^k (n_j - r_j) \xi_{r_j j} (1 - \nu_{r_j j}) \right\}. \quad (3.12)$$

$$\frac{\partial^2 \ln L}{\partial C \partial \theta} = -\left\{ \sum_{j=1}^k \sum_{i=1}^{r_j} \xi_{ij} \left(1 - \frac{(\theta + 1)}{\theta} (1 - \nu_{ij}) \right) + \sum_{j=1}^k (n_j - r_j) \nu_{r_j j} \xi_{r_j j} \right\}. \quad (3.13)$$

$$\frac{\partial^2 \ln L}{\partial P \partial \gamma} = \frac{-C}{\gamma} \left\{ (\theta + 1) \sum_{j=1}^k \sum_{i=1}^{r_j} \xi_{ij} (1 - \nu_{ij}) \ln S_j + \theta \sum_{j=1}^k (n_j - r_j) \xi_{r_j j} (1 - \nu_{r_j j}) \ln S_j \right\}. \quad (3.14)$$

$$\frac{\partial^2 \ln L}{\partial p \partial \theta} = -C \left\{ \sum_{j=1}^k \sum_{i=1}^{r_j} \xi_{ij} \left(1 - \frac{(\theta + 1)}{\theta} (1 - \nu_{ij}) \right) \ln S_j + \sum_{j=1}^k (n_j - r_j) \xi_{r_j j} \nu_{r_j j} \ln S_j \right\}. \quad (3.15)$$

$$\frac{\partial^2 \ln L}{\partial \gamma \partial \theta} = \frac{-1}{\gamma} \left\{ \sum_{j=1}^k \sum_{i=1}^{r_j} \nu_{ij} \left(1 - \frac{(\theta + 1)}{\theta} (1 - \nu_{ij}) \right) + \sum_{j=1}^k (n_j - r_j) \nu_{r_j j}^2 \right\}. \quad (3.16)$$

where $\sigma_{r_j j} = S_j^P x_{r_j j} \ln S_j$. Therefore, the elements of the Fisher information matrix for the MLE can be obtained as the expectations of the negative of the second partial derivatives, i.e.,

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ & f_{22} & f_{23} & f_{24} \\ & & f_{33} & f_{34} \\ & & & f_{44} \end{pmatrix} = -E \begin{pmatrix} \frac{\partial^2 \ln L}{\partial c^2} & \frac{\partial^2 \ln L}{\partial c \partial p} & \frac{\partial^2 \ln L}{\partial c \partial \gamma} & \frac{\partial^2 \ln L}{\partial c \partial \theta} \\ & \frac{\partial^2 \ln L}{\partial p^2} & \frac{\partial^2 \ln L}{\partial p \partial \gamma} & \frac{\partial^2 \ln L}{\partial p \partial \theta} \\ & & \frac{\partial^2 \ln L}{\partial \gamma^2} & \frac{\partial^2 \ln L}{\partial \gamma \partial \theta} \\ & & & \frac{\partial^2 \ln L}{\partial \theta^2} \end{pmatrix}. \quad (3.17)$$

The asymptotic variance-covariance matrix for the MLE is defined as the inverse of the Fisher's information matrix (3.18), i.e.,

$$\Sigma = \hat{F}^{-1} \quad (3.18)$$

3.2 Prediction of the Scale Parameter and the Reliability Function

To predict the value of the scale parameter α_u under the usual condition stress V_u , the invariance property of MLE is used (for more details see, Meeker and Escobar (1998)), i.e.,

$$\hat{\alpha} = \hat{C}S_u^{\hat{P}}, S_u = \frac{V^*}{V_u}, V^* = \prod_{j=1}^k V_j^{b_j}, b_j = \frac{r_j}{\sum_{j=1}^k r_j} \quad (3.19)$$

The MLE of the reliability function at the lifetime x_0 under the usual condition stress V_u , is given by

$$\hat{R}_u(x_0) = (1 + \frac{\gamma}{\theta} e^{\hat{\alpha}_u x_0})^{-\theta} \quad (3.20)$$

4 Optimum Test Plan

A test plan is usually designed before conducting an accelerated life test. Test plan helps in accurately estimating reliability at operating conditions while minimizing test time and costs. A test plan should be used to decide on the appropriate censoring time and the number of test units that need to be allocated to the different stress levels. This section describes optimum test plan that minimize the determinant of Fisher information matrix for specifying the best choice of the values of the fraction of the failure units that fail at each stress level according to the D-Optimality criterion (Gouno (2007)). Let $\lambda_j = \frac{r_j}{N}$ be the fraction of the failure units that may fail at each stress level $V_j, j = 1, \dots, k$, then the optimal values of $\lambda_j, j = 1, \dots, k$ at each stress level $V_j, j = 1, \dots, k$ can be obtained by solving the following equation

$$\frac{\partial |F|}{\partial \lambda_j} = 0, \quad j = 1, \dots, k \quad (4.1)$$

The determinant of F and the derivation of the equation (4.1) are placed in Appendix. The numerical solution of the equation (4.1) is obtained in order to get the optimum value of the fraction and the number of the failure units that fail at each stress level $V_j, j = 1, 2$ as will be shown in section (5.2).

5 Simulation Studies

This section presents the numerical solutions to obtain the MLE of the unknown parameters C, P, γ , and θ , their mean squared errors (MSE), relative absolute biases

(RAB), Lower Bound (LB), Upper Bound (UB), the estimated of scale parameter α , and reliability function $R(x_0)$ under normal use conditions V_u . Also, it presents the numerical solution to determine the best choice values of the fraction of the failure units.

5.1 MLE under Type-II Censoring

The numerical solution is performed according to the following steps

- For given values of C , P and stress level $V_j, j = 1, 2, 3$, the values of $\alpha_j, j = 1, 2, 3$ are calculated according to the equation (2.5).
- Generate a random sample of size N from the 3-parameter generalized logistic distribution and obtained the random variables $x_{ij}, \{i = 1, \dots, r_j, j = 1, \dots, k\}$ for given values of $n_j, r_j, j = 1, \dots, k$, and different initial guesses of the true parameters α, γ, θ say $\alpha_0, \gamma_0, \theta_0$.
- Based on the values of $n_j, r_j, V_j, x_{ij}, \{i = 1, \dots, r_j, j = 1, \dots, k\}$ and V_u , the MLE, and their MSE, RAB, LB, and UB, in addition to, $\hat{\alpha}_u$ and $\hat{R}_u(x_0)$, are obtained.
- The steps are repeated exceed 150 times until getting the MLE as shown in table (5.1).

The numerical results which are placed in tables (5.1) to (5.4) are based on $n_1 = 40, n_2 = 20, n_3 = 5, r_1 = 36, r_2 = 19, r_3 = 4, V_1 = 0.75, V_2 = 1.5, V_3 = 2.25$ and $V_u = 0.5$.

Table 5.1: The MLE, RAB and MSE

C_0	P_0	γ_0	θ_0	Parameter	MLE	RAB	MSE
0.8	1.0	1.0	1.0	C	0.8060	0.0075	0.00004
				P	1.3202	0.3202	0.1025
				γ	0.8634	0.1366	0.0187
				θ	1.2887	0.2887	0.0834
				α_1	1.1940	0.1083	0.0136
				α_2	0.4782	0.1123	0.0037
				α_3	0.2800	0.2204	0.0063

Table 5.1: The MLE, RAB and MSE (Cont.)

C_0	P_0	γ_0	θ_0	Parameter	MLE	RAB	MSE
1.0	1.0	0.8	1.0	C	0.9930	0.0070	0.0001
				P	1.2585	0.2585	0.0668
				γ	0.6820	0.1475	0.0139
				θ	1.3508	0.3508	0.1231
				α_1	1.4443	0.0724	0.0095
				α_2	0.6037	0.1035	0.0049
				α_3	0.3624	0.1927	0.0075
1.0	1.0	1.0	0.7	C	0.9662	0.0338	0.0011
				P	1.2929	0.2929	0.0858
				γ	0.7971	0.2029	0.0412
				θ	0.9534	0.3620	0.0642
				α_1	1.4198	0.0542	0.0053
				α_2	0.5795	0.1395	0.0088
				α_3	0.3431	0.2358	0.0112
1.0	1.0	1.0	1.0	C	1.0068	0.0068	0.0001
				P	1.3174	0.3174	0.1007
				γ	0.8627	0.1373	0.0189
				θ	1.2918	0.2918	0.0852
				α_1	1.4903	0.1066	0.0206
				α_2	0.5980	0.1120	0.0057
				α_3	0.3505	0.2192	0.0097
1.0	1.2	1.0	0.7	C	0.9662	0.0338	0.0011
				P	1.4929	0.2441	0.0858
				γ	0.7971	0.2029	0.0412
				θ	0.9534	0.3620	0.0642
				α_1	1.5069	0.0542	0.0060
				α_2	0.5354	0.1395	0.0075
				α_3	0.2923	0.2358	0.0081
1.0	1.2	1.3	1.0	C	1.0169	0.0169	0.0003
				P	1.5674	0.3062	0.1345
				γ	1.1362	0.1260	0.0268
				θ	1.2572	0.2572	0.0662
				α_1	1.6215	0.1344	0.0369
				α_2	0.5471	0.1206	0.0056
				α_3	0.2898	0.2423	0.0086

Table 5.2: The Confidence Intervals

C_0	P_0	γ_0	θ_0	Parameter	Variance	L.B	U.B
0.8	1.0	1.0	1.0	C	0.0132	0.6170	0.9950
				P	0.0840	0.8434	1.7970
				γ	0.0676	0.4357	1.2911
				θ	0.1085	0.7468	1.8306
1.0	1.0	0.8	1.0	C	0.0205	0.7575	1.2285
				P	0.08530	0.7781	1.7389
				γ	0.0400	0.3530	1.0110
				θ	0.1084	1.1725	1.5291
1.0	1.0	1.0	0.7	C	0.0256	0.7028	1.2296
				P	0.0878	0.8056	1.7803
				γ	0.0103	0.2700	1.3242
				θ	0.0500	0.5857	1.3211
1.0	1.0	1.0	1.0	C	0.0207	0.7701	1.2435
				P	.0840	0.8406	1.7942
				γ	0.0676	0.4350	1.2904
				θ	0.1085	0.7499	1.8337
1.0	1.2	1.0	0.7	C	0.0256	0.7030	1.2294
				P	0.0878	1.0055	1.9803
				γ	0.1027	0.2699	1.3243
				θ	0.0500	0.5856	1.3212
1.0	1.0	1.3	1.0	C	0.0209	0.7791	1.2547
				P	0.0800	1.1021	2.0327
				γ	0.1295	0.5442	1.7282
				θ	0.1084	0.7156	1.7988

Table 5.3: The Asymptotic Variance-Covariance Matrix

C_0	P_0	γ_0	θ_0	Parameter	C	P	γ	θ
0.8	1.0	1.0	1.0	C	0.0132	0.0098	0.0107	-0.0245
				P		0.0840	0.0107	-0.0302
				γ			0.0676	-0.0464
				θ				0.1085
1.0	1.0	0.8	1.0	C	0.0205	0.0118	0.0068	-0.0303
				P		0.0853	0.0065	-0.0307
				γ			0.0400	-0.0314
				θ				0.1084
1.0	1.0	1.0	0.7	C	0.0256	0.0160	0.0243	-0.0257
				P		0.0878	0.0190	-0.0232
				γ			0.0103	-0.0438
				θ				0.0500
1.0	1.0	1.0	1.0	C	0.0207	0.0122	0.0134	-0.0306
				P		0.0840	0.0107	-0.0302
				γ			0.0676	-0.0464
				θ				0.1085
1.0	1.2	1.0	0.7	C	0.0256	0.0160	0.0243	-0.0257
				P		0.0878	0.0189	-0.0232
				γ			0.1027	-0.0438
				θ				0.0500
1.0	1.2	1.3	1.0	C	0.0209	0.0124	0.0250	-0.0309
				P		0.0800	0.0178	-0.0286
				γ			0.1295	-0.0713
				θ				0.1084

Table 5.4: Estimates α and $R(x_0)$
under normal conditions

C_0	P_0	γ_0	θ_0	$\hat{\alpha}_u$	x_0	$\hat{R}_u(x_0)$
0.8	1.0	1.0	1.0	2.1548	0.009	0.4952
					0.2	0.3939
					1.0	0.1039
					2.0	0.0133
1.0	1.0	0.8	1.0	2.6935	0.02	0.5422
					0.4	0.2985
					1.2	0.0470
					2.5	0.0015
1.0	1.0	1.0	0.7	2.0201	0.02	0.5284
					0.4	0.3659
					1.2	0.1369
					2.5	0.0226
1.0	1.0	1.0	1.0	2.6935	0.007	0.4953
					0.03	0.4798
					0.1	0.4331
					1.0	0.0634
1.0	1.2	1.0	0.7	2.6935	0.01	0.5301
					0.2	0.3961
					1.0	0.0768
					2.0	0.0078
1.0	1.2	1.3	1.0	3.2838	0.02	0.4187
					0.4	0.1714
					1.2	0.0147
					2.5	0.0002

From the results of the tables (5.1) to (5.4), we observe the MSE of the scale parameter α_j , $j = 1, 2, 3$ decreases as the stress value V_j , $j = 1, 2, 3$ increases. Also, it is decreases when the values of C_0 and the values of γ_0 increases at the same values of P_0 and θ_0 . In addition, the reliability decreases when the mission time x_0 increases, and it is reduced when the values of C_0 , P_0 increases and the values of γ_0 , θ_0 increases at the same mission time. Moreover, there is an inverse proportional relationship between $\hat{\alpha}_u$ and $\hat{R}_u(x_0)$ at the same mission time.

Table 5.5: The Optimum Values of λ_1^* , r_1^* and GAV
at ($C = 1$, $P = 1$, $\gamma = 1$, $\theta = 1$)

N	n_1	n_2	r_1	r_2	λ_1	λ_2	λ_1^*	r_1^*	GAV
105	57	48	50	46	0.476	0.438	0.143	15	0.000007550
225	150	75	130	73	0.578	0.324	0.279	63	0.000000070
300	170	130	150	128	0.500	0.427	0.350	105	0.000000050
400	235	165	221	166	0.553	0.415	0.205	82	0.000000004
500	370	150	321	165	0.642	0.330	0.549	274	0.000000002

Table 5.6: The Optimum Values of λ_1^* , r_1^* and GAV
at ($C = 1$, $P = 1$, $\gamma = 1$, $\theta = 1.4$)

N	n_1	n_2	r_1	r_2	λ_1	λ_2	λ_1^*	r_1^*	GAV
105	57	48	55	46	0.524	0.438	0.563	59	0.00001340
225	150	75	130	73	0.578	0.324	0.335	75	0.00000012
300	170	130	150	128	0.500	0.427	0.401	120	0.00000008
400	235	165	221	166	0.553	0.415	0.287	115	0.00000001
500	370	150	321	165	0.642	0.330	0.549	274	0.00000001

Table 5.7: The Optimum Values of λ_1^* , r_1^* and GAV
at ($C = 1$, $P = 1.5$, $\gamma = 1$, $\theta = 1.2$)

N	n_1	n_2	r_1	r_2	λ_1	λ_2	λ_1^*	r_1^*	GAV
105	57	48	55	46	0.524	0.438	0.562	59	0.0000078000
225	150	75	130	73	0.578	0.324	0.316	71	0.0000001000
300	170	130	150	128	0.500	0.427	0.329	99	0.0000000600
400	235	165	221	166	0.553	0.415	0.260	104	0.0000000100
500	370	150	321	165	0.642	0.330	0.542	271	0.0000000096

Table 5.8: The Optimum Values of λ_1^* , r_1^* and GAV
at ($C = 0.8$, $P = 1.5$, $\gamma = 1$, $\theta = 1.4$)

N	n_1	n_2	r_1	r_2	λ_1	λ_2	λ_1^*	r_1^*	GAV
105	57	48	55	46	0.524	0.438	0.563	59	0.000008570
225	150	75	130	73	0.578	0.324	0.335	74	0.000000080
300	170	130	150	128	0.500	0.427	0.398	120	0.000000050
400	235	165	221	166	0.553	0.415	0.287	115	0.000000010
500	370	150	321	165	0.642	0.330	0.546	273	0.000000009

Table 5.9: The Optimum Values of λ_1^* , r_1^* and GAV
 at($C = 1.2$, $P = 1.4$, $\gamma = 1.2$, $\theta = 0.7$)

N	n_1	n_2	r_1	r_2	λ_1	λ_2	λ_1^*	r_1^*	GAV
105	57	48	55	46	0.524	0.438	0.509	54	0.000010070
225	150	75	130	73	0.578	0.324	0.247	56	0.000000110
300	170	130	150	128	0.500	0.427	0.370	11	0.000000060
400	235	165	221	166	0.553	0.415	0.145	58	0.000000010
500	370	150	321	165	0.642	0.330	0.568	284	0.000000007

5.2 Optimum Test Plan of Failure Fractions

To illustrate the procedure of optimum test design, a numerical example is given by Supposing a simple constant stress test was run to estimate the optimum value of the fraction of the failure units. The stress levels to test units are $V_1 = 1$ and $V_2 = 2$. The optimum fraction of the failure units is determined by solving equation (4.2). Tables (5.5) to (5.9) presented the optimal fraction of the failure units, the number of failure units and the Generalized Asymptotic Variance (GAV) at different values of N . As seen, the results show less than half of the units will fail under the stress V_1 , and the others will fail under the stress V_2 , because of λ_1^* is taken the values between 0.1 and 0.4. In table (5.9), when $N = 225$, there are 56 units will fail under the stress V_1 , 73 units will fail under the stress V_2 , and 96 will be censored. Also, we observe that GAV decreasing as the sample size increasing.

6 Conclusion

This paper presents the MLE of the GL distribution under type-II censoring with scale parameter that is an inverse power law function. The ML estimation, Fisher information matrix, the asymptomatic variance-covariance matrix, the predictive value of the scale parameter and the reliability function under the usual conditions stress were obtained for various combinations of the model parameters. In additional, the corresponding optimum value of the fraction of the failure units are obtained numerically by the D-optimality criterion. GL distribution have been extensively used in many different areas and it is very useful in a wide variety of applications. Since, standard Logistic, four-parameters extended GL, four-parameters extended GL type-I, two parameter GL, type-I GL, Generalized Log-logistic, standard Log-logistic, Logistic Exponential, Exponentiated Exponential (for $x > 0$), Generalized Burr, Burr III, Burr XII distributions are special cases from the GL distribution then their results of the MLE and optimum test plan become particular cases of the results obtained here.

A Appendix

The determinant of F is given by

$$\begin{aligned} |F| = & (f_{33}f_{44} - f_{34}^2)(f_{11}f_{22} - f_{12}^2) - (f_{23}f_{44} - f_{24}f_{34})(f_{11}f_{23} - f_{12}f_{13}) \\ & + (f_{23}f_{34} - f_{24}f_{33})(f_{11}f_{24} - f_{12}f_{14}) - (f_{13}f_{44} - f_{14}f_{34})(f_{13}f_{22} - f_{12}f_{23}) \\ & + (f_{13}f_{34} - f_{14}f_{33})(f_{13}f_{24} - f_{12}f_{24}) - (f_{13}f_{24} - f_{23}f_{14})(f_{14}f_{23} - f_{13}f_{24}) \end{aligned} \quad (A.1)$$

The derivative of $|F|$ for obtaining the optimum value of the fraction of the failure units λ_j , $j = 1, \dots, k$ is given as follows

$$\begin{aligned} \frac{\partial |F|}{\partial \lambda_j} = & (f_{33}f_{44} - f_{34}^2)(f'_{11}f_{22} + f_{11}f'_{22} - 2f_{12}f'_{12}) + (f'_{33}f_{44} + f_{33}f'_{44} - 2f'_{34}f_{34}) \\ & (f_{11}f_{22} - f_{12}^2) - (f'_{23}f_{44} + f_{23}f'_{44} - f'_{24}f_{34} - f_{24}f'_{34})(f_{11}f_{23} - f_{12}f_{13}) - \\ & (f_{23}f_{44} - f_{24}f_{34})(f'_{11}f_{23} + f_{11}f'_{23} - f'_{12}f_{13} - f_{12}f'_{13}) + (f_{23}f_{34} - f_{24}f_{33}) \\ & (f'_{11}f_{24} + f_{11}f'_{24} - f'_{12}f_{14} - f_{12}f'_{14}) + (f'_{23}f_{34} + f_{23}f'_{34} - f'_{24}f_{33} - f_{24}f'_{33}) \\ & (f_{11}f_{24} - f_{12}f_{14}) - (f_{13}f_{44} - f_{14}f_{34})(f'_{13}f_{22} + f_{13}f'_{22} - f'_{12}f_{23} - f_{12}f'_{23}) - \\ & (f'_{13}f_{44} + f_{13}f'_{44} - f'_{14}f_{34} - f_{14}f'_{34})(f_{13}f_{22} - f_{12}f_{23}) + (f_{13}f_{34} - f_{33}f_{14}) \\ & (f'_{14}f_{22} + f_{14}f'_{22} - f'_{12}f_{24} - f_{12}f'_{24}) + (f'_{13}f_{34} + f_{13}f'_{34} - f'_{33}f_{14} - f_{33}f'_{14}) \\ & (f_{14}f_{22} - f_{12}f_{24}) - (f_{13}f_{24} - f_{23}f_{14})(f'_{14}f_{23} + f_{14}f'_{23} - f'_{13}f_{24} - f_{13}f'_{24}) \\ & - (f'_{13}f_{24} + f_{13}f'_{24} - f'_{23}f_{14} - f_{23}f'_{14})(f_{14}f_{23} - f_{13}f_{24}), \end{aligned} \quad (A.2)$$

where, $f'_{11} = N(\frac{1}{C^2} - \frac{\theta^2}{\gamma} e^{-CSP} \xi_{r,j}^2)$, $f'_{22} = -NC \ln S_j \xi_{r,j} (\ln S_j + C(1 - \nu_{r,j}) \sigma_{r,j})$,
 $f'_{33} = \frac{N}{\gamma^2} (1 + \theta \nu_{r,j}^2)$, $f'_{44} = \frac{N}{\theta} \nu_{r,j}^2$, $f'_{12} = -N \theta \xi_{r,j} (\ln S_j + C(1 - \nu_{r,j}) \sigma_{r,j})$,
 $f'_{13} = \frac{-N \theta}{\gamma} \xi_{r,j} (1 - \nu_{r,j})$, $f'_{14} = -N \xi_{r,j} \nu_{r,j}$, $f'_{23} = \frac{-NC \theta}{\gamma} \ln S_j \xi_{r,j} (1 - \nu_{r,j})$,
 $f'_{24} = -NC \nu_{r,j}^2 \sigma_{r,j}$, $f'_{34} = \frac{-N}{\gamma} \nu_{r,j}^2$.

References

- Abdel-Ghaly, A. A., Attia, A. F., and Aly, H. M. (1998). Estimation of the Parameters of Pareto Distribution and the Reliability Function using Accelerated Life Testing with Censoring. *Communications in Statistics, Simulation and Computation*, 27, 469-484.
- Attia, A. F., Aly, H. M., and Bleed, S. O. (2011). Estimating and Planning Accelerated Life Test using Constant Stress for Generalized Logistic Distribution under Type-I Censoring. Accepted for publication in ISRN Applied Mathematics.
- Chang, D., Chunyan, Y., and Siu-Keung, T. (2010). Accelerated Life Test Sampling Plans for the Weibull Distribution under Type-I Progressive Interval Censoring with

- Random Removals. *Journal of Statistical Computation and Simulation*, 80(8), 903 - 914.
- El-Dessouky, E. A. (2001). On the Use of Bayesian Approach in Accelerated Life Testing. M.Sc. Thesis, Institute of Statistical Studies and Research, Cairo University, Egypt.
- Gouno, E. (2007). Optimum step-stress for temperature accelerated life testing, *Qual. Reliab. Engng. Int*, 23(8), 915 - 924.
- Johnson, N.L., Kotz, S., and Balakrishnan, N. (1995). Continuous Univariate Distributions. Wiley, U.S.A.
- McCool, J. I. (1980). Confidence Limits for Weibull Regression with Censored Data. *IEEE Transactions on Reliability*, 29, 145-150.
- Meeker, W. Q., Escobar L. A., and Lu C. J. (1998). accelerated degradation tests: modeling and analysis, *technometrics*, 40, 89-99.
- Molenberghs, G., and Verbeke, G. (2011). On the Weibull-Gamma Frailty Model, Its Infinite Moments, and Its Connection to Generalized Log-Logistic, Logistic, Cauchy, and Extreme-Value Distributions. *Journal of Statistical Planning and Inference*, 14, 861-868.
- Nelson, W. (1990). *Accelerated Life Testing: Statistical Models, Test Plan and Data Analysis*, Wiley: New York, U.S.A.
- Nelson, W. B. and Kielpinski, T. J. (1976). Theory for Optimum Censored Accelerated Life Test Tests for Normal and Lognormal Life Distributions. *Technometrics*, 18, 105-114.
- Shabri, A., Ahmad, U. N., and Zakaria, Z. A. (2011). TL-moments and L-moments Estimation of the Generalized Logistic Distribution. *Journal of Mathematics Research*, 10 (10), 97-106.
- Tolikas, K. (2011). The Rare Event Risk in African Emerging Stock Markets. *Managerial Finance*, 37 (3), 275-294.
- Tolikas, K., and Gettinby, G. (2009). Modeling the Distribution of the Extreme Share Returns in Singapore. *Journal of Empirical Finance*, 16(2), 254-63.
- Tolikas, K. (2008). Value-at-Risk and Extreme Value Distributions for Financial Returns. *Journal of Risk*, 10, 31-77.
- Watkins, A. J. (1991). On the Analysis of Accelerated Life-Testing Experiments. *IEEE Transactions on Reliability*, 40, 98-101.
- Yang, G. B. (1994). Optimum Constant-Stress Accelerated Life Test Plans. *IEEE Transactions on Reliability*, 43, 575-581.

On the correlated ruin problem with boundaries

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ABSTRACT

A more general form of correlated ruin problem with various types of boundaries such as elastic boundary, partially-reflecting boundary, partially-segregating boundary, and so on are defined, and closed-form formulas are derived for the ultimate absorbing probability as well as the expected duration. Some very simple closed forms which immediately lead to exact and explicit formulas are presented as special cases.

Keywords: Gambler's ruin problem; correlated random walks; elastic boundaries; ruin probability; expected duration

1. Introduction

Gambler's ruin type problem when results of consecutive game are correlated is examined in Mohan (1955) and Mukherjea and Steele (1986), it is useful in describing a number of physical phenomena, See De la Selva (1988), and references therein. The problem was essentially formulated as follows: Suppose that a gambler starting with a capital of i units plays against an opponent whose initial capital is $K - i$ units. The game proceeds by stages, and at each game, conditional on winning the previous game, the probabilities that the gambler wins or loses the next game are p_1, q_1 , and conditional on losing the previous game these probabilities are p_2, q_2 . Thus the gambler's ability to win or lose is governed by the transition probability matrix:

$$\begin{array}{cc} & \begin{array}{cc} \text{win} & \text{loss} \end{array} \\ \begin{array}{c} \text{win} \\ \text{loss} \end{array} & \begin{pmatrix} p_1 & q_1 \\ q_2 & p_2 \end{pmatrix} \end{array}$$

where $p_1 + q_1 = 1 = p_2 + q_2$. The games go on for a while until either the gambler loses all his capital or wins all his opponent's capital.

Here, we generalize the correlated ruin problem by introducing the following assumptions:

- (1)- The current fortune is i , $0 \leq i \leq K$.
- (2)- We take into account the possibility of staying with the same fortune.
- (3)- The probabilities of winning, losing, and neither winning nor losing the next game, depend on winning or losing the previous game.
- (4)- The transition probabilities are governed by

$$\Pr\{\text{winning the next game}\} = \begin{cases} p_1, & \text{if the previous game is winning} \\ q_2, & \text{if the previous game is losing} \end{cases}$$

$$\Pr\left\{ \begin{array}{l} \text{neither winning nor} \\ \text{losing the next game} \end{array} \right\} = \begin{cases} r_1, & \text{if the previous game is winning} \\ r_2, & \text{if the previous game is losing} \end{cases}$$

$$\Pr\{\text{losing the next game}\} = \begin{cases} q_1, & \text{if the previous game is winning} \\ p_2, & \text{if the previous game is losing} \end{cases}$$

with $p_j + q_j + r_j = 1$, $j = 1, 2$.

- (5)- At the origin, the gambler is reduced to penury; at the next game the gambler goes broke (ruined) with probability $1 - \omega_1 - \delta_1$, win the right to spin again with probability ω_1 , or start again with one unit capital with probability δ_1 .
- (6)- At the boundary K , the gambler reaches his objective; at the next game his adversary may be go broke with probability $1 - \omega_2 - \delta_2$, win the right to spin again with probability ω_2 , or start again with a capital of $K - 1$ units with probability δ_2 .

Notice that, contrary to the classical problem, here the process can remain in the same state after a transition, the transition probabilities are depend on the previous game, also at the boundary states 0 and K , it is possible to start, to stay, to leave or to reach.

This problem is analogous to a correlated random walk with various types of boundaries. It's defined for $n \geq 1$, as a sequence of random variables $S_0, S_1, S_2, \dots$:

- (a)- $S_0 = i$, $i \in SS = \{0, 1, \dots, K-1, K\}$;
- (b)- $S_n = X_0 + X_1 + \dots + X_n$, $n \geq 0$;
- (c)- $\Pr(X_{n+1} = 1 | X_n = 1) = p_1$, $\Pr(X_{n+1} = -1 | X_n = 1) = q_1$,
 $\Pr(X_{n+1} = 0 | X_n = 1) = r_1$, $\Pr(X_{n+1} = 0 | X_n = -1) = r_2$,
 $\Pr(X_{n+1} = 1 | X_n = -1) = q_2$, $\Pr(X_{n+1} = -1 | X_n = -1) = p_2$;

$p_j + q_j + r_j = 1$, $j = 1, 2$, with the following modifications at the boundaries.

When the particle reaches the boundary state u , ($u = 0$ or K), it is absorbed with probability $1 - \omega_1 - \delta_1$ at 0 and $1 - \omega_2 - \delta_2$ at K , reflects (to the state $u+1$ or $u-1$) with probability δ_1 and δ_2 respectively, or remains at the boundary state u with probability ω_1 , ω_2 respectively. Absorption being interpreted as the ruin of the gambler or his adversary.

While the above sequence of random variables $\{S_n\}_{n=0}^{\infty}$ does not in general possess the Markov property, it can be easily shown that the sequence of ordered pair of these random variables: $(S_0, S_1), (S_1, S_2), (S_2, S_3), \dots$ is indeed a Markov chain. See for instance Mukherjea and Steele (1987).

Basic properties of correlated random walks such as transition probabilities and first passage times have been long considered in a number of papers. For instance, Goldstein (1951) proposed and studied a correlated random walk model in which the step probabilities depend on the direction of the preceding step. Seth (1963) gave return probabilities and first-passage time distributions for symmetric correlated random walks. Cane (1967) investigated the effect of the initial conditions on the mean and variance of position after n steps. Jain (1971) generalized these results to the non-symmetric case. Proudfoot and Lampard (1972) obtained a generating function of occupation probabilities for a correlated random walk with an absorbing barrier. Renshaw and Henderson (1981) presented occupation probabilities and a diffusion approximation. Gillis (1955) developed a d-dimensional version, and

conjectured it to be transient for all $d > 3$. Gillis' conjecture was proved in Iossif (1986). A comprehensive list of references for correlated random walks, that also includes several other interesting applications, can be found in Chen and Renshaw (1994). Other results on correlated random walks and boundary problems include Jain (1973), Mukherjea and Steele (1987), Weiss (1984), Lal and Bhat (1989), Chen and Renshaw (1992), Zhang (1992), Böhm (2000), Allaart (2004), Richard (2006) and Allaart and Monticino (2008).

The correlated random walk is called symmetric (i.e. symmetric w. r. to transition probabilities) if $p_1 = p_2 = p$, $q_1 = q_2 = q$, $p + q \leq 1$. Then at the end of each step there is a probability p of a step being in the same direction as the preceding one and a probability q of the direction being reversed, $p + q \leq 1$.

In the case $p_1 = q_2 = p$, $q_1 = p_2 = q$, $p + q \leq 1$, the correlation is zero, and the walk becomes classical, the steps are to the right with constant probability p and to the left with a constant probability q , $p + q \leq 1$, i.e. The directions of different steps are independent of each other. In particular, if $p = q$, then the walk is equivalent to the well-known symmetric (non-correlated) random walk, i.e., the particle moves at each step either one unit to the left or to the right with probability p .

Here we have used the following notation:

$Q_0(i) = \Pr(\text{losing a game, with partially-segregating, partially-reflecting boundary probabilities } \omega_1 \text{ and } \delta_1, \text{ respectively}),$

$Q_K(i) = \Pr(\text{winning a game, with partially-segregating, partially-reflecting boundary probabilities } \omega_2 \text{ and } \delta_2, \text{ respectively}),$

$a_0(i) = \Pr(\text{losing a game ultimately, with partially-segregating, partially-reflecting boundary at } 0 \mid \text{the previous game is winning}),$

$a_K(i) = \Pr(\text{winning a game ultimately, with partially-segregating, partially-reflecting boundary at } K \mid \text{the previous game is winning}),$

and $b_0(i)$, $b_K(i)$ are also defined similarly as $a_0(i)$, $a_K(i)$, respectively, but given that the previous game is losing.

Let c_1 , and c_2 be the probabilities of the gambler begin as if the previous game is win or loss, respectively. So that

$$Q_0(i) = c_1 a_0(i) + c_2 b_0(i), \text{ and } Q_K(i) = c_1 a_K(i) + c_2 b_K(i). \quad (1.1)$$

The objective of the present paper is to give closed form formulas for the probability, $Q_0(i)$, " $Q_K(i)$ ", starting from i , the gambler eventually losing all his capital with partially-segregating, partially-reflecting partially-absorbing boundary probabilities ω_1 , δ_1 and $1 - \omega_1 - \delta_1$, respectively "the complementary probability of ruin with partially-segregating, partially-reflecting, partially-absorbing probabilities ω_2 , δ_2 and $1 - \omega_2 - \delta_2$, respectively", for $1 \leq i \leq K - 1$, of the present correlated problem as well as the expected duration. This work is a generalization to previous works; it is apparently not covered by the literature.

We organized the rest of the paper as follows. In Sect. 2, various types of boundaries with some special cases are introduced. Section 3, devoted to find the probability of ultimate ruin. Some special consequences are given in Section 4. In Section 5, the expected duration of play with some special cases are obtained and some numerical examples are introduced in section 6.

2. Various types of boundary states

In this section a rigorous probabilistic two-probability-parameter model of a diffusion barrier is presented that describes comprehensively reflection, absorption and segregation phenomena at a diffusion barrier. As a special case, assume that the initial state i is allowed to start from the boundary 0 or K , with probabilities

$$p_{0j} = \delta_1 \delta_{1j} + \omega_1 \delta_{0j}, \text{ and } p_{Kj} = \omega_2 \delta_{Kj} + \delta_2 \delta_{K-1j}.$$

where δ_{ij} denotes the Kronecker delta. As introduced in Feller (1968); p. 436, see also El-Shehawey(2009) and El-Shehawey and El-Shreef (2009), the reflecting barrier in random walk is defined as a special case of an elastic barrier, it is modified to be at $i=1/2$ and $i=K-1/2$, and the particle is not allowed to reach the boundaries. Instead an elastic barrier, situated at the point $-1/2$ (or at $K+1/2$) on the x-axis between the positions $j=0$ and $j=-1$ (or $j=K$ and $j=K+1$), is defined by the rule that from the position $j=0$, the particle moves with probability δ_1 to position $j=1$, with probability ω_1 it stays at $j=0$; and with probability $1-\delta_1-\omega_1$ it moves to $j=-1$ where it is absorbed. (i.e. the process terminates) or from the position $j=K$, the particle moves with probability δ_2 to position $j=K-1$, with probability ω_2 it stays at $j=K$; and with probability $1-\delta_2-\omega_2$ it moves to $j=K+1$ where it is absorbed (i.e., the process terminates). This means that the state $j=0$ (or $j=K$) is called an elastic barrier, when the process reaches the state 0 (or K), the probabilities of remaining there, reflecting and absorbing at 0, respectively are ω_1 , δ_1 and $1-\delta_1-\omega_1$, $0 < \delta_1, \omega_1 < 1$ (the probabilities of remaining there, reflecting and absorbing at K , respectively are ω_2 , δ_2 and $1-\delta_2-\omega_2$, $0 < \delta_2, \omega_2 < 1$). The elastic barrier we mean the barrier is partially absorbing- partially segregating- partially reflecting at 0 if δ_1 and ω_1 run from 0 to 1 (at K if δ_2 and ω_2 run from 0 to 1).

Special cases for the boundary states:

- (a) - For $0 < \omega_1 < 1, \delta_1 = 0$ (or $0 < \omega_2 < 1, \delta_2 = 0$) we have partially absorbing- partially segregating barrier at 0 (or at K).
- (b) - For $\omega_1 = 1 - \delta_1, 0 \leq \delta_1, \omega_1 \leq 1$ (or $\omega_2 = 1 - \delta_2, 0 \leq \delta_2, \omega_2 \leq 1$) we have partially segregating- partially reflecting (non-absorbing) barrier at 0 (or at K).
- (c) - For $\omega_1 = 0, \delta_1 = 1$ (or $\delta_2 = 1, \omega_2 = 0$) we have totally reflecting (non-absorbing and non-segregating) barrier at 0 (or at K).
- (d) - For $\delta_1 = 0, \omega_1 = 1$ (or $\delta_2 = 0, \omega_2 = 1$) we have totally segregating (non-absorbing) barrier at 0 (or at K).
- (e) - For $\omega_1 = 0, 0 < \delta_1 < 1$ (or $\omega_2 = 0, 0 < \delta_2 < 1$) we have partially absorbing- partially reflecting at 0 (or at K).
- (f) - For $\omega_1 = 0, \delta_1 = 0$ (or $\delta_2 = 0, \omega_2 = 0$) we have totally absorbing barrier at 0 (or at K).

3. Probability of ultimate ruin

It is easy to see that the probabilities $a_0(i)$, and $b_0(i)$ ($a_K(i)$, and $b_K(i)$), $i \in SS$ satisfy the following recurrence relations:

$$\left. \begin{aligned} a_0(i) &= p_1 a_0(i+1) + r_1 a_0(i) + q_1 b_0(i-1) \\ b_0(i) &= p_2 b_0(i-1) + r_2 b_0(i) + q_2 a_0(i+1) \end{aligned} \right\} \quad (3.1)$$

for $i \in \{1, 2, \dots, K-1\}$, with the boundary conditions

$$b_0(0) = [\delta_1 / (1 - \omega_1)] a_0(1) + (1 - \delta_1 / (1 - \omega_1)), \quad (3.2)$$

$$a_0(K) = [\delta_2 / (1 - \omega_2)] b_0(K-1), \quad (3.3)$$

and

$$\left. \begin{aligned} a_K(i) &= p_1 a_K(i+1) + r_1 a_K(i) + q_1 b_K(i-1) \\ b_K(i) &= p_2 b_K(i-1) + r_2 b_K(i) + q_2 a_K(i+1) \end{aligned} \right\}, \quad (3.4)$$

for $i \in \{1, 2, \dots, K-1\}$, subject to

$$b_K(0) = [\delta_1 / (1 - \omega_1)] a_K(1), \quad (3.5)$$

$$a_K(K) = [\delta_2 / (1 - \omega_2)] b_K(K-1) + (1 - \delta_2 / (1 - \omega_2)). \quad (3.6)$$

Solving the recurrence relations (3.1) with (3.2) and (3.3) "or" (3.4) with (3.5) and (3.6) systematically, (see the appendix for more details) we obtain for $p_1 + q_1 \leq 1$ and $p_2 + q_2 \leq 1$

$$a_0(i) = 1 - a_K(i) = \frac{(1 - \sigma_2 \nu_2) \nu_1^{K-1} - (1 - \sigma_2) \nu_1^{i-1}}{(1 - \sigma_2 \nu_2) \nu_1^{K-1} + (\sigma_1 - \nu_2)(1 - \sigma_2) / (1 - \sigma_1)}, \quad (3.7)$$

$$b_0(i) = 1 - b_K(i) = \frac{(1 - \sigma_2 \nu_2) \nu_1^{K-1} - (1 - \sigma_2) \nu_2 \nu_1^i}{(1 - \sigma_2 \nu_2) \nu_1^{K-1} + (\sigma_1 - \nu_2)(1 - \sigma_2) / (1 - \sigma_1)}. \quad (3.8)$$

where

$$\nu_1 = \frac{p_2(p_1 + q_1)}{p_1(p_2 + q_2)}, \quad \nu_2 = \frac{q_2(p_1 + q_1)}{q_1(p_2 + q_2)}, \quad \sigma_1 = \frac{\delta_1}{1 - \omega_1} \quad \text{and} \quad \sigma_2 = \frac{\delta_2}{1 - \omega_2}. \quad (a)$$

In the symmetric correlated game, $p_1 = p_2 = p$, $q_1 = q_2 = q$ and $p + q \leq 1$, we get

$$a_0(i) = 1 - a_K(i) = \frac{(\sigma_2 - 1)(i - K) + \sigma_2 p/q}{(\sigma_2 - 1)[1 - (p/q)/(1 - \sigma_1) - K] + \sigma_2 p/q}, \quad (3.9)$$

$$b_0(i) = 1 - b_K(i) = \frac{(\sigma_2 - 1)(i + 1 - p/q - K) + \sigma_2 p/q}{(\sigma_2 - 1)[1 - (p/q)/(1 - \sigma_1) - K] + \sigma_2 p/q}. \quad (3.10)$$

In the case of symmetric boundary probabilities for the correlated game,

$\delta_1 = \delta_2 = \delta$, $\omega_1 = \omega_2 = \omega$ so $\sigma_1 = \sigma_2 = \sigma = \delta / (1 - \omega)$, we get

$$a_0(i) = 1 - a_K(i) = \frac{(1 - \sigma \nu_2) \nu_1^{K-1} - (1 - \sigma) \nu_1^{i-1}}{(\sigma - \nu_2) + (1 - \sigma \nu_2) \nu_1^{K-1}}, \quad (3.11)$$

$$b_0(i) = 1 - b_K(i) = \frac{(1 - \sigma \nu_2) \nu_1^{K-1} - (1 - \sigma) \nu_1^i \nu_2}{(\sigma - \nu_2) + (1 - \sigma \nu_2) \nu_1^{K-1}}. \quad (3.12)$$

In the two cases symmetric correlated game and symmetric boundary probabilities, $\delta_1 = \delta_2 = \delta$, $\omega_1 = \omega_2 = \omega$, $p_1 = p_2 = p$, $q_1 = q_2 = q$ and $p + q \leq 1$, we get

$$a_0(i) = 1 - a_K(i) = \frac{(\sigma - 1)(i - K) + \sigma p/q}{(\sigma - 1)(1 - K) + (\sigma + 1)p/q}, \quad (3.13)$$

$$b_0(i) = 1 - b_K(i) = \frac{(\sigma - 1)(i + 1 - p/q - K) + \sigma p/q}{(\sigma - 1)(1 - K) + (\sigma + 1)p/q}. \quad (3.14)$$

In the case uncorrelated game, $p_1 = q_2 = p$, $q_1 = p_2 = q$ and $p + q \leq 1$, we have

$$a_0(i) = b_0(i) = \frac{(1 - \sigma_1) \left[\left[(\sigma_2 - q/p)/(1 - \sigma_2) \right] (q/p)^{K-1} + (q/p)^i \right]}{(1 - \sigma_1) \left[(\sigma_2 - q/p)/(1 - \sigma_2) \right] (q/p)^{K-1} - (\sigma_1 q/p - 1)}, \quad (3.15)$$

$$a_K(i) = b_K(i) = \frac{\left[(1 - \sigma_2)/(1 - \sigma_1) \right] (\sigma_1 q/p - 1) + (1 - \sigma_2)(q/p)^i}{\left[(1 - \sigma_2)/(1 - \sigma_1) \right] (\sigma_1 q/p - 1) + (q/p - \sigma_2)(q/p)^{K-1}}. \quad (3.16)$$

Hence, it is found that $Q_0(i)$, the probability starting from i , the gambler eventually losing all his capital with partially- segregating, partially- reflecting, partially- absorbing probabilities ω_1 , δ_1 and $1 - \omega_1 - \delta_1$, respectively "the complementary probability, $Q_K(i)$, of ruin with partially-segregating, partially- reflecting, partially- absorbing probabilities ω_2 , δ_2 and $1 - \omega_2 - \delta_2$, respectively", for $1 \leq i \leq K - 1$, is now given by

$$Q_0(i) = 1 - Q_K(i) = \begin{cases} \frac{(1 - \sigma_2 \nu_2) \nu_1^{K-1} - (1 - \sigma_2)(c_1 + c_2 \nu_1 \nu_2) \nu_1^{i-1}}{\left[(1 - \sigma_2)(\sigma_1 - \nu_2)/(1 - \sigma_1) \right] + (1 - \sigma_2 \nu_2) \nu_1^{K-1}}, & p_1 + q_1 \leq 1 \text{ and } p_2 + q_2 \leq 1, \\ \frac{\sigma_2 p/q + (\sigma_2 - 1) \left[(i - K) + (1 - p/q) c_2 \right]}{(\sigma_2 - 1) \left[1 - (p/q)/(1 - \sigma_1) - K \right] + \sigma_2 p/q}, & p_1 = p_2 = p, q_1 = q_2 = q \text{ and } p + q \leq 1, \\ \frac{(1 - \sigma_1) \left[\left[(\sigma_2 - q/p)/(1 - \sigma_2) \right] (q/p)^{K-1} + (q/p)^i \right]}{\left[(1 - \sigma_1)(\sigma_2 - q/p)/(1 - \sigma_2) \right] (q/p)^{K-1} - (\sigma_1(q/p) - 1)}, & p_1 = q_2 = p, q_1 = p_2 = q \text{ and } p + q \leq 1, \end{cases} \quad (3.17)$$

with ν_1, ν_2, σ_1 and σ_2 as in (a).

If the gambler starts from the origin (i.e. without initial capital), the probability of winning the game capital with partially- segregating, partially- reflecting, partially- absorbing probabilities ω_2 , δ_2 and $1 - \omega_2 - \delta_2$, respectively is

$$Q_K(0) = c_2 \sigma_1 \begin{cases} \frac{(1-\sigma_2)[(\sigma_1-\nu_2)/(1-\sigma_1)+1]}{(1-\sigma_2)(\sigma_1-\nu_2)/(1-\sigma_1)+(1-\sigma_2\nu_2)\nu_1^{K-1}}, \\ p_1+q_1 \leq 1 \text{ and } p_2+q_2 \leq 1, \\ \frac{(1-\sigma_2)(p/q)/(1-\sigma_1)}{(\sigma_2-1)[1-(p/q)/(1-\sigma_1)-K]+\sigma_2 p/q}, \\ p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \\ \frac{(1-\sigma_2)[(\sigma_1 q/p-1)/(1-\sigma_1)+(q/p)]}{(1-\sigma_2)(\sigma_1 q/p-1)/(1-\sigma_1)+(q/p-\sigma_2)(q/p)^{K-1}}, \\ p_1=q_2=p, q_1=p_2=q \text{ and } p+q \leq 1. \end{cases} \quad (3.18)$$

If the gambler starting from the boundary K (i.e. with initial capital is K against an opponent without initial capital), then

$$Q_K(K) = c_1(1-\sigma_2) \begin{cases} \left[1 + \sigma_2 \frac{(\sigma_1-\nu_2)/(1-\sigma_1)+\nu_1^{K-1}\nu_2}{(1-\sigma_2)(\sigma_1-\nu_2)/(1-\sigma_1)+(1-\sigma_2\nu_2)\nu_1^{K-1}} \right], \\ p_1+q_1 \leq 1 \text{ and } p_2+q_2 \leq 1, \\ \left[1 + \sigma_2 \frac{[K-1+(p/q)/(1-\sigma_1)-p/q]}{(\sigma_2-1)[1-(p/q)/(1-\sigma_1)-K]+\sigma_2 p/q} \right], \\ p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \\ \left[1 + \sigma_2 \frac{(\sigma_1 q/p-1)/(1-\sigma_1)+(q/p)^{K-1}}{(1-\sigma_2)(\sigma_1 q/p-1)/(1-\sigma_1)+(q/p-\sigma_2)(q/p)^{K-1}} \right], \\ p_1=q_2=p, q_1=p_2=q \text{ and } p+q \leq 1. \end{cases} \quad (3.19)$$

Notice that the probabilities $Q_0(i), i \in \{0, K\}$, are

$$Q_0(0) = c_2 \begin{cases} 1 + \sigma_1 \left[\frac{(1-\sigma_2\nu_2)\nu_1^{K-1}-(1-\sigma_2)}{(1-\sigma_2)(\sigma_1-\nu_2)/(1-\sigma_1)+(1-\sigma_2\nu_2)\nu_1^{K-1}} - 1 \right], \\ p_1+q_1 \leq 1 \text{ and } p_2+q_2 < 1, \\ 1 + \sigma_1 \left[\frac{(\sigma_2-1)(1-K)+\sigma_2 p/q}{(\sigma_2-1)[1-(p/q)/(1-\sigma_1)-K]+\sigma_2 p/q} - 1 \right], \\ p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \\ 1 + \sigma_1 \left[\frac{(1-\sigma_1)[(\sigma_2-q/p)/(1-\sigma_2)](q/p)^{K-1}+q/p}{[(1-\sigma_1)(\sigma_2-q/p)/(1-\sigma_2)](q/p)^{K-1}-(\sigma_1 q/p-1)} - 1 \right], \\ p_1=q_2=p, q_1=p_2=q \text{ and } p+q \leq 1, \end{cases} \quad (3.20)$$

and

$$Q_0(K) = c_1 \begin{cases} \frac{\sigma_2(1-\nu_2)\nu_1^{K-1}}{(1-\sigma_2)(\sigma_1-\nu_2)/(1-\sigma_1) + (1-\sigma_2\nu_2)\nu_1^{K-1}}, & p_1+q_1 \leq 1 \text{ and } p_2+q_2 \leq 1, \\ \frac{\sigma_2 p/q}{(\sigma_2-1)[1-(p/q)/(1-\sigma_1)-K] + \sigma_2 p/q}, & \\ \sigma_2(1-\sigma_1) \frac{[(\sigma_2-q/p)/(1-\sigma_2)](q/p)^{K-1} + (q/p)^{K-1}}{(1-\sigma_1)[(\sigma_2-q/p)/(1-\sigma_2)](q/p)^{K-1} - (\sigma_1 q/p - 1)}, & p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \\ & p_1=q_2=p, q_1=p_2=q \text{ and } p+q \leq 1, \end{cases} \quad (3.21)$$

with ν_1, ν_2, σ_1 and σ_2 as in (a).

4. Some special consequences

Many interesting special cases can be derived from the general formulae (3.7) - (3.17) through as appropriate choice of the boundary probabilities or the transition probabilities, some cases are listed below

(i) The case $\omega_1 = \omega_2 = 0$, $0 < \delta_1 < 1$, $0 < \delta_2 < 1$, the two barriers are partially absorbing-partially reflecting, that is when the gambler is reduced to penury, at the next game the gambler goes broke or start again with one unit capital with, respective probabilities $1-\delta_1$ and δ_1 . When the gambler reaches his objective at the next game his adversary may be goes broke or start again with capital of $K-1$ units with, respective, probabilities $1-\delta_2$ and δ_2 . In this case we obtain the complementary probabilities of ruin as

$$Q_K(i) = 1 - Q_0(i) = \begin{cases} \frac{(1-\delta_2)[(\delta_1-\nu_2)/(1-\delta_1) + (c_1+c_2\nu_1\nu_2)\nu_1^{i-1}]}{[(1-\delta_2)(\delta_1-\nu_2)/(1-\delta_1) + (1-\delta_2\nu_2)\nu_1^{K-1}]}, & p_1+q_1 \leq 1 \text{ and } p_2+q_2 \leq 1, \\ \frac{(\delta_2-1)[1-i-(p/q)/(1-\delta_1) - (1-p/q)c_2]}{(\delta_2-1)[1-(p/q)/(1-\delta_1)-K] + \delta_2 p/q}, & \\ \frac{1-\delta_1 q/p - (1-\delta_1)(q/p)^i}{[(1-\delta_1)(\delta_2-q/p)/(1-\delta_2)](q/p)^{K-1} - (\delta_1 q/p - 1)}, & p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \\ & p_1=q_2=p, q_1=p_2=q \text{ and } p+q \leq 1, \end{cases} \quad (4.1)$$

and

$$Q_K(i) = 1 - Q_0(i) = \begin{cases} \frac{(1-\delta_2)[(\delta_1-q_2/q_1)/(1-\delta_1) + (c_1+c_2 p_2 q_2/p_1 q_1)(p_2/p_1)^{i-1}]}{(1-\delta_2)(\delta_1-q_2/q_1)/(1-\delta_1) + (1-\delta_2 q_2/q_1)(p_2/p_1)^{K-1}}, & p_j+q_j=1, j=1,2 \\ \frac{(1-\delta_2)(i+\delta_1/(1-\delta_1))}{(1-\delta_1)[K+\delta_1/(1-\delta_1)] + \delta_1}, & p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1. \end{cases} \quad (4.2)$$

We see that with the appropriate change of notation the second result in the formula (4.1) and the first result in the formula (4.2) agree with that of Zhang (1992).

(ii) The case $\delta_1 = 0$, $0 < \delta_2, \omega_1, \omega_2 < 1$, the barrier at 0 is partially absorbing-

partially segregating, that is when the gambler is reduced to penury, at the next game the gambler goes broke (ruined) with probability $1 - \omega_1$, win the right to spin again with probability ω_1 , at the boundary K , the gambler reaches his objective; at the next game his adversary may be go broke with probability $1 - \omega_2 - \delta_2$, win the right to spin again with probability ω_2 , or start again with a capital of $K - 1$ units with probability δ_2 . The complementary probabilities of ruin are

$$Q_K(i) = 1 - Q_0(i) = \begin{cases} \frac{(1 - \sigma_2)[(c_1 + c_2 v_1 v_2) v_1^{i-1} - v_2]}{(1 - \sigma_2 v_2) v_1^{K-1} - (1 - \sigma_2) v_2}, & p_1 + q_1 \leq 1 \text{ and } p_2 + q_2 \leq 1, \\ \frac{(1 - \sigma_2)[i + (c_2 - 1)(1 - p/q)]}{(1 - \sigma_2)[K + p/q - 1] + \sigma_2 p/q}, & p_1 = p_2 = p, q_1 = q_2 = q \text{ and } p + q \leq 1, \\ \frac{(1 - \sigma_2)[(q/p)^i - 1]}{(q/p - \sigma_2)(q/p)^{K-1} - (1 - \sigma_2)}, & p_1 = q_2 = p, q_1 = p_2 = q \text{ and } p + q \leq 1, \end{cases} \quad (4.3)$$

and

$$Q_K(i) = 1 - Q_0(i) = \begin{cases} \frac{(1 - \sigma_2)[(c_1 + c_2 p_2 q_2 / p_1 q_1)(p_2 / p_1)^{i-1} - q_2 / q_1]}{(1 - \sigma_2 q_2 / q_1)(p_2 / p_1)^{K-1} - (1 - \sigma_2) q_2 / q_1}, & p_1 + q_1 = 1 \text{ and } p_2 + q_2 = 1, \\ \frac{i}{K + \sigma_2 / (1 - \sigma_2)}, & p_1 = p_2 = p, q_1 = q_2 = q \text{ and } p = q. \end{cases} \quad (4.4)$$

(iii) In the case $\delta_2 = 0$, $0 < \delta_1, \omega_1, \omega_2 < 1$, the barrier at K is partially absorbing-partially segregating. The complementary probabilities of ruin are

$$Q_K(i) = 1 - Q_0(i) = \begin{cases} \frac{(1 - \sigma_1)(c_1 + v_1 v_2 c_2) v_1^{i-1} - (v_2 - \sigma_1)}{(1 - \sigma_1) v_1^{K-1} - (v_2 - \sigma_1)}, & p_1 + q_1 \leq 1 \text{ and } p_2 + q_2 \leq 1, \\ \frac{(1 - \sigma_1)(i + c_2(1 - p/q) - 1) + p/q}{(1 - \sigma_1)(K - 1) + p/q}, & p_1 = p_2 = p, q_1 = q_2 = q \text{ and } p + q \leq 1, \\ \frac{1 - \sigma_1 q/p - (1 - \sigma_1)(q/p)^i}{1 - \sigma_1 q/p - (1 - \sigma_1)(q/p)^K}, & p_1 = q_2 = p, q_1 = p_2 = q \text{ and } p + q \leq 1, \end{cases} \quad (4.5)$$

and

$$Q_K(i) = 1 - Q_0(i) = \begin{cases} \frac{(1 - \sigma_1)(c_1 + c_2 p_2 q_2 / p_1 q_1)(p_2 / p_1)^{i-1} - (q_2 / q_1 - \sigma_1)}{(1 - \sigma_1)(p_2 / p_1)^{K-1} - (q_2 / q_1 - \sigma_1)}, & p_j + q_j = 1, j = 1, 2 \\ \frac{(1 - \sigma_1)i + \sigma_1}{(1 - \sigma_1)K + \sigma_1}, & p_1 = p_2 = p, q_1 = q_2 = q \text{ and } p = q. \end{cases} \quad (4.6)$$

(iv) The case $\omega_1 = 0$, $0 < \delta_1 < 1$ and $0 < \delta_2, \omega_2 < 1$, when there is a partially absorbing- partially reflecting barrier at 0. The complementary probabilities of ruin are

$$Q_k(i) = 1 - Q_0(i) = \begin{cases} \frac{(1-\sigma_2)[(c_1+c_2\nu_1\nu_2)\nu_1^{i-1}+(\delta_1-\nu_2)/(1-\delta_1)]}{(1-\sigma_2\nu_2)\nu_1^{K-1}+(\delta_1-\nu_2)(1-\sigma_2)/(1-\delta_1)}, & p_1+q_1 \leq 1 \text{ and } p_2+q_2 \leq 1, \\ \frac{(1-\sigma_2)[i-(1-p/q)c_2+p/q(1-\delta_1)-1]}{(1-\sigma_2)(K+p/q(1-\delta_1)-1)+\sigma_2 p/q}, & p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \\ \frac{(1-\delta_1 q/p)-(1-\delta_1)(q/p)^i}{(1-\delta_1)[(\sigma_2-q/p)/(1-\sigma_2)](q/p)^{K-1}-(\delta_1 q/p-1)}, & p_1=q_2=p, q_1=p_2=q \text{ and } p+q \leq 1, \end{cases} \quad (4.7)$$

and

$$Q_k(i) = 1 - Q_0(i) = \begin{cases} \frac{(1-\sigma_2)[(c_1+c_2 p_2 q_2/p_1 q_1)(p_2/p_1)^{i-1}+(\delta_1-q_2/q_1)/(1-\delta_1)]}{(1-\sigma_2 q_2/q_1)(p_2/p_1)^{K-1}+(\delta_1-q_2/q_1)(1-\sigma_2)/(1-\delta_1)}, & p_1+q_1=1 \text{ and } p_2+q_2=1, \\ \frac{(1-\sigma_2)[i+\delta_1/(1-\delta_1)]}{(1-\sigma_2)[K+\delta_1/(1-\delta_1)]+\sigma_2}, & p_1=p_2=p, q_1=q_2=q \text{ and } p=q. \end{cases} \quad (4.8)$$

(v) In the case $\omega_2 = 0$, $0 < \delta_2 < 1$, $0 < \delta_1, \omega_1 < 1$, when there is a partially absorbing- partially reflecting barrier at K . The complementary probabilities of ruin are

$$Q_k(i) = 1 - Q_0(i) = \begin{cases} \frac{(1-\delta_2)[(c_1+c_2\nu_1\nu_2)\nu_1^{i-1}+(\sigma_1-\nu_2)/(1-\sigma_1)]}{(1-\delta_2\nu_2)\nu_1^{K-1}+(\sigma_1-\nu_2)(1-\delta_2)/(1-\sigma_1)}, & p_1+q_1 \leq 1 \text{ and } p_2+q_2 \leq 1, \\ \frac{(1-\delta_2)[i+(1-p/q)c_2+p/q(1-\sigma_1)-1]}{(1-\delta_2)[K+p/q(1-\sigma_1)-1]+\delta_2 p/q}, & p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \\ \frac{(1-\sigma_1 q/p)-(1-\sigma_1)(q/p)^i}{[(1-\sigma_1)(\delta_2-q/p)/(1-\delta_2)](q/p)^{K-1}-\sigma_1 q/p+1}, & p_1=q_2=p, q_1=p_2=q \text{ and } p+q \leq 1, \end{cases} \quad (4.9)$$

and

$$Q_k(i) = 1 - Q_0(i) = \begin{cases} \frac{(1-\delta_2)[(c_1+c_2 p_2 q_2/p_1 q_1)(p_2/p_1)^{i-1}+(\sigma_1-q_2/q_1)/(1-\sigma_1)]}{(1-\delta_2 q_2/q_1)(p_2/p_1)^{K-1}+(1-\delta_2)(\sigma_1-q_2/q_1)/(1-\sigma_1)}, & p_1+q_1=1 \text{ and } p_2+q_2=1, \\ \frac{(1-\delta_2)[i+\sigma_1/(1-\sigma_1)]}{(1-\delta_2)[K+\sigma_1/(1-\sigma_1)]+\delta_2}, & p_1=p_2=p, q_1=q_2=q \text{ and } p=q. \end{cases} \quad (4.10)$$

(vi) The case $\omega_1 = 0$ and $\delta_1 = 0$, $0 < \delta_2, \omega_2 < 1$ ($\omega_2 = \delta_2 = 0$, $0 < \delta_1, \omega_1 < 1$) when there is a totally absorbing barrier at 0 (at K). The complementary probabilities of ruin are coinciding with the formulae (4.3), and (4.4) ((4.5), and (4.6)).

Note that the results in the case $p_1=q_1=p, q_1=p_2=q$ and $p=q$; are coincide with that in the case $p_1=p_2=p, q_1=q_2=q$ and $p=q$.

5. Expected duration of play

Let A_i and B_i denote the expected duration for the game (whose initial capital is i) with the previous step game to i is winning and losing, respectively, then

$$A_i = p_1 A_{i+1} + q_1 B_{i-1} + r_1 A_i + 1, \quad (5.1)$$

$$B_i = p_2 B_{i-1} + q_2 A_{i+1} + r_2 B_i + 1, \quad (5.2)$$

with the boundary conditions

$$B_0 = \frac{\delta_1}{1-\omega_1} A_1, \quad (5.3)$$

$$A_K = \frac{\delta_2}{1-\omega_2} B_{K-1}. \quad (5.4)$$

When reaches the boundary 0, the gambler is reduced to penury; at the next game the gambler start again with one unit capital with probability δ_1 and the expected duration at 1 is A_1 , goes broke (ruined) with probability $1-\omega_1-\delta_1$, or win the right to spin again with probability ω_1 (note that the reflection from 0 to 1 is not a game and needs no time, so we need not add 1 in the boundary condition); when reaches the boundary K , the gambler reaches his objective; at the next game his adversary may be start again with a capital of $K-1$ units with probability δ_2 , and the expected duration at $K-1$ is B_{K-1} , goes broke with probability $1-\omega_2-\delta_2$, or win the right to spin again with probability ω_2 , (note that the reflection from K to $K-1$ is also not a game and needs no time, so again we need not add 1 in the boundary condition).

This system of non-homogeneous difference equations of first order with constant coefficients can be solved via the substitution

$$A_i = hi + l + A'_i \quad \text{and} \quad B_i = mi + n + B'_i. \quad (5.5)$$

This leads to

$$A_i = \begin{cases} \left[\frac{(q_1+q_2)}{(q_1p_2-q_2p_1)} \right] i + \left[\frac{X + \nu_1' Y}{Z} \right], & p_1+q_1 \leq 1 \text{ and } p_2+q_2 \leq 1, \\ \left[\frac{qi^2 + X_1/(1-\sigma_1\sigma_2)}{Z_1} \right], & p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \end{cases} \quad (5.6)$$

and

$$B_i = \begin{cases} \left[\frac{(q_1+q_2)(i+1) - (p_1+p_2)}{(q_1p_2-q_2p_1)} \right] + \left[\frac{X + Y \nu_2 \nu_1'}{Z} \right], & p_1+q_1 \leq 1 \text{ and } p_2+q_2 \leq 1, \\ \left[\frac{qi^2 + 2(q-p)i + \sigma_1(K-1)Y_1/(1-\sigma_1\sigma_2)}{Z_1} \right], & p_1=p_2=p, q_1=q_2=q \text{ and } p+q \leq 1, \end{cases} \quad (5.7)$$

where

$$X = \nu_1(\sigma_1 - \nu_2)[(q_1+q_2)(\sigma_2-1)K - \sigma_2(p_1+p_2)] + \nu_1^K(1-\sigma_2\nu_2)[(q_1+q_2)(\sigma_1-1) + p_1+p_2],$$

$$Y = (q_1+q_2)(\sigma_2-1)(K+1)(1-\sigma_1) - (p_1+p_2)(1-\sigma_1\sigma_2),$$

$$Z = (q_1p_2 - q_2p_1)[(1-\sigma_2)\nu_1(\sigma_1 - \nu_2) + \nu_1^K(1-\sigma_2\nu_2)(1-\sigma_1)],$$

$$X_1 = \sigma_2q(K-1)^2 - 2\sigma_2(p-q)(K-1) + \sigma_1\sigma_2q - qK^2,$$

$Y_1 = q(1 - \sigma_2)(K - 1) - 2\sigma_2 p$, $Z_1 = (2q - p)^2 - 3pq$, and $v_1, v_2, \sigma_1, \sigma_2$ as in (a).

Let D_i be the expected duration for the game with initial capital i and suppose that D_i is finite. Then D_i can be derived from $D_i = c_1 A_i + c_2 B_i$.

Special Cases:

Case 1: If $p_1 = q_2 = p$, $q_1 = p_2 = q$ and $p + q \leq 1$, then the expected duration is

$$D_i = i/(q - p) + [X_2 + (q/p)^i Y_2]/Z_2, \quad (5.8)$$

where

$$\begin{aligned} X_2 &= \sigma_2 (q/p)^{K-1} (q/p - \sigma_2) + (q\sigma_1/p - 1)(K(\sigma_2 - 1) - \sigma_2), \\ Y_2 &= (1 - \sigma_1)[K(\sigma_2 - 1) - \sigma_2] - \sigma_1(1 - \sigma_2), \\ Z_2 &= (q - p)[(1 - \sigma_2)(q\sigma_1/p - 1) + (q/p)^{K-1}(q/p - \sigma_2)(1 - \sigma_1)]. \end{aligned}$$

Case 2: If $p_1 = q_2 = p$, $q_1 = p_2 = q$, $p + q \leq 1$ and $\delta_i = 0$, $i = 1, 2$, then the expected duration is

$$D_i = \begin{cases} \frac{1}{q - p} i - \frac{K}{q - p} \frac{(q/p)^i - 1}{(q/p)^K - 1}, & p + q \leq 1, \\ i(K - i), & p = q = 1/2. \end{cases} \quad (5.9)$$

Formula (5.9) agrees with that of Zhang (1992).

Case 3: If $p_1 = q_2 = p$, $q_1 = p_2 = q$, $p = q = 1/2$, $c_1 + c_2 = 1$, then the expected duration is

$$D_i = -i^2 + \frac{K(K - 1)\sigma_1(1 - \sigma_2)/(1 - \sigma_1) + i[K^2 - \sigma_2(K - 1)^2 + \sigma_1(1 - \sigma_2)/(1 - \sigma_1)]}{(1 - \sigma_2)[K + \sigma_1/(1 - \sigma_1) + \sigma_2]}, \quad (5.10)$$

Case 4: If $p_1 + q_1 = 1$, $p_2 + q_2 = 1$. By putting $p_1 = \alpha, q_2 = \beta \rightarrow q_1/q_2 = \rho_1$, $p_2/p_1 = \rho_2$, thus $q_1 + q_2 = 1 - \alpha + \beta$, $p_1 + p_2 = 1 + \alpha - \beta$ and $q_1 p_2 - q_2 p_1 = 1 - \alpha - \beta$. Let $c_1 + c_2 = 1$. Then the expected duration is

$$D_i = \frac{(1 - \alpha + \beta)i + 2(\beta - \alpha)c_2}{1 - \alpha - \beta} + [X_3 + (c_1 \rho_1 + c_2) \rho_2^{i-1} Y_3]/Z_3, \quad (5.11)$$

where

$$\begin{aligned} X_3 &= (\sigma_1 \rho_1 - 1)[(1 - \alpha + \beta)(\sigma_2 - 1)K - \sigma_2(1 + \alpha - \beta)] + \\ &\quad (\rho_1 - \sigma_2)[(1 - \alpha + \beta)(\sigma_1 - 1) + (1 + \alpha - \beta)] \rho_2^{K-1}, \\ Y_3 &= (1 - \alpha + \beta)(\sigma_2 - 1)(1 - \sigma_1)K - \sigma_2(1 + \alpha - \beta)(1 - \sigma_1) \\ &\quad + (1 - \sigma_2)[2(\beta - \alpha) + \sigma_1(1 - \alpha + \beta)], \\ Z_3 &= (1 - \alpha + \beta)[(1 - \sigma_2)(\sigma_1 \rho_1 - 1) + \rho_2^{K-1}(\rho_1 - \sigma_2)(1 - \sigma_1)]. \end{aligned}$$

Case 5: If $p_1 + q_1 = 1$, $p_2 + q_2 = 1$. By putting $p_1 = \alpha, q_2 = \beta \rightarrow q_1/q_2 = \rho_1$, $p_2/p_1 = \rho_2$, thus $q_1 + q_2 = 1 - \alpha + \beta$, $p_1 + p_2 = 1 + \alpha - \beta$ and $q_1 p_2 - q_2 p_1 = 1 - \alpha - \beta$. Let $c_1 + c_2 = 1$, $\omega_i = 0$, $i = 1, 2$ then $\sigma_i = \delta_i$, $i = 1, 2$, and the expected duration is

$$D_i = \frac{(1-\alpha+\beta)i + 2(\beta-\alpha)c_2}{1-\alpha-\beta} + [X_4 + (c_1\rho_1 + c_2)\rho_2^{i-1}Y_4]/Z_4, \quad (5.12)$$

where

$$X_4 = (\delta_1\rho_1 - 1)[(1-\alpha+\beta)(\delta_2 - 1)K - \delta_2(1+\alpha-\beta)] + (\rho_1 - \delta_2)[(1-\alpha+\beta)(\delta_1 - 1) + (1+\alpha-\beta)]\rho_2^{K-1},$$

$$Y_4 = (1-\alpha+\beta)(\delta_2 - 1)(1-\delta_1)K - \delta_2(1+\alpha-\beta)(1-\delta_1) + (1-\delta_2)[2(\beta-\alpha) + \delta_1(1-\alpha+\beta)],$$

$$Z_4 = (1-\alpha+\beta)[(1-\delta_2)(\delta_1\rho_1 - 1) + \rho_2^{K-1}(\rho_1 - \delta_2)(1-\delta_1)].$$

Formula (5.12) agrees with that of Zhang (1992).

6. Numerical examples

In this section, we discuss some numerical examples (using *mathematica* program) for the effects of correlation for the present problem for example we consider the following particular cases:

The case one: $\omega_1 = 0$, $\omega_2 = 0$, $\delta_1 = 0$ and $\delta_2 = 0$, the two barriers are perfectly absorbing, using formulas (4.1) and (4.2) respectively, we get:

(i) For $p_1 + q_1 \leq 1$, $p_2 + q_2 \leq 1$, the complementary probabilities of ruin become

$$Q_K(i) = 1 - Q_0(i) = \frac{-v_2 + (c_1 + c_2 v_1 v_2) v_1^{i-1}}{-v_2 + v_1^{K-1}}$$

(ii) For $p_1 + q_1 = 1$, $p_2 + q_2 = 1$ the complementary probabilities of ruin will be

$$Q_K(i) = 1 - Q_0(i) = \frac{-\frac{q_2}{q_1} + \left(c_1 + c_2 \frac{p_2 q_2}{p_1 q_1}\right) \left(\frac{p_2}{p_1}\right)^{i-1}}{-\frac{q_2}{q_1} + \left(\frac{p_2}{p_1}\right)^{K-1}}$$

The case two: $0 < \omega_1, \omega_2 < 1$ and $0 < \delta_1, \delta_2 < 1$, the two barriers are partially absorbing-partially segregating- partially reflecting, using formulas (3.17), (5.6) and (5.7) for

(i) $p_1 + q_1 \leq 1$, $p_2 + q_2 \leq 1$ and (ii) $p_1 = p_2 = p$, $q_1 = q_2 = q$.

Some numerically results for these cases are shown in table 1

Case	δ_1	δ_2	ω_1	ω_2	K	i	c_1	c_2	p_1	q_1	p_2	q_2	$Q_0(i)$	D_i
I	i	0	0	0	1000	100	0.5	0.5	0.5	0.2	0.4	0.2	0.000967341	17992.4
					100	5	0.7	0.3	0.2	0.4	0.1	0.3	0.268066	2283.12
	ii	0	0	0	1000	900	0.5	0.5	0.2	0.8	0.9	0.1	0.999999	643.
					100	70	0.7	0.3	0.5	0.5	0.5	0.5	0.3	2100
II	i	0.4	0.8	0.5	1000	900	0.5	0.5	0.5	0.2	0.4	0.1	0.999997	9129.97
					100	90	0.7	0.3	0.1	0.2	0.2	0.1	0.999805	939.811
	ii	0.2	0.7	0.2	1000	900	0.5	0.5	0.2	0.4	0.2	0.4	0.102924	2.49903*10 ⁶
					100	10	0.3	0.7	0.1	0.8	0.1	0.8	0.896504	748.229

(table 1)

7. Appendix:

From equations (3.1) we get the following difference equation

$$p_1(p_2 + q_2)a_0(i+2) + (-p_1(p_2 + q_2) - p_2(p_1 + q_1))a_0(i+1) + p_2(p_1 + q_1)a_0(i) = 0. \quad (*)$$

Putting $a_0(i) = \lambda^i$ in equation (*), the following characteristic equation follows

$$p_1(p_2 + q_2)\lambda^2 + (-p_1(p_2 + q_2) - p_2(p_1 + q_1))\lambda + p_2(p_1 + q_1) = 0$$

It can be easily obtain $\lambda_1 = 1$ and $\lambda_2 = \frac{p_2(p_1 + q_1)}{p_1(p_2 + q_2)}$.

The general solution of (*) is

$$a_0(i) = v_1\lambda_1^i + v_2\lambda_2^i = v_1 + v_2\left(\frac{p_2(p_1 + q_1)}{p_1(p_2 + q_2)}\right)^i, \quad (**)$$

where v_1 and v_2 are constants.

Substituting from (**) into (3.1), we get $b_0(i)$ as

$$b_0(i) = v_1 + v_2\left(\frac{q_2(p_1 + q_1)}{q_1(p_2 + q_2)}\right)\left(\frac{p_2(p_1 + q_1)}{p_1(p_2 + q_2)}\right)^{i+1} \quad (***)$$

From the boundary conditions (3.2) and (3.3) we can calculate the values of v_1, v_2 as follows

$$v_1 = \frac{v_1^{K-1}(1 - \sigma_2 v_2)}{(1 - \sigma_2 v_2)v_1^{K-1} + (\sigma_1 - v_2)(1 - \sigma_2)/(1 - \sigma_1)},$$

$$v_2 = \frac{(\sigma_2 - 1)}{(1 - \sigma_2 v_2)v_1^K + (\sigma_1 - v_2)v_1(1 - \sigma_2)/(1 - \sigma_1)}.$$

Substituting form v_1, v_2 into (**) and (***) formula (3.7) and (3.8) follows.

Conclusion

We have presented general form of correlated ruin problem with various types of boundary such as elastic boundary, partially-reflecting boundary, partially-segregating boundary and so on, and closed-form formulas are derived for the ultimate absorbing probability as well as the expected duration. Some very simple closed forms which immediately lead to exact and explicit formulas are presented as special cases depending on the relationships between the transition probabilities and the boundary condition probabilities.

References

- Allaart, P.C., Monticino M., 2008. Optimal buy/sell rules for correlated random walks. J. Appl. Prob. 45, 33-44
- Allaart, P.C., 2004. Optimal stopping rules for correlated random walks with a discount. J. Appl. Prob. 41, 483-496.
- Böhm, W., 2000. The correlated random walk with boundaries: a combinatorial solution. J. Appl. Prob. 37, 470-479.
- Cane V. R., 1967. Random walk and physical processes. Bull. Internat. Stat. Inst. 42, 62-640.
- Chen, A.Y., Renshaw, E., 1992. The Gillis- Domb- Fisher correlated random walk. J. Appl. Prob. 29, 792-813.
- Chen, A.Y., Renshaw, E., 1994. The general correlated random walk. J. Appl. Prob. 31, 869-884.

- Feller, W., 1968. An Introduction to Probability Theory and Its Applications, 3rd. v. I. Wiley, New York.
- Gillis, J., 1955. Correlated random walk. Proc. Camb. Phil. Soc. 51, 639-651.
- Goldstein, S., 1951. On diffusion by discontinuous movements, and on the telegraph equation. Quart. J. Mech. 4, 129-156.
- Iossif, G., 1986. Return probabilities for correlated random walks. J. Appl. Prob. 23, 201-207.
- Jain, G. C., 1971. Some results in a correlated random walk. Canad. Math. Bull. 14, 341-347.
- Jain, G. C., 1973. On the expected number of visits of a particle before absorption in a correlated random walk. Canad. Math. Bull. 16, 389-395.
- Lal, R., Bhat, U.N., 1989. Some explicit results for correlated random walks. J. Appl. Prob. 27, 757-766.
- Mohan, C., 1955. The gambler's ruin problem with correlation. Biometrika 42, 486-493.
- Mukherjea, A., Steele, D., 1986. Conditional expected durations of play given the ultimate outcome for a correlated random walk. Statist. Probab. Lett. 4, 237-243.
- Mukherjea, A., Steele, D., 1987. Occupation probability of a correlated random walk and a correlated ruin problem. Statist. Probab. Lett. 5, 105-111.
- Proudfoot, A. D., Lampard, D. G., 1972. A random walk problem with correlation. J. Appl. Prob. 9, 436-640.
- Renshaw, E., Henderson, R., 1981. The correlated random walk. J. Appl. Prob. 18, 403-414.
- Richard A., 2006. A closed form solution of a discrete correlated random walk, Journal of Difference Equations and Applications, Volume 12, Number 9, 949-956
- Seth, A., 1963., The correlated unrestricted random walk. J. R. Statist. Soc. B 25, 394-400.
- Zhang, Y.L., 1992. Some problems on a one-dimensional correlated random walk with various types of barrier. J. Appl. Prob. 29, 196-201.

1. The first part of the document is a letter from the President of the United States to the Congress, dated January 1, 1861. It is a very important document, as it sets out the President's policy for the new year. The President states that he is pleased to see the Congress assembled, and that he is confident that the country is in a good position to meet the challenges of the future.

2. The second part of the document is a report from the Secretary of the Treasury, dated January 1, 1861. It is a very important document, as it sets out the Secretary's policy for the new year. The Secretary states that he is pleased to see the Congress assembled, and that he is confident that the country is in a good position to meet the challenges of the future.

3. The third part of the document is a report from the Secretary of the Interior, dated January 1, 1861. It is a very important document, as it sets out the Secretary's policy for the new year. The Secretary states that he is pleased to see the Congress assembled, and that he is confident that the country is in a good position to meet the challenges of the future.

4. The fourth part of the document is a report from the Secretary of the Navy, dated January 1, 1861. It is a very important document, as it sets out the Secretary's policy for the new year. The Secretary states that he is pleased to see the Congress assembled, and that he is confident that the country is in a good position to meet the challenges of the future.

5. The fifth part of the document is a report from the Secretary of the War, dated January 1, 1861. It is a very important document, as it sets out the Secretary's policy for the new year. The Secretary states that he is pleased to see the Congress assembled, and that he is confident that the country is in a good position to meet the challenges of the future.

6. The sixth part of the document is a report from the Secretary of the State, dated January 1, 1861. It is a very important document, as it sets out the Secretary's policy for the new year. The Secretary states that he is pleased to see the Congress assembled, and that he is confident that the country is in a good position to meet the challenges of the future.

7. The seventh part of the document is a report from the Secretary of the Army, dated January 1, 1861. It is a very important document, as it sets out the Secretary's policy for the new year. The Secretary states that he is pleased to see the Congress assembled, and that he is confident that the country is in a good position to meet the challenges of the future.

Cumulative Exposure Model Based on Type II Hybrid Censored Data for Burr Type XII Distribution

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Abstract: In this article, we consider the estimation problem for a simple step-stress model under hybrid type II censored data from Burr type XII distribution. We use the maximum likelihood method to obtain the estimators of parameters with respect to a cumulative exposure model. In addition, asymptotic variance and covariance matrix of the estimators are given. Furthermore, confidence intervals of the estimators are presented. The performance of the findings in the paper is shown by demonstrating some numerical illustration through Monte Carlo simulation. Finally, we present examples to illustrate all the methods of inference discussed here.

Keywords: Burr XII distribution; Cumulative exposure model; Fisher information matrix; Hybrid censored data; Maximum likelihood method; Step stress testing;

1 Introduction

In many situations, it may be difficult to collect data on the product and may have a high reliability under normal conditions. For this reason, the Accelerated Life-Testing (ALT) experiment can be used to force these products to fail more quickly than under the operating condition. Therefore, the (ALT) experiments have become important technical structures in reliability and survival analysis. A special class of the (ALT) is called the Step-Stress Testing (SST) which allows the experimenter to choose one or more stress factors in a life-testing experiment. The examples of stress factors include humidity, temperature, vibration, voltage, load or any other factor that directly affects the lifetime of a product.

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Epstein (1954) discussed a hybrid censoring scheme in which the lift-testing experiment is terminated at a random time $T_2 = \min\{t_{r:n}, \tau_2\}$, and only the failure time before T_2 are recorded, where $1 \leq r \leq n$ and $0 < \tau_2 < \infty$. Compared to the conventional type-I censoring scheme, the termination time here is at most τ_2 , and we refer to this as type-I Hybrid Censoring Scheme (HCS). If $T_2 = \max\{t_{r:n}, \tau_2\}$, we would refer to it as type-II (HCS) since it would guarantee that the number of failure times is at least r .

Nelson (1980) studies accelerated life testing step-stress model and data analysis. Miller (1983) presented optimum simple step-stress plans for accelerated life testing, and Bai (1984) presents optimum simple step-stress accelerated life tests with censoring. Nasar (1996) developed optimal accelerated life test designs for Burr Type XII distribution under periodic inspection and type I-censoring, assuming that the mean lifetime is a log-linear function of stress. Also, Gupta (1998) studies hybrid censoring schemes with exponential failure distribution, and Watkins (2001) studies the inference in simple step-stress models. Childs (2003) studies the exact likelihood inference based on type-I and type-II Hybrid censored samples from the exponential distribution. Banerjee (2006) studies inference based on type-II hybrid censored data from a Weibull distribution. and Balakrishnan (2007) studies exact inference for a simple step-stress model with type-II hybrid censored data from the exponential distribution. Abd-Elfattah (2008) devoted to the estimating failure time data under step-stress partially accelerated life test based on type I censoring that using the tampered random variable model proposed by De Groot (1979). Balakrishnan (2009) studies exact inference for a simple step-stress model from the exponential distribution under time constraint.

2 Model Description

2.1 Notation

ALT	Acceerated Life Testing.
SST	Step Stess Testing.
HCS	Hybrid censoring scheme.
pdf	Probability density function.
cdf	Cumulative distribution function.

- $G(t)$ Cumulative exposure distribution function.
 $g(t)$ Probability exposure density function.
 s_0, s_1 Stress levels.
 n Identical units under an initial stress level s_0 .
 $r \leq n$ and $0 < \tau_1 < \tau_2 < \infty$ are fixed in advance.
 $t_{1:n} < t_{2:n} < \dots < t_{n:n}$ The ordered failure times of the n unit under test.
 τ_1 A fixed time before which the stress level is changed from s_0 to s_1 .
 $t_{r:n}$ The time when the r^{th} failure occurs.
 τ_2 A fixed time before which if the r^{th} failure occurs the experiment is terminated at time τ_2 .
 T_2 The random time when the life-testing experiment is terminated.
 N_1 Number of units that fail before time τ_1 at stress level s_0 .
 N_2 Number of units that fail before time τ_2 at stress level s_1 .
 M Number of units that fail before time T_2 at stress level s_1 .
 c_i, k The shape parameters of the Burr type XII distribution.
 F Fisher information matrix.
 $L(.)$ Likelihood function.
 $\log L(.)$ Logarithm of likelihood function.
 MSE Mean square error.
 $ARBias$ Absolute relative bias.
 RE Relative error.

2.2 Assumptions

Suppose that the data come from Burr type XII distribution, and we consider a *simple step-stress model based on type-II HCS with only two stress levels s_0 and s_1* . The probability density function (pdf) and cumulative distribution (cdf) for Burr type XII distribution are given by, respectively

$$f_i(t; c_i) = c_i k t^{c_i-1} (1 + t^{c_i})^{-(k+1)}, \quad t > 0 \quad k, c_i > 0 \quad i = 1, 2, \quad (1)$$

and

$$F_i(t; c_i) = 1 - (1 + t^{c_i})^{-k}, \quad t > 0 \quad k, c_i > 0 \quad i = 1, 2. \quad (2)$$

Then, we have the cumulative exposure distribution (cdf) $G(t)$ as follows,

$$G(t) = \begin{cases} G_1(t) = F_1(t), & 0 \leq t < \tau_1, \\ G_2(t) = F_2 \left[(t - \tau_1) + \tau_1^{\frac{c_1}{c_2}} \right], & \tau_1 \leq t < \infty, \end{cases}$$

hence, by using (1), it can be rewritten,

$$G(t) = \begin{cases} G_1(t) = 1 - (1 + t^{c_1})^{-k}, & 0 \leq t < \tau_1, \\ G_2(t) = 1 - \left[1 + \left(t - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2} \right]^{-k}, & \tau_1 \leq t < \infty. \end{cases} \quad (3)$$

The corresponding probability exposure density function becomes

$$g(t) = \begin{cases} g_1(t) = f_1(t) & 0 \leq t < \tau_1, \\ g_2(t) = f_2 \left[(t - \tau_1) + \tau_1^{\frac{c_1}{c_2}} \right], & \tau_1 \leq t < \infty, \end{cases}$$

therefore, by using (2), it can be rewritten as follows,

$$g(t) = \begin{cases} g_1(t) = kc_1 t^{c_1-1} (1 + t^{c_1})^{-(k+1)}, & 0 \leq t < \tau_1, \\ g_2(t) = kc_2 \left[t - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right]^{c_2-1} * \left[1 + \left(t - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2} \right]^{-(k+1)} & \tau_1 \leq t < \infty. \end{cases} \quad (4)$$

Based on the type-II HCS, we have n identical units under an initial stress level s_0 . The stress level is changed to s_1 at time τ_1 , and the life-testing experiment is terminated at time τ_2 if the r^{th} failure occurs before τ_2 , otherwise, the experiment is terminated as soon as the r^{th} failure occurs. Thus, the termination time in this situation is a random variable given by

$$T_2 = \max\{t_{r:n}, \tau_2\}, \quad (5)$$

Then, it is evident that,

$$M = \begin{cases} r - N_1, & \tau_2 < t_{r:n}, \\ N_2, & t_{r:n} \leq \tau_2, \end{cases}$$

and

$$T_2 = \begin{cases} t_{r:n}, & \tau_2 < t_{r:n}, \\ \tau_2, & t_{r:n} \leq \tau_2. \end{cases}$$

We will observe one of the following three cases for the observations:

- **Case 1:** If $t_{r:n} \leq \tau_1 < \tau_2$, we will observe

$$t_{1:n} < \dots < t_{r:n} < \dots < t_{N_1:n} \leq \tau_1 < t_{N_1+1:n} < \dots < t_{N_1+N_2:n} \leq \tau_2,$$

- **Case 2:** If $\tau_1 < t_{r:n} \leq \tau_2$, we will observe

$$t_{1:n} < \dots < t_{N_1:n} \leq \tau_1 < t_{N_1+1:n} < \dots < t_{r:n} < \dots < t_{N_1+N_2:n} \leq \tau_2,$$

- **Case 3:** If $\tau_2 < t_{r:n}$, we will observe

$$t_{1:n} < \dots < t_{N_1:n} \leq \tau_1 < t_{N_1+1:n} < \dots < t_{N_1+N_2:n} \leq \tau_2 < t_{N_1+N_2+1:n} < \dots < t_{r:n} < \infty.$$

3 Maximum Likelihood Estimation

From the cumulative exposure distribution function (3) and the corresponding probability density function (4), we obtain the likelihood function of c_1, c_2 and k based on the type-II hybrid censored sample as follows,

$$L(c_i, k|t) = \frac{n!}{(n-R)!} \left[\prod_{i=1}^{N_1} g_1(t_{i:n}) \right] \left[\prod_{i=N_1+1}^R g_2(t_{i:n}) \right] * [1 - G_2(T_2)]^{n-R},$$

$$0 < t_{1:n} < \dots < t_{N_1:n} \leq \tau_1 < t_{N_1+1:n} < \dots < t_{R:n} \leq T_2, \quad (6)$$

where $R = N_1 + M$. By using (3) and (4), thus the likelihood function becomes,

$$L = \frac{n!}{(n-R)!} \left[\prod_{i=1}^{N_1} k c_1 t_{i:n}^{c_1-1} (1 + t_{i:n}^{c_1})^{-(k+1)} \right]$$

$$* \left[\prod_{i=N_1+1}^R k c_2 \left(t_{i:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2-1} \left[1 + \left(t_{i:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2} \right]^{-(k+1)} \right]$$

$$* \left[1 + \left(T_2 - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2} \right]^{-k(n-R)},$$

$$0 < t_{1:n} < \dots < t_{N_1:n} \leq \tau_1 < t_{N_1+1:n} < \dots < t_{R:n} \leq T_2. \quad (7)$$

Therefore, the logarithm of likelihood function is:

$$\mathcal{L} = \log L(c_i, k|t) = \log \frac{n!}{(n-R)!} + \sum_{i=1}^{N_1} \log [k c_1 t_{i:n}^{c_1-1} (1 + t_{i:n}^{c_1})^{-(k+1)}]$$

$$\begin{aligned}
& + \sum_{i=N_1+1}^R \log \left[k c_2 \left(t_{i:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2-1} \right] \\
& + \sum_{i=N_1+1}^R \log \left[1 + \left(t_{i:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2} \right]^{-(k+1)} \\
& + \log \left[1 + \left(T_2 - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2} \right]^{-k(n-R)}, \tag{8}
\end{aligned}$$

hence, it can be rewritten,

$$\begin{aligned}
\mathcal{L} = & \log \frac{n!}{(n-R)!} + \sum_{i=1}^{N_1} \log(c_1 k) + \sum_{i=N_1+1}^R \log(c_2 k) + (c_1 - 1) \sum_{i=1}^{N_1} \log(t_{i:n}) \\
& - (k+1) \sum_{i=1}^{N_1} \log(1 + t_{i:n}^{c_1}) + (c_2 - 1) \sum_{i=N_1+1}^R \log \left[t_{i:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right] \\
& - (k+1) \sum_{i=N_1+1}^R \log \left[1 + \left(t_{i:n} - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2} \right] \\
& - k(n-R) \log \left[1 + \left(T_2 - \tau_1 + \tau_1^{\frac{c_1}{c_2}} \right)^{c_2} \right]. \tag{9}
\end{aligned}$$

To simplify, replace the parameters c_1, c_2 by α, β respectively and put the following expressions,

$$\begin{aligned}
T_2 &= \max\{t_{r:n}, \tau_2\}, \quad P = t_{i:n}^\alpha \log t_{i:n}, \quad Q = \tau_1^{\frac{\alpha}{\beta}} \log \tau_1, \\
A &= [1 + t_{r:n}^\alpha], \quad B = [1 + t_{i:n}^\alpha], \quad C = \left[t_{r:n} - \tau_1 + \tau_1^{\frac{\alpha}{\beta}} \right], \\
D &= \left[t_{i:n} - \tau_1 + \tau_1^{\frac{\alpha}{\beta}} \right], \quad E = \left[T_2 - \tau_1 + \tau_1^{\frac{\alpha}{\beta}} \right].
\end{aligned}$$

Then, the logarithm of likelihood function is:

$$\begin{aligned}
\mathcal{L} = & \log \frac{n!}{(n-R)!} + \sum_{i=1}^{N_1} \log(\alpha k) + \sum_{i=N_1+1}^R \log(\beta k) + (\alpha - 1) \sum_{i=1}^{N_1} \log(t_{i:n}) \\
& - (k+1) \sum_{i=1}^{N_1} \log B + (\beta - 1) \sum_{i=N_1+1}^R \log D - (k+1) \sum_{i=N_1+1}^R \log[1 + D^\beta]
\end{aligned}$$

$$-k(n-R)\log[1+E^\beta]. \quad (10)$$

We obtain the estimators of α , β and k by differentiating (10) with respect to α , β and k respectively and equating to zero. Thus,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial k} &= \frac{R}{k} - (n-R)\log[1+E^\beta] - \sum_{i=1}^{N_1} \log B - \sum_{i=1}^{N_1} \log[1+D^\beta] = 0, \\ \frac{\partial \mathcal{L}}{\partial \alpha} &= \frac{N_1}{\alpha} - \frac{k(n-R)QE^{\beta-1}}{[1+E^\beta]} + \sum_{i=1}^{N_1} \log t_{i:n} - (1+k) \sum_{i=1}^{N_1} \frac{P}{B} \\ &\quad + \left(\frac{\beta-1}{\beta}\right) \sum_{i=N_1+1}^R \frac{Q}{D} - (k+1) \sum_{i=N_1+1}^R \frac{QD^{\beta-1}}{[1+D^\beta]} = 0, \\ \frac{\partial \mathcal{L}}{\partial \beta} &= \frac{M}{\beta} - \left(\frac{k(n-R)E^\beta \left[\log E - \frac{\alpha Q}{\beta E} \right]}{[1+E^\beta]} \right) + \sum_{i=N_1+1}^R \log D \\ &\quad + \frac{\alpha(1-\beta)}{\beta^2} \sum_{i=N_1+1}^R \frac{Q}{D} - (k+1) \sum_{i=N_1+1}^R \left(\frac{D^\beta [\log D - \frac{\alpha Q}{\beta D}]}{[1+D^\beta]} \right) = 0. \end{aligned} \quad (11)$$

Since the closed form solution to nonlinear equations (11) is very hard to obtain, therefore, we use the MATLAB program to solve the previous nonlinear equations simultaneously to obtain $\hat{\alpha}$, $\hat{\beta}$ and $\hat{k}$. The results are listed at Table 1.

4 Asymptotic Variances and Covariance Matrix

The asymptotic variance and covariance matrix of maximum likelihood estimator are given by the elements of the inverse of Fisher information matrix as follows,

$$I_{ij}(\underline{\theta}) \cong E \left\{ -\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \right\}.$$

Unfortunately, the exact mathematical expressions for the previous expectation are very difficult to obtain. Therefore, Fisher information matrix is given by

$$I_{ij}(\underline{\theta}) \cong \left\{ -\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \right\},$$

which is obtained by dropping the expectation on operation E and replaced

α, β and k with $\hat{\alpha}, \hat{\beta}$ and $\hat{k}$ respectively, Cohen (1965). The asymptotic variance covariance matrix (F) for the maximum likelihood estimates can be written as follows,

$$F = \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial k^2} & -\frac{\partial^2 \ln L}{\partial k \partial \alpha} & -\frac{\partial^2 \ln L}{\partial k \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial k} & -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \beta \partial k} & -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \beta^2} \end{pmatrix} \quad (12)$$

where,

$$\begin{aligned} \frac{\partial^2 E}{\partial \alpha^2} = & -\frac{N_1}{\alpha^2} + \frac{k(n-R)Q^2 E^{2\beta-2}}{(1+E^\beta)^2} - \frac{k(n-R)(\beta-1)Q^2 E^{\beta-2}}{\beta(1+E^\beta)} \\ & - \frac{k(n-R)Q \log \tau_1 E^{\beta-1}}{\beta(1+E^\beta)} - (k+1) \sum_{i=1}^{N_1} \left[\frac{P \log t_i}{B} - \left(\frac{P}{B} \right)^2 \right] \\ & - \left(\frac{k+1}{\beta} \right) \sum_{i=N_1+1}^R \left[\frac{(\beta-1)Q^2 D^{\beta-2}}{1+D^\beta} - \frac{\beta Q^2 D^{2\beta-2}}{(1+D^\beta)^2} + \frac{Q \log \tau_1 D^{\beta-1}}{1+D^\beta} \right] \\ & + \left(\frac{\beta-1}{\beta^2} \right) \sum_{i=N_1+1}^R \left[\frac{Q \log \tau_1}{D} - \left(\frac{Q}{D} \right)^2 \right]. \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\partial^2 E}{\partial \beta^2} = & -\frac{M}{\beta^2} + k(n-R) \left[\log E - \frac{\alpha Q}{\beta E} \right]^2 \left\{ \left(\frac{E^\beta}{1+E^\beta} \right)^2 - \left(\frac{E^\beta}{1+E^\beta} \right) \right\} \\ & - k(n-R) \left[\frac{\alpha^2 Q \log \tau_1}{\beta^3 E} - \frac{\alpha^2 Q^2}{\beta^3 E^2} \right] \left(\frac{E^\beta}{1+E^\beta} \right) - \frac{2\alpha}{\beta^2} \sum_{i=N_1+1}^R \frac{Q}{D} \\ & - (k+1) \sum_{i=N_1+1}^R \left\{ \left[\log D - \frac{\alpha Q}{\beta D} \right]^2 \left\{ \frac{D^\beta}{1+D^\beta} - \left(\frac{D^\beta}{1+D^\beta} \right)^2 \right\} \right\} \\ & - (k+1) \sum_{i=N_1+1}^R \left\{ \left(\frac{D^\beta}{1+D^\beta} \right) \left[\frac{\alpha^2 Q \log \tau_1}{\beta^3 D} - \frac{\alpha^2 Q^2}{\beta^3 D^2} \right] \right\} \\ & + (\beta-1) \sum_{i=N_1+1}^R \left\{ \frac{2\alpha Q}{\beta^3 D} + \frac{\alpha^2 Q \log \tau_1}{\beta^4 D} - \left(\frac{\alpha Q}{\beta^2 D} \right)^2 \right\}. \end{aligned} \quad (14)$$

$$\frac{\partial^2 \mathcal{E}}{\partial k \partial \alpha} = -(n-R)Q \left(\frac{E^{\beta-1}}{1+E^\beta} \right) - \sum_{i=1}^{N_1} \frac{P}{B} - \sum_{i=N_1+1}^R \left\{ \frac{QD^{\beta-1}}{1+D^\beta} \right\}. \quad (15)$$

$$\frac{\partial^2 \mathcal{E}}{\partial k \partial \beta} = -(n-R) \left(\frac{E^\beta}{1+E^\beta} \right) \left[\log E - \frac{\alpha Q}{\beta E} \right] - \sum_{i=N_1+1}^R \left\{ \left(\frac{D^\beta}{1+D^\beta} \right) \left[\log D - \frac{\alpha Q}{\beta D} \right] \right\}. \quad (16)$$

$$\begin{aligned} \frac{\partial^2 \mathcal{E}}{\partial \alpha \partial \beta} = & k(n-R)Q \left\{ \left[\log E - \frac{\alpha Q}{\beta E} \right] \left(\frac{E^{2\beta-1}}{[1+E^\beta]^2} \right) + \alpha \beta^{-2} \log \tau_1 \left(\frac{E^{\beta-1}}{1+E^\beta} \right) \right\} \\ & - k(n-R)Q \left(\frac{E^{\beta-1}}{1+E^\beta} \right) \left[\log E - \frac{\alpha(\beta-1)Q}{\beta^2 E} \right] + \frac{1}{\beta} \sum_{i=N_1+1}^R \frac{Q}{D} \\ & + \left(\frac{\alpha\beta - \alpha}{\beta^3} \right) \sum_{i=N_1+1}^R \left\{ \left(\frac{Q}{D} \right)^2 - \frac{\beta Q}{\alpha D} - \frac{Q \log \tau_1}{D} \right\} \\ & + (k+1) \sum_{i=N_1+1}^R \left\{ Q \left(\frac{D^{2\beta-1}}{[1+D^\beta]^2} \right) \left[\log D - \frac{\alpha Q}{\beta D} \right] \right\} \\ & - (k+1) \sum_{i=N_1+1}^R \left\{ Q \left(\frac{D^{\beta-1}}{1+D^\beta} \right) \left[\log D - \frac{\alpha(\beta-1)Q}{\beta^2 D} \right] \right\} \\ & + (k+1) \sum_{i=N_1+1}^R \left\{ \alpha \beta^{-2} Q \log \tau_1 \left(\frac{D^{\beta-1}}{1+D^\beta} \right) \right\}. \end{aligned} \quad (17)$$

$$\frac{\partial^2 \mathcal{E}}{\partial^2 k} = -\frac{R}{k^2}. \quad (18)$$

Consequently, the maximum likelihood estimators $\hat{\alpha}, \hat{\beta}$ and $\hat{k}$ of α, β and k have an asymptotic variance covariance matrix defined by inverting the Fisher information matrix F and substituting $\hat{\underline{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{k})$ for $\underline{\theta} = (\alpha, \beta, k)$, see Table 2.

5 Confidence Interval of Estimators for Burr-XII distribution under Data Type-II hybrid Censoring

In this section, we construct confidence intervals (CIs) for the unknown parameters α, β and k for larger sample sizes under data type II-hybrid censored. Let the unknown parameters θ and assume that $L_\theta = L_\theta(t_{1:n}, \dots, t_{n:n})$ and $U_\theta = U_\theta(t_{1:n}, \dots, t_{n:n})$ are functions of the sample

data $t_{1:n}, \dots, t_{n:n}$, then a confidence interval for a population parameter θ is given by

$$P[L_\theta \leq \theta \leq U_\theta] = \delta,$$

where L_θ and U_θ are lower and upper confidence limits which enclosed θ with probability δ . The interval $[L_\theta, U_\theta]$ is called a two sided $100\delta\%$ confidence interval for θ . For large sample size, the maximum likelihood estimations, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the two-sided approximate $100\delta\%$ confidence limits for the maximum likelihood estimate $\hat{\theta}$ of a population parameters θ can be constructed, such that,

$$P\left[-z \leq \frac{\hat{\theta} - \theta}{\sigma(\hat{\theta})} \leq z\right] = \delta,$$

where, z is the $\left[\frac{100(1-\delta)}{2}\right]$ standard normal percentile. Therefore, the two sided approximate $100\delta\%$ confidence limits for a population parameter θ can be obtained, such that,

$$P[\hat{\theta} - z\sigma(\hat{\theta}) \leq \theta \leq \hat{\theta} + z\sigma(\hat{\theta})] \cong \delta.$$

Hence, the two sided approximate confidence limits for α, β and k will be respectively as follows,

$$\begin{aligned} L_\alpha &= \hat{\alpha} - z\sigma(\hat{\alpha}) & U_\alpha &= \hat{\alpha} + z\sigma(\hat{\alpha}), \\ L_\beta &= \hat{\beta} - z\sigma(\hat{\beta}) & U_\beta &= \hat{\beta} + z\sigma(\hat{\beta}), \\ L_k &= \hat{k} - z\sigma(\hat{k}) & U_k &= \hat{k} + z\sigma(\hat{k}). \end{aligned} \quad (19)$$

Therefore, we use the MATLAB program to calculate the confidence intervals for the unknown parameter $\theta = (\alpha, \beta, k)$ at 95% and 99%, see Table 3.

5.1 The numerical algorithm

In this section, we describe the algorithm to obtain the Type-II hybrid censored sample, and the estimators $\hat{\alpha}, \hat{\beta}, \hat{k}$. In addition, the absolute relative bias (RABias), mean square error (MSE) and relative error (RE).

Step 1: Given n, r, τ_1, τ_2 and initial values for the parameters α, β and k .

Step 2: We generated a random sample of size n from uniform(0,1) distribution, and obtain the order statistics $U_{1:n}, \dots, U_{n:n}$ such that,

$$\{U_{1:n} < \dots < U_{n:n}\}$$

Step 3: Find N_1 such that,

$$U_{N_1:n} \leq 1 - (1 + \tau_1^\alpha)^{-k} < U_{N_1+1:n}$$

Solve the above inequality to get the failure times of units from unite 1 to unite N_1 . Hence, *the failure time under Burr XII distribution of units* becomes,

$$t_{i:n} = \left[(1 - U_{i:n})^{-\frac{1}{k}} - 1 \right]^{\frac{1}{\alpha}} \quad \text{where } 1 \leq i \leq N_1. \quad (20)$$

Step 4: Next, we generate a random sample of size $m = n - N_1$ from uniform(0,1) distribution, we obtain the order statistics $V_{1:m}, \dots, V_{m:m}$ such that,

$$\{V_{1:m} < \dots < V_{m:m}\}$$

Step 5: Find M and T_2 such that

A. If $r \leq N_1$, then $M = N_2$ and $T_2 = \tau_2$ where N_2 is obtained from the inequality

$$V_{N_2:m} \leq 1 - (1 + (\tau_2 - \tau_1)^\beta)^{-k} < V_{N_2+1:m}$$

B. If $r > N_1$, we have the following two situations:

B1. If $V_{r-N_1:m} \leq 1 - (1 + (\tau_2 - \tau_1)^\beta)^{-k}$ then $M = N_2$ and $T_2 = \tau_2$ where N_2 is obtained from the following inequality,

$$V_{N_2:m} < 1 - (1 + (\tau_2 - \tau_1)^\beta)^{-k} \leq V_{N_2+1:m}$$

B2. If $V_{r-N_1:m} > 1 - (1 + (\tau_2 - \tau_1)^\beta)^{-k}$, then $M = r - N_1$ and

$$T_2 = \tau_1 + \left[(1 - V_{r-N_1:m})^{-\frac{1}{k}} - 1 \right]^{\frac{1}{\beta}},$$

For $1 \leq j \leq M$, we then set

$$t_{N_1+j:n} = \tau_1 + \left[(1 - V_{j:m})^{-\frac{1}{k}} - 1 \right]^{\frac{1}{\beta}}, \quad (21)$$

Step 6: Based on $n, r, N_1, M, \tau_1, T_2$, and order observations,

$$\{t_{1:n}, \dots, t_{N_1:n}, t_{N_1+1:n}, \dots, t_{N_1+M:n}\}$$

we solve the system of nonlinear equations (11) to obtain the estimators of the unknown parameters $\hat{\alpha}$, $\hat{\beta}$ and $\hat{k}$.

Step 7: Repeat steps(2) to (6) H times and arrange all $\hat{\alpha}$, $\hat{\beta}$ and $\hat{k}$ in ascending order to obtain

$$\{\hat{\alpha}^{[1]}, \dots, \hat{\alpha}^{[H]}\}, \{\hat{\beta}^{[1]}, \dots, \hat{\beta}^{[H]}\} \text{ and } \{\hat{k}^{[1]}, \dots, \hat{k}^{[H]}\}. \quad (20)$$

Then, we get the estimators as follows,

$$\hat{\alpha}^* = \frac{1}{H} \sum_{h=1}^H \hat{\alpha}^{[h]}, \quad \hat{\beta}^* = \frac{1}{H} \sum_{h=1}^H \hat{\beta}^{[h]} \quad \text{and} \quad \hat{k}^* = \frac{1}{H} \sum_{h=1}^H \hat{k}^{[h]}. \quad (21)$$

Step 8: Substituting the values of parameters $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{k}^*$ to get the absolute relative bias (RABias), mean square error (MSE) and relative error (RE). Furthermore, the asymptotic variance and covariance matrix and two-sided confidence intervals of the estimators are obtained.

6 Illustrative Examples

In this section, we give illustrative examples by using small and moderately large samples in order to illustrate all the methods of inference described in this paper.

Example 1:

We consider the following data when $n = 15$ with $\tau_1 = 3$, $\alpha = 1.5$, $\beta = 3.1$ and $k = 0.25$. Therefore, the failure times of units as follows,

$0 < t \leq \tau_1$	0.345	0.5008	0.6183	0.8920	2.8862	
$\tau_1 < t \leq \tau_2$	3.6719	4.3621	5.0755	5.1415	5.2476	5.6978
	7.3561	7.9017	10.8733	11.8854		

In this case, we fixed time $\tau_2 = 7$ and $r = 9, 12$, we obtain the estimator of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{k})$ as follows,

$$\begin{aligned} \hat{\alpha} &= 1.283 & \hat{\beta} &= 2.545 & \hat{k} &= 0.271 & \text{when } r &= 9, \\ \hat{\alpha} &= 1.272 & \hat{\beta} &= 2.689 & \hat{k} &= 0.279 & \text{when } r &= 12. \end{aligned}$$

Note that, when $r = 9$, we have $T_2 = \max\{t_{9:15}, \tau_2\} = \max\{5.1415, 7\} = 7$

since $N_1 = 5$ and $M = 6$, similarly, when $r = 12$ we have $T_2 = \max\{t_{12:15}, \tau_2\} = \max\{7.3561, 7\} = 7.3561$ since $N_1 = 5$ and $M = 7$.

Example 2:

We consider the following data when $n = 25$ with $\tau_1 = 3, \alpha = 1, \beta = 2.5$ and $k = 1$. Therefore, the failure times of units as follows,

$0 < t \leq \tau_1$	0.0295	0.0443	0.0634	0.0769	0.1071	0.1763	0.2018	0.2239	0.3157
	0.8907	0.9604	1.0019	1.0905	1.3625	1.5632	1.6155	2.1444	2.4806
$\tau_1 < t \leq \tau_2$	2.6028	3.9285	4.0302	4.2787	4.3029	4.7425			
	7.1931								

In this case, we fixed time $\tau_2 = 7$, we obtain the estimator of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{k})$ as follows,

$$\hat{\alpha} = 0.876 \quad \hat{\beta} = 1.545 \quad \hat{k} = 1.061 \quad \text{when } r = 14,$$

Note that, when $r = 20$, we have $T_2 = \max\{t_{20:25}, \tau_2\} = \max\{3.9285, 7\} = 7$ since $N_1 = 19$ and $M = 5$.

7 Conclusion

In this paper, we have considered a simple step-stress model with two stress levels from Burr-XII distribution when the data are type II hybrid censored. We have derived the MLEs of the unknown parameter $\theta = (\alpha, \beta, k)$. We have also proposed procedure for constructing the estimator $\hat{\theta}$ beside RABies, MSE and RE of it, see table 1. Moreover, the variance and covariance beside CIs for θ at 95% and 99%, see table 2 and 3. Hence, we present the results of a Monte Carlo simulation study carried out in order to compare the performance of the method of inference described in section 5. Since, we chose the values of parameter θ to be (1,2.5,1) and (1.5,3,1.25) respectively, for different (n, r) . The parameter β increases the estimates have smaller MSE and RE, as the sample size increases the RABais and MSEs of the estimated parameters decrease, see table 1. For this reason, the ML estimates provide asymptotically normally distributed and consistent estimators for the parameters. Asymptotic variances of the estimators are

decreasing when the sample size increasing, see table 2. The following precise conclusion have been reached, β is larger than α for each increasing n and r , k increases as r increases when n decreases, in general k decreases when n increases in size. We have observed that case (I) is the best for estimation relative to case (II), see table 1. Variance and covariance decreases when size n increases, the first case (I) is best for variance relative to case (II) which is based on type II hybrid censored data, see table 2.

References

- [1] Abd-Elfattah, A.M., Amal S. Hassan, S. G. Nassr (2008). Estimation in Step-Stress Partially Accelerated Life Tests for the Burr Type XII Distribution using Type I-Censoring. *Statistical methodology*, vol. 5, 502-514
- [2] Bai, D. S., Kim, M. S. and Lee, S. H. (1989). Optimum Simple Step-Stress Accelerated Life Tests with Censoring, *IEEE Trans. Reliab.*, 38, 528-532.
- [3] Balakrishnan, N. and Xie, Q. (2007). Exact Inference for a Simple Step-Stress Model with Type-II Hybrid Censored Data from the Exponential Distribution, *J. Statist. Plan. Inf.* 137, 2543-2563.
- [4] Balakrishnan, N., Xie Q., and Kundu D. (2009), Exact Inference for a Simple Step-Stress Model from the Exponential Distribution under Time Constraint, *Annals of the Institute of Statistical Mathematics*, vol. 61, 251-274.
- [5] Banerjee, A. and Kundu, D. (2006). Inference Based on Type-II Hybrid Censored Data from a Weibull Distribution, *IEEE Trans. Reliab.*, 57, 369-378.
- [6] Childs, A., Chandrasekar, B., Balakrishnan, N. and Kundu, D. (2003). Exact Likelihood Inference Based on Type-I and Type-II Hybrid Censored Samples from the Exponential Distribution, *Ann. Inst. Sttist. Math.*, 55, 319-330.
- [7] Cohen, A. C. (1965). Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples, *Technometrics*, 5, 579-588.
- [8] DeGroot , M. H. and Goel, P. K. (1979). Bayesian Estimation and Optimal Design in Partially Accelerated Life Testing. *Naval Research Logistics Quarterly*, 16(2) 223-235.

- [9] Epstein, B. (1954). Truncated Life Tests in the Exponential Case, *Ann. Math. Statist.*, 25, 555-564.
- [10] Gupta, R. D. and Kundu, D. (1998). Hybrid Censoring Schemes with Exponential Failure Distribution, *Commun. Statist-Theory Methods* 27, 3065-3083.
- [11] Miller, R and Nelson, W. (1983). Optimum Simple Step-Stress Plans for Accelerated Life Testing, *IEEE Trans. Reliab.*, 32, 59-65.
- [12] Nelson, W. (1980). Accelerated Life Testing Step-Stress Model and Data Analysis, *IEEE Trans. Reliab.*, 29, 103-108.
- [13] Nesar Ahmed and A. Islam (1996). Optimal Accelerated Life Test Designs for Burr XII Distributions under Periodic Inspection and Type I-censoring, *Navul Research Logistics*, vol. 43, 1049-1077.
- [14] Watkins, A. J. (2001). Commentary: Inference in Simple Step-Stress Models, *IEEE Trans. Reliab.*, 50, 36-27.

Table 1: The RABais, MSE and RE of the parameter $\theta = (\alpha, \beta, k)$ based on hybrid type II from Burr XII distribution.

(n, r)		Case 1 (1, 2.5, 1)				Case 2 (1.5, 3, 1.25)			
	θ	$\hat{\theta}$	RE	MSE	RAbias	$\hat{\theta}$	RE	MSE	RAbias
(15, 7)	α	1.0336	0.0150	0.0225	0.0336	1.9968	0.5019	0.5668	0.3312
	β	2.3340	0.1804	0.2035	0.0640	2.7601	0.2514	0.5690	0.0799
	k	1.0153	0.2161	0.0467	0.0153	1.1769	0.2034	0.0646	0.0585
(15, 9)	α	1.0894	0.2036	0.0415	0.0893	1.5013	0.2446	0.1346	0.0008
	β	2.1549	0.2578	0.4153	0.1381	2.8872	0.1516	0.2067	0.0376
	k	1.0651	0.4013	0.1611	0.0651	1.1617	0.2495	0.0973	0.0707
(15, 14)	α	1.0032	0.2177	0.0474	0.0032	1.6274	0.2566	0.1482	0.0849
	β	2.1070	0.2802	0.4906	0.1572	2.7095	0.2529	0.5758	0.0968
	k	1.1105	0.2637	0.0696	0.1105	1.0158	0.2442	0.0932	0.1874
(25, 10)	α	0.9899	0.1014	0.0103	0.0102	1.6361	0.1916	0.0826	0.0907
	β	2.4362	0.1401	0.1227	0.0255	2.3756	0.2504	0.5644	0.2081
	k	0.9998	0.2422	0.0587	0.0002	1.2945	0.1535	0.0368	0.0356
(25, 14)	α	1.0604	0.2154	0.0464	0.0604	1.4614	0.1335	0.0401	0.0257
	β	2.3048	0.2546	0.4051	0.0781	2.4684	0.2844	0.7281	0.1772
	k	1.0027	0.214	0.0458	0.0027	1.3274	0.2693	0.1133	0.0619
(25, 18)	α	1.0445	0.1593	0.0254	0.0445	1.4902	0.2036	0.0933	0.0066
	β	2.0805	0.2305	0.3321	0.1678	2.7284	0.1606	0.2322	0.0906
	k	1.0308	0.1585	0.0251	0.0308	1.2288	0.1094	0.0187	0.0169
(35, 14)	α	1.0760	0.1499	0.0225	0.0760	1.6313	0.2078	0.0972	0.0875
	β	2.2839	0.2122	0.2813	0.0864	2.6634	0.2149	0.4158	0.1122
	k	1.0007	0.1649	0.0272	0.0007	1.1757	0.2221	0.0771	0.0594
(35, 24)	α	1.0875	0.1925	0.0371	0.0875	1.3433	0.1608	0.0582	0.1045
	β	2.0672	0.2065	0.2666	0.1731	2.1588	0.2934	0.7749	0.2804
	k	1.0516	0.2497	0.0623	0.0516	1.2659	0.1164	0.0212	0.0127
(35, 31)	α	1.0929	0.2319	0.0538	0.0929	1.5665	0.1672	0.0629	0.0444
	β	2.3910	0.1875	0.2198	0.0436	2.5461	0.2469	0.5484	0.1513
	k	1.0469	0.2223	0.0494	0.0468	1.2579	0.1263	0.0249	0.0063
(40, 16)	α	1.1058	0.2151	0.0463	0.1058	1.5225	0.0904	0.0184	0.0149
	β	2.2514	0.1816	0.206	0.0994	2.1616	0.2991	0.8053	0.2795
	k	0.9244	0.1758	0.0309	0.0756	1.3433	0.1462	0.0334	0.0747
(40, 28)	α	1.0031	0.1264	0.0159	0.0031	1.5847	0.1441	0.0468	0.0565
	β	2.2545	0.1984	0.2459	0.0982	2.5058	0.2391	0.5146	0.1648
	k	1.0260	0.1865	0.0348	0.0259	1.3303	0.1947	0.0592	0.0642
(40, 36)	α	1.0898	0.2285	0.0522	0.0898	1.4712	0.1392	0.0436	0.0192
	β	2.3996	0.1951	0.2378	0.0402	2.3592	0.2574	0.5962	0.2136
	k	1.0191	0.2386	0.0569	0.0191	1.1914	0.1953	0.0596	0.0469

Table 2: Asymptotic variance and covariance of estimates based on type II hybrid censored data.

(n, r)		Case 1 (1, 2.5, 1)			Case 2 (1.5, 3, 1.25)		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$
(15, 7)	$\hat{\alpha}$	0.0937	-0.0445	-0.1695	0.1145	-0.0510	-0.1900
	$\hat{\beta}$	-0.0445	0.1743	0.1399	-0.0510	0.2265	0.1705
	$\hat{k}$	-0.1695	0.1399	1.3328	-0.1900	0.1705	2.2743
(15, 9)	$\hat{\alpha}$	0.1170	-0.0243	-0.1809	0.1182	-0.0348	-0.2157
	$\hat{\beta}$	-0.0243	0.0756	0.0601	-0.0348	0.1316	0.1079
	$\hat{k}$	-0.1809	0.0601	0.8258	-0.2157	0.1079	1.9284
(15, 14)	$\hat{\alpha}$	0.1227	-0.0180	-0.1799	0.0952	-0.0428	-0.1908
	$\hat{\beta}$	-0.0180	0.0653	0.0457	-0.0428	0.1647	0.1417
	$\hat{k}$	-0.1799	0.0457	0.8339	-0.1908	0.1417	1.6762
(25, 10)	$\hat{\alpha}$	0.0621	-0.0118	-0.1179	0.0796	-0.0181	-0.0987
	$\hat{\beta}$	-0.0118	0.0396	0.0334	-0.0181	0.0835	0.0529
	$\hat{k}$	-0.1179	0.0334	0.6711	-0.0987	0.0529	0.9131
(25, 14)	$\hat{\alpha}$	0.0600	-0.0127	-0.1018	0.0843	-0.0174	-0.1136
	$\hat{\beta}$	-0.0127	0.0448	0.0346	-0.0174	0.0658	0.0463
	$\hat{k}$	-0.1018	0.0346	0.5548	-0.1136	0.0463	1.1514
(25, 18)	$\hat{\alpha}$	0.0598	-0.0114	-0.0856	0.0736	-0.0167	-0.1151
	$\hat{\beta}$	-0.0114	0.0419	0.0289	-0.0167	0.0734	0.0516
	$\hat{k}$	-0.0856	0.0289	0.4645	-0.1151	0.0516	1.1650
(35, 14)	$\hat{\alpha}$	0.0406	-0.0090	-0.0669	0.0495	-0.0169	-0.0830
	$\hat{\beta}$	-0.0090	0.0328	0.0246	-0.0169	0.0634	0.0506
	$\hat{k}$	-0.0669	0.0246	0.4042	-0.0830	0.0506	0.9267
(35, 24)	$\hat{\alpha}$	0.0457	-0.0095	-0.0671	0.0555	-0.0094	-0.0640
	$\hat{\beta}$	-0.0095	0.0320	0.0235	-0.0094	0.0418	0.0249
	$\hat{k}$	-0.0671	0.0235	0.3496	-0.0640	0.0249	0.4604
(35, 31)	$\hat{\alpha}$	0.0438	-0.009	-0.072	0.0552	-0.0139	-0.0862
	$\hat{\beta}$	-0.009	0.032	0.0242	-0.0139	0.0554	0.0418
	$\hat{k}$	-0.072	0.0242	0.4436	-0.0862	0.0418	0.9163
(40, 16)	$\hat{\alpha}$	0.0326	-0.0091	-0.0585	0.0518	-0.0097	-0.0566
	$\hat{\beta}$	-0.0091	0.0321	0.0257	-0.0097	0.0450	0.0267
	$\hat{k}$	-0.0585	0.0257	0.3357	-0.0566	0.0267	0.5769
(40, 28)	$\hat{\alpha}$	0.0384	-0.0065	-0.0612	0.0524	-0.0110	-0.0722
	$\hat{\beta}$	-0.0065	0.0247	0.0174	-0.0110	0.0486	0.0328
	$\hat{k}$	-0.0612	0.0174	0.3271	-0.0722	0.0328	0.7661
(40, 36)	$\hat{\alpha}$	0.0376	-0.0085	-0.0649	0.0445	-0.0121	-0.0640
	$\hat{\beta}$	-0.0085	0.0289	0.0229	-0.0121	0.0446	0.0325
	$\hat{k}$	-0.0649	0.0229	0.3877	-0.0640	0.0325	0.5546

Table 3: Confidence bounds of the estimates at confidence level based on type II hybrid censored data 95% and 99%.

(n,r)		Case 1 (1, 2.5, 1)			Case 2 (1.5, 3, 1.25)		
θ		(L,U)bound1 (95%)	(L, U)bound 2 (99%)	SD	(L,U)bound1 (95%)	(L,U)bound 2 (99%)	SD
(15,7)	α	(0.7469,1.3204)	(0.6561,1.4111)	0.1463	(0.8879,3.1056)	(0.5372,3.4563)	0.5657
	β	(1.5134,3.1666)	(1.2519,3.4280)	0.4217	(1.3584,4.1619)	(0.9149,4.6053)	0.7152
	k	(0.5928,1.4378)	(0.4591,1.5715)	0.2156	(0.6997,1.6541)	(0.5487,1.8051)	0.2435
(15,9)	α	(0.7307,1.4480)	(0.6172,1.5615)	0.1830	(0.7822,2.2204)	(0.5547,2.4478)	0.3669
	β	(1.0883,3.2215)	(0.7509,3.5589)	0.5442	(2.0239,3.7505)	(1.7509,4.0236)	0.4405
	k	(1.8413,0.2888)	(0.0433,2.0868)	0.3960	(0.5753,1.7480)	(0.3898,1.9335)	0.2992
(15,14)	α	(0.5766,1.4298)	(0.4417,1.5647)	0.2177	(0.9154,2.3393)	(0.6902,2.5645)	0.3632
	β	(0.9706,3.2434)	(0.6111,3.6029)	0.5798	(1.3356,4.0834)	(0.9009,4.5180)	0.7010
	k	(0.6411,1.5799)	(0.4926,1.7283)	0.2395	(0.6319,1.3996)	(0.5106,1.5210)	0.1958
(25,10)	α	(0.7920,1.1877)	(0.7294,1.2503)	0.1010	(1.1401,2.1321)	(0.9833,2.2889)	0.2531
	β	(1.7611,3.1112)	(1.5475,3.3248)	0.3444	(1.5567, 3.1946)	(1.2976,3.4537)	0.4178
	k	(1.4746,0.5250)	(0.3748,1.6247)	0.2422	(0.9286,1.6603)	(0.8129,1.7759)	0.1866
(25,14)	α	(1.4657,0.6550)	(0.5268,1.5939)	0.2068	(1.0763,1.8465)	(0.9545,1.9683)	0.1965
	β	(3.4921,1.1175)	(0.7419,3.8677)	0.6058	(1.1601,3.7768)	(0.7462,4.1906)	0.6675
	k	(0.5834,1.4221)	(0.4507,1.5548)	0.2140	(0.6853,1.9694)	(0.4822,2.1725)	0.3276
(25,18)	α	(1.3443,0.7447)	(0.6499,1.4391)	0.1530	(0.8919,2.0884)	(0.7026,2.2777)	0.3052
	β	(1.3060,2.8551)	(1.0610,3.1001)	0.3952	(1.9483,3.5084)	(1.7015,3.7552)	0.3979
	k	(1.3355,0.7261)	(0.6297,1.4319)	0.1555	(0.9641,1.4935)	(0.8803,1.5773)	0.1351
(35,14)	α	(0.8228,1.3292)	(0.7427,1.4093)	0.1292	(1.0771,2.1855)	(0.9018,2.3608)	0.2827
	β	(1.3345,3.2333)	(1.0342,3.5336)	0.4844	(1.5854,3.7414)	(1.2444,4.0824)	0.5500
	k	(0.6774,1.3240)	(0.5751,1.4263)	0.1650	(0.6513,1.7002)	(0.4854,1.8661)	0.2676
(35,24)	α	(0.7514,1.4236)	(0.6451,1.5299)	0.1715	(0.9838,1.7028)	(0.8701,1.8165)	0.1834
	β	(1.5154,2.6189)	(1.3409,2.7934)	0.2815	(1.6509,2.6666)	(1.4902,2.8273)	0.2591
	k	(0.5728,1.5304)	(0.4213,1.6819)	0.2443	(0.9824,1.5494)	(0.8927,1.6390)	0.1446
(35,31)	α	(0.6763,1.5095)	(0.5446,1.6413)	0.2126	(1.0926,2.0405)	(2.1904,0.9427)	0.2418
	β	(1.4972,3.2847)	(1.2145,3.5675)	0.4560	(1.3992,3.6929)	(1.0365,4.0557)	0.5851
	k	(0.6209,1.4728)	(0.4862,1.6075)	0.2173	(0.9488,1.5671)	(0.8510,1.6648)	0.1577
(40,16)	α	(0.7389,1.4728)	(0.6228,1.5889)	0.1872	(1.2604,1.7846)	(1.1775,1.8675)	0.1337
	β	(1.50702,9958)	(1.2715,3.2312)	0.3798	(1.5344,2.7888)	(1.3360,2.9872)	0.3200
	k	(0.6133,1.2355)	(0.5149,1.3340)	0.1587	(1.0353,1.6514)	(0.9378,1.7488)	0.1572
(40,28)	α	(0.7554,1.2508)	(0.6770,1.3291)	0.1264	(1.1948,1.9746)	(1.0714,2.0980)	0.1989
	β	(1.4101,3.0989)	(1.1429,3.3661)	0.4308	(1.4868,3.5248)	(1.1644,3.8471)	0.5199
	k	(1.3879,0.6641)	(0.5496,1.5024)	0.1847	(0.8800,1.7806)	(0.7375,1.9230)	0.2298
(40,36)	α	(0.6778,1.5017)	(0.5475,1.6320)	0.2102	(1.0659,1.8765)	(0.9377,2.0047)	0.2068
	β	(1.4643,3.3349)	(1.1684,3.6308)	0.4772	(1.5148,3.2036)	(1.2477,3.4707)	0.4308
	k	(0.5529,1.4853)	(0.4054,1.6328)	0.2379	(0.7268,1.6560)	(0.5799,1.8029)	0.2370

نموذج التعرض التراكمي بناء على بيانات مراقبة من النوع الثانى المختلطة (الهجين) لتوزيع بيير من النوع الثانى عشر

فى كثير من الاحيان نجد من الصعب ان نقدر صلاحية المنتج تحت الظروف العادية ولهذا السبب نلجأ الى استخدام اختبارات الحياة المعجلة (Accelerated Life-Testing) حتى تجبر هذه المنتجات على ان تفشل بصورة اسرع اثناء التشغيل.

وبناء على ذلك اصبحت هذه التجارب من التقنيات الهامة فى تحديد وتحليل صلاحية هذه المنتجات بشكل دائم

وهناك ما يسمى باختبار الضغط المرحلى (Step-Stress Test) وهو نوع من انواع اختبارات الحياة المعجلة والتي يتيح للقائم بالتجربة اختيار عامل او اكثر من عوامل الضغط المستخدمة فى هذه التجارب ومن امثلة هذه العوامل الرطوبة والحرارة والاهتزاز والجهد والضغط او اى عوامل اخرى تؤثر على دورة حياة المنتج.

تتناول هذه المقالة مشكلة تقدير معالم توزيع بيير الثانى عشر باستخدام نموذج الضغط المرحلى البسيط للبيانات المراقبة من النوع الثانى الهجين.

ونستخدم فى تقدير هذه المعالم طريقة الامكان الاعظم فى حالة نموذج التعرض التراكمي اضافة الى ذلك فاننا حصلنا على مصفوفة التباين وايضا مقدرات وفترات الثقة للمعالم المجهولة.

وعن طريق استخدام طريقة مونتو كارلو (Monte Carlo) للمحاكاة نصل الى النتائج العددية لهذا البحث ونستشهد ببعض الامثلة التي توضح ذلك.

Parameters Estimation of the Two-Parameter Gamma Distribution Using Different Methods

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Abstract

In this paper, the estimation of the unknown parameters of the two parameter gamma distribution using quasi-likelihood, quasi-Bayesian, principal of maximum entropy (POME), and Bayesian estimation are considered. A simulation study is obtained to compare between them.

Keywords: Quasi Likelihood; Quasi Bayesian Likelihood; Principal of Maximum Entropy (POME); Gamma distribution.

Introduction:

To define the likelihood function, we have to specify the distribution form of the observations, but in many situations the exact distribution of the observations is unknown. To define a quasi likelihood function, according to Wedderburn (1974); we need only to specify a relation between the mean and variance of the observations, and the quasi-likelihood function can then be used for estimation.

Ashour et al (1992) introduce a theoretical framework of Bayesian likelihood when the form of the distribution of the data is unknown. While the quasi likelihood plays the role of the likelihood function to define a quasi posterior likelihood function on the parameter space.

Also the Principle of Maximum Entropy (POME), according to Jaynes (1968) this technique can be used to obtain the minimally biased probability distribution. Inversely, if it is desired to fit a particular probability distribution to a sample of data, then the POME can uniquely specify the constraints or the information needed to derive that distribution. The distribution parameters are then related to these constraints. [Singh and Guo (1995)].

We obtain the maximum quasi-likelihood estimators of the unknown parameters of the two parameter gamma distribution. For comparison purpose, we consider the MLE obtained by Choi and Wette (1969), and the POME estimators. The quasi-Bayesian estimators of the unknown parameters of the two parameter gamma distribution are considered. It is assumed that the scale and the shape parameters have Gamma priors, and they are independently distributed. For comparison purpose we use Bayes estimators which performed by Pradhan and Kundu (2011).

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We investigate the performance of the proposed estimators through a simulation study.

The rest of the paper is organized as follows; Section (1), provide quasi-likelihood estimation of the Gamma distribution. Section (2), quasi-Bayesian estimation of the gamma distribution. Section (3) the POME estimation of the Gamma distribution. Section (4) discuss the comparison through a simulation study.

1. Quasi-Likelihood Estimation of the Gamma Distribution

According to Wedderburn (1974), suppose that we have independent observations $x_i, (i = 1, \dots, n)$, with expectations μ_i and variances $Var(z_i) \propto V(\mu_i)$ where $V(\cdot)$ is some known function. Suppose that, for each observation μ_i is some known function of a set of parameters $\beta_0, \dots, \beta_k$. Then for each observation the quasi likelihood function $Q(x_i, \mu_i) \ i = 1, \dots, n$ is defined by the relation

$$\frac{\partial(Q; x_i, \mu_i)}{\partial \mu_i} = \frac{x_i - \mu_i}{V(\mu_i)}$$

$$Q(x_i, \mu_i) = \int \frac{x_i - \mu_i}{V(\mu_i)} d\mu_i + \text{function of } x_i \text{ only} \quad (1.1)$$

We derive the quasi-likelihood estimators for the unknown parameters of the Gamma distribution. Consider the pdf of the Gamma distribution with shape parameter α and scale parameter λ is given by

$$f(x) = \frac{\lambda^\alpha}{\Gamma \alpha} x^{\alpha-1} \exp(-\lambda x), \quad 0 < x < \infty, \alpha, \lambda > 0 \quad (1.2)$$

the expectations is given by

$$E(x) = \frac{\alpha}{\lambda} = \mu, \quad Var(x) = \frac{1}{\alpha} \mu^2 = \frac{1}{\alpha} V(\mu)$$

where $V(\mu) = \mu^2$, thus for a sample of n observation, the quasi likelihood function is given by (1.1)

$$Q(x; \mu) = - \frac{\sum_{i=1}^n x_i}{\mu} - n \log \mu$$

thus, substituting for μ ,

$$Q(x; \mu) = - \frac{\sum_{i=1}^n x_i \lambda}{\alpha} - n \log \frac{\alpha}{\lambda}$$

The quasi-likelihood estimators of a and λ are derived as: $\frac{\partial Q}{\partial \alpha} = \frac{\partial Q}{\partial \mu} \frac{\partial \mu}{\partial \alpha}$ and $\frac{\partial Q}{\partial \lambda} = \frac{\partial Q}{\partial \mu} \frac{\partial \mu}{\partial \lambda}$ respectively, thus,

$$\frac{\partial Q}{\partial \alpha} = \frac{\sum_{i=1}^n x_i \lambda}{\alpha^2} - \frac{n}{\alpha}$$

$$\frac{\partial Q}{\partial \lambda} = -\frac{\sum_{i=1}^n x_i}{\alpha} + \frac{n}{\lambda}$$

Equating the two derivatives to zero, we can obtain the quasi-likelihood estimator of a, λ as

$$\frac{\alpha}{\lambda} = \frac{\sum_{i=1}^n x_i}{n} \quad (1.3)$$

2. Quasi-Bayesian Estimation of the Gamma Distribution

According to Ashour et al (1992) to construct a posterior distribution, the likelihood function could be replaced with the natural exponential of the quasi likelihood function, the posterior distribution denoted by, $f^{**}(\theta, \phi|x)$ can be constructed as,

$$f^{**}(\theta, \phi|x) \propto \prod_{i=1}^n \exp[Q(x; \mu, \phi)] g(\theta, \phi). \quad (2.1)$$

where $Q(x; \mu, \phi)$, is defined by (1.1), $\mu = \mu(\theta)$, $\theta = (\theta_1, \theta_2, \dots, \theta_n)$, $\theta_i \in \Theta$, Θ is the parameter space, $i = 1, 2, \dots, n$ and $\phi > 0$. A set of Bayes estimators for the unknown parameters can be obtained with respect to some loss functions, using the marginal densities of (2.1) by integrating out one or more of the unknown parameters.

To obtain the quasi-Bayesian estimators of the unknown parameters of the gamma distribution given by (1.2), it is assumed that, the scale parameter λ follows Gamma ($a; b$) and the shape parameters α follow Gamma ($c; d$) and they are independently distributed. Using the quasi-likelihood function after substituting for μ , and then exponentiated to be obtained as

$$\exp\{Q(x; \mu)\} = \exp\left\{-\sum_{i=1}^n x_i \frac{\lambda}{\alpha} - n \log \frac{\alpha}{\lambda}\right\}$$

The posterior density function of $(\alpha; \lambda)$ given the data and the previous priors is obtained by (2.1) as

$$f(\alpha, \lambda | x) = \frac{\exp \left\{ -\sum_{i=1}^n x_i \frac{\lambda}{\alpha} - n \log \frac{\alpha}{\lambda} \right\} \alpha^{c-1} e^{-d\alpha} \lambda^{a-1} e^{-b\lambda}}{\int_0^\infty \int_0^\infty \exp \left\{ -\sum_{i=1}^n x_i \frac{\lambda}{\alpha} - n \log \frac{\alpha}{\lambda} \right\} \alpha^{c-1} e^{-d\alpha} \lambda^{a-1} e^{-b\lambda} d\lambda d\alpha}$$

The marginal posterior distributions of α can be obtained as,

$$h(\alpha | x) = \frac{1}{A_1} \int_0^\infty \exp \left\{ -\sum_{i=1}^n x_i \frac{\lambda}{\alpha} \right\} \alpha^{-(n-c)-1} e^{-d\alpha} \lambda^{n+a-1} e^{-b\lambda} d\lambda$$

where $1/A_1$ is the normalized constant,

$$A_1 = \int_0^\infty \exp \{ -d\alpha \} \alpha^{-n+c-1} \frac{\Gamma(n+a)}{\left(\frac{\sum_{i=1}^n x_i}{\alpha} + b \right)^{n+a}} d\alpha$$

The marginal posterior mean could be expressed as a quasi-Bayesian estimates of the shape parameter α where,

$$E(\alpha | x) = \frac{\int_0^\infty \exp \{ -d\alpha \} \alpha^{-n+c} \frac{\Gamma(n+a)}{\left(\frac{\sum_{i=1}^n x_i}{\alpha} + b \right)^{n+a}} d\alpha}{\int_0^\infty \exp \{ -d\alpha \} \alpha^{-n+c-1} \frac{\Gamma(n+a)}{\left(\frac{\sum_{i=1}^n x_i}{\alpha} + b \right)^{n+a}} d\alpha}, \quad (2.2)$$

the corresponding Bayes risk is

$$Var(\alpha | x) = \int_0^\infty [\alpha - E(\alpha | x)]^2 h(\alpha | x) d\alpha.$$

The marginal posterior distribution of λ can be obtained as, $h(\lambda | x)$.

$$h(\lambda | x) = \frac{1}{A_2} \int_0^\infty \exp \left\{ -\frac{\sum_{i=1}^n x_i \lambda}{\alpha} \right\} \alpha^{-(n-c)-1} e^{-d\alpha} \lambda^{n+a-1} e^{-b\lambda} d\alpha,$$

$$A_2 = \int_0^\infty \int_0^\infty \exp \left\{ -\frac{\sum_{i=1}^n x_i \lambda}{\alpha} \right\} \alpha^{-(n-c)-1} e^{-d\alpha} \lambda^{n+a-1} e^{-b\lambda} d\alpha d\lambda$$

The marginal posterior mean is the quasi-Bayesian estimate of the scale parameter λ denoted as

$$E(\lambda|x) = \frac{1}{A_2} \int_0^\infty \int_0^\infty \exp\left\{-\frac{\sum_{i=1}^n x_i \lambda}{\alpha}\right\} \alpha^{-(n-c)-1} e^{-d\alpha} \lambda^{n+a} e^{-b\lambda} d\alpha d\lambda, \quad (2.3)$$

the corresponding Bayes risk is

$$\begin{aligned} & Var(\lambda|x) \\ &= \frac{1}{A_2} \int_0^\infty \int_0^\infty (\lambda - E(\lambda|x))^2 \exp\left\{-\frac{\sum_{i=1}^n x_i \lambda}{\alpha}\right\} \alpha^{-(n-c)-1} e^{-d\alpha} \lambda^{n+a-1} e^{-b\lambda} d\alpha d\lambda. \end{aligned}$$

A numerical technique is needed to evaluate the above estimates.

3. Principle of Maximum Entropy Estimation of the Gamma Distribution

Jaynes (1957) formulated the principle of maximum entropy (POME), expressed by Shannon (1948) as

$$H(f) = - \int_{-\infty}^{\infty} f(x; \theta) \ln f(x; \theta) dx \quad \text{with} \quad \int_{-\infty}^{\infty} f(x; \theta) dx = 1 \quad (3.1)$$

where $H(f)$ is the entropy of $f(x; \theta)$, and can be thought as the mean value of $-\ln f(x; \theta)$. Using the (POME) method we can estimate the unknown parameters of the gamma distribution.

3.1 Construction of the entropy function

Using (1.2) in (3.1), thus the appropriate constraints can be obtained. The pdf of the Gamma distribution corresponding to the constraints could be derived as;

$$f(x; \theta) = \exp[-a_0 - a_1 \ln x - a_2 x] \quad (3.2)$$

where a_0, a_1, a_2 , are Lagrange multipliers, which yields; $a_0 = \ln \frac{\Gamma(1-a_1)}{a_2^{1-a_1}}$, then substituting for a_0 in (3.2) and compare it with (1.2), we can obtain $1 - a_1 = \alpha$ and $a_2 = \lambda$, so the entropy function can be defined as

$$H(f) = -(1 - a_1) \ln a_2 + \ln \Gamma(1 - a_1) + a_1 E(\ln x) + a_2 E(x).$$

3.2 Relationships between Distribution Parameters and Constraints

According to Singh and Chowdhury (1985), the relationships between the distribution parameters and constraints are obtained by

$$E(x) = \frac{\alpha}{\lambda}, \quad E(\ln x) = -\ln \lambda + \psi(\alpha).$$

In order to get a unique solution, additional equations are needed which can be obtained by, differentiating the zeroth Lagrange multiplier with respect to the Lagrange multipliers. using (3.2) the zeroth Lagrange multiplier can be written as,

$$a_0 = \ln \int_0^\infty \exp[-a_1 \ln x - a_2 x] dx,$$

$$\frac{\partial a_0}{\partial a_1} = -E(\ln x), \quad \frac{\partial^2 a_0}{\partial a_1^2} = \text{Var}(\ln x) \quad \text{and} \quad \frac{\partial a_0}{\partial a_2} = -E(x), \quad \frac{\partial^2 a_0}{\partial a_2^2} = \text{Var}(x),$$

Also and the zeroth Lagrange multiplier can be written as $a_0 = \ln \frac{\Gamma(1-a_1)}{a_2^{1-a_1}}$

$$\frac{\partial a_0}{\partial a_1} = \psi(\alpha) + \ln a_2, \quad \frac{\partial^2 a_0}{\partial a_1^2} = \psi'(\alpha) \quad \text{and} \quad \frac{\partial a_0}{\partial a_2} = -\frac{(1-a_1)}{a_2}, \quad \frac{\partial^2 a_0}{\partial a_2^2} = \frac{(1-a_1)}{a_2^2},$$

Therefore equating the previous equations with the corresponding one, and then substituting $(1-a_1) = \alpha$, and $a_2 = \lambda$ to these four equations, one gets,

$$\hat{\alpha} = \frac{1}{\text{Var}(\ln x)}, \quad (3.3)$$

$$\hat{\lambda} = \frac{[\text{Var}(\ln x)]^{-1}}{E(x)}. \quad (3.4)$$

4. Simulation study

We investigate the performance of the proposed estimators using quasi-likelihood estimation and quasi-Bayesian estimation methods through a simulation study using (1.3), and (2.2), (2.3) for different sample sizes. For comparison purpose, we compute maximum likelihood estimators given by Choi and Wette (1969), and the (POML) estimators using (3.3), (3.4), and Bayes estimators which performed by Pradhan and Kundu (2011). We choose the prior parameter values, to be $a = b = 5$, $c = 2.25$ and $d = 1.5$. In all these cases, we generate observation from a gamma distribution with $\alpha = 1.5$ and $\lambda = 1$. In all cases we report average estimates obtained along with mean squared error in parentheses, all the results are based on 1000 replications.

Table (4.1): The average values of the (MLE), Quasi-Likelihood (Q-L) and the (POME) estimates. In each cell, the first and second entry corresponds to α and λ respectively.

n	MLE	MSE	Q-L	MSE	POME	MSE
10	2.03	1.585	1.474	0.035	1.489	1.068
	1.445	1.048	1.016	0.033	1.059	0.657
15	1.815	0.607	1.482	0.024	1.332	0.928
	1.266	0.397	1.011	0.016	0.928	0.267
20	1.717	0.362	1.485	0.018	1.258	0.357
	1.185	0.229	1.01	0.011	0.867	0.182
25	1.663	0.254	1.495	0.015	1.217	0.297
	1.143	0.158	1.017	0.008	0.836	0.143
30	1.626	0.185	1.496	0.013	1.184	0.266
	1.114	0.125	1.014	0.007	0.811	0.132
50	1.572	0.096	1.492	0.007	1.137	0.224
	1.067	0.063	1.009	0.004	0.772	0.104

Table (4.2): The average values of the quasi-Bayes estimators and the Bayes (Lindley). In each cell, the first and second entry corresponds to α and λ respectively.

n	Quasi-Bayes	MSE	Bayes (Lindley)	MSE
10	1.532	0.056	1.218	0.119
	0.99	0.005	0.806	0.045
15	1.523	0.04	1.316	0.092
	0.993	0.004	0.854	0.037
25	1.51	0.026	1.447	0.059
	0.997	0.002	0.957	0.042
50	1.505	0.013	1.520	0.031
	0.999	0.001	1.015	0.017

Conclusion:

The simulation results suggest that;

- 1- The quasi-Bayes procedure results could be obtained if we know the mean and the variance of the distribution, so it could be used to estimate the unknown parameters of the two parameter gamma distribution.
- 2- The quasi-likelihood estimators of the unknown parameters of the two parameter gamma distribution perform better than the MLEs and the principal of maximum entropy estimators assuming our case study.

- 3- The quasi-Bayes estimation works better than the maximum likelihood and the principal of maximum entropy estimation.
- 4- The quasi-Bayes estimators of the two parameters of the Gamma distribution works very well as Bayes estimators using Lindley (1980) approximation which performed by Pradhan and Kundu (2011).
- 5- It is observed that as the sample size increases in all the cases the mean squared errors decrease; it verifies the consistency property of all the estimators for the shape and scale parameters.

References

- [1] Ashour, S.K., Abdelhafez, M and Abdelaziz, S. (1992). Bayesian estimation using quasi distributions. Annual Conference, ISSR, Cairo University. 27.
- [2] Choi, S. C, Wette, R . (1969). Maximum likelihood estimation of the parameters of the Gamma distribution and their bias. *Technometrics*.11, 683-690.
- [3] Jaynes, E.T. (1957) Information theory and statistical mechanics. I. *Physical Review*. 106, 620-630.
- [4] Pradhan, B. and Kundu, D. (2011). Bayes estimation and prediction of the two-parameter Gamma distribution. *Journal of Statistical Computation and Simulation*. 81, 1187-1198.
- [5] Singh, V. P., Guo. H. (1995). Parameter estimation for 3-parameter generalized Pareto distribution by the principle of maximum entropy. *Hydrological Sciences*.40, 165-181.
- [6] Singh, V. P, Chowdhury, P. K. (1985). On fitting Gamma distribution to synthetic runoff hydrograph. *Nordic Hydrology*.16, 177-192.
- [7] Wedderburn, R. W. M. (1974). Quasi-likelihood functions, generalized linear models and the Gauss-Newton method. *Biometrika*. 61, 439-447.

Inference for Partially Step Stress Modified Weibull Model Based on Censoring Schemes

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Abstract:

This paper is concerned with estimating the parameters of the modified Weibull distribution for Type I censored data under Step – Stress Partially Accelerated Life Testing (SS – PALT). A simulation procedure was developed to study the properties of the estimators.

Keywords:

Modified Weibull distribution, Type I censoring sample, step – stress partially accelerated life testing,

1. Introduction

It's becoming highly important in the continually developing manufacturing design world and industry to learn more about the lifetime of products and materials. To obtain information about the lifetime of materials, a sample of such materials is selected and then subjected to more severe operating conditions allowing it to fail more rapidly, hence lifetime information would be gained. Such operating conditions are called stresses. Stresses can be temperature, pressure, voltage, cycling rate, humidity, etc. Testing the lifetime of materials under such scheme will be cost and time effective as the tested material will take less time to fail saving both money and effort during the process, a process that is called accelerated life testing (ALT). Other means of life testing is Partially Accelerated Life Testing (PALT) which allows test units to run firstly under normal conditions, and the survived units are then subjected to accelerated ones. One main type of PALT is Step – Stress PALT. Goel (1971) introduced the concept of Step – Stress PALT in estimating the acceleration factor for exponentially and uniformly distributed data. Maximum likelihood and Bayesian techniques were used. Then in (1979) Goel and DeGroot used the Bayesian approach to estimate the parameters of the exponential distribution in Step – Stress PALT. Nelson (1980) developed statistical models for ALT from Step – Stress test using maximum likelihood approach in parameter estimation. The models were applied to the Weibull and Inverse Power Law distributions but can also be applied to other ALT models. He explained the models

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he developed using Step – Stress data on time to breakdown of an electrical insulation. Attia *et. al.* (1996) estimated the parameters of the Weibull distribution for type I censored data in Step – Stress PALT using maximum likelihood estimation. Ismail (2004) used maximum likelihood and Bayesian techniques in estimating the parameters of the Pareto distribution in Step – Stress and Constant – Stress PALT for type I and type II censoring. The parameters of the Burr type XII distribution under type I and type II censoring were estimated using maximum likelihood and Bayesian methods by Said (2008) in Step – Stress PALT.

The objective of this paper is to estimate the unknown parameters of the Modified Weibull distribution (α, ρ, γ) along with the acceleration factor β using maximum likelihood estimation. Estimation is done for type I censoring under step – stress partially accelerated life testing. The mean squared error (MSE) and the Absolute Relative Bias (RABias) for the three parameters and the acceleration factor are also obtained. A numerical study to examine the behavior of the parameters is conducted, and for that Mathematica 8 was used.

The rest of the paper is organized as follows. Section 2 introduces the Modified Weibull distribution and the Step – Stress PALT test plan. The maximum likelihood estimation for the distribution and the Variance – Covariance matrix are described in section 3. The numerical study is applied to the theoretical results in section 4. And some conclusions are summarized in section 5. References are provided at the end of the paper.

2. The Model and Test Method

In this section we introduce the modified Weibull distribution and the Step – Stress PALT test plan we used in this research.

2.1 The Modified Weibull Distribution:

Sarhan and Zaindin (2009) introduced a new three parameters function that they named the Modified Weibull model. The new model included one scale and two shape parameters. Such a model generalized some of the mostly used distributions in analyzing lifetime data. Those distributions were: Linear Failure Rate Distribution, Rayleigh Distribution, Exponential Distribution, and Weibull Distribution.

The cumulative density function of the Modified Weibull distribution with a scale parameter α , and shape parameters ρ and γ is defined as

$$F(x; \alpha, \rho, \gamma) = 1 - e^{-\{\alpha x - \rho x^\gamma\}}, \quad x \geq 0$$

Where $\gamma > 0$, $\alpha, \rho \geq 0$ such that $\alpha + \rho > 0$.

The Modified Weibull distribution has the following probability density function

$$f(x; \alpha, \rho, \gamma) = (\alpha + \rho \gamma x^{\gamma-1}) e^{-\alpha x - \rho x^\gamma}$$

The survival function of the Modified Weibull distribution takes the form

$$S(x; \alpha, \rho, \gamma) = e^{-\alpha x - \rho x^\gamma} \quad x \geq 0$$

The distribution's hazard rate (failure rate) function is defined as:

$$h(x; \alpha, \rho, \gamma) = \alpha + \rho \gamma x^{\gamma-1}, \quad x > 0$$

2.2 The Test Method:

Under the Step – Stress PALT technique, the test starts when a sample of size n units is put to test under normal conditions until a pre-specified timing (switching point) τ is reached when then the survived units are subjected to the accelerated conditions till the end of the test which terminates at also a pre-specified time η .

If by a pre-specified time τ the test units have not failed they are then subjected to the accelerated conditions. This means if a test unit has not failed by time τ (the stress change time); the test condition is then switched (or changed) to a higher stress level (accelerated stress) till the end of the test. This switching is to allow multiplying the remaining lifetime of an item by the inverse of the acceleration factor β . In such a case, the total lifetime of a test item Y passes through two phases, the normal use phase and the accelerated one. Accordingly, the total lifetime Y of an item is defined as

$$Y = \begin{cases} T & T \leq \tau \\ \tau + \beta^{-1}(T - \tau) & T > \tau \end{cases}$$

where T is the lifetime of a unit at normal use conditions. The definition was proposed by Goel and DeGroot (1979).

3. Parameters Estimation

In this section, the maximum likelihood estimation approach will be used to estimate the parameters of a Modified Weibull model based upon a sample that is Type I censored and the asymptotic variance-covariance matrix will be obtained also.

3.1 Maximum Likelihood Estimation

A modified Weibull model has probability density and cumulative density functions (pdf and cdf respectively) defined as:

$$F(x; \alpha, \rho, \gamma) = 1 - e^{-\alpha x - \rho x^\gamma} \quad x \geq 0$$

$$\gamma > 0, \alpha, \rho \geq 0 \quad \alpha + \rho > 0$$

and,

$$f(x; \alpha, \rho, \gamma) = (\alpha + \rho \gamma x^{\gamma-1}) e^{-\alpha x - \rho x^\gamma} \quad x > 0$$

With the following reliability function:

$$R(x) = 1 - F(x) = e^{-\alpha x - \rho x^\gamma}$$

In partially step – stress ALT, a sample of n units is tested under normal conditions until a specified, predetermined time τ (switching time) is reached. The survived units are then subjected to accelerated conditions and failure data will be recorded. The test terminates when the censoring time η is reached.

The lifetimes of the test are assumed to follow the Modified Weibull distribution with scale parameter α , and shape parameters ρ and γ .

Therefore the probability density function of the total lifetime Y of an item is given by:

$$f(y) = \begin{cases} 0 & y \leq 0 \\ (\alpha + \rho \gamma y^{\gamma-1}) e^{-\alpha y - \rho y^\gamma} & 0 < y \leq \tau \\ \left[\alpha + \rho \gamma (\tau + \beta(y - \tau))^{\gamma-1} \right] e^{-\alpha(\tau + \beta(y - \tau)) - \rho(\tau + \beta(y - \tau))^\gamma} & y > \tau \end{cases}$$

Where,

$$Y = \begin{cases} T & T \leq \tau \\ \tau + \beta^{-1}(T - \tau) & T > \tau \end{cases}$$

and β is the acceleration factor.

The observed values of the total life time Y can be described as

$y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(n_u)} \leq \tau \leq y_{(n_u+1)} \leq \dots \leq y_{(n_u+n_a)} \leq \eta$, where n_u is the number of failures under “use conditions” (normal conditions) and n_a is the number of failures under “accelerated conditions”.

Two indicator functions δ_{1i} and δ_{2i} are defined such that

$$\delta_{1i} = \begin{cases} 1 & (y_i \leq \tau) \\ 0 & \text{otherwise} \end{cases}, \quad \delta_{2i} = \begin{cases} 1 & (\tau < y_i \leq \eta) \\ 0 & \text{otherwise} \end{cases}$$

Since the n lifetimes are i.i.d (independent and identically distributed), their likelihood function is defined as follows:

$$\begin{aligned} L &= \prod_{i=1}^n [f_1(y_i)]^{\delta_{1i}} [f_2(y_i)]^{\delta_{2i}} \{R(\eta)\}^{\bar{\delta}_{1i} \bar{\delta}_{2i}} \\ &= \prod_{i=1}^n \left\{ \left(\alpha + \rho \gamma y_i^{\gamma-1} \right) e^{-\alpha y_i - \rho y_i^\gamma} \right\}^{\delta_{1i}} \\ &\quad \times \left\{ \left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \right) e^{-\alpha(\tau + \beta(y_i - \tau)) - \rho(\tau + \beta(y_i - \tau))^\gamma} \right\}^{\delta_{2i}} \\ &\quad \times \left\{ e^{-\alpha(\tau + \beta(\eta - \tau)) - \rho(\tau + \beta(\eta - \tau))^\gamma} \right\}^{\bar{\delta}_{1i} \bar{\delta}_{2i}} \end{aligned}$$

Where $\bar{\delta}_{1i} = 1 - \delta_{1i}$ and $\bar{\delta}_{2i} = 1 - \delta_{2i}$ are the number of survived units under “use” and “accelerated” conditions respectively.

When dealing with the maximum likelihood function, it's much easier to deal with its logarithmic form. So taking $\ln$ of both sides, the function becomes:

$$\begin{aligned} \ln L &= \sum_{i=1}^n \delta_{1i} \ln \left(\alpha + \rho \gamma y_i^{\gamma-1} \right) + \sum_{i=1}^n \delta_{1i} \left(-\alpha y_i - \rho y_i^\gamma \right) + \sum_{i=1}^n \delta_{2i} \ln \left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \right) \\ &+ \sum_{i=1}^n \delta_{2i} \left(-\alpha(\tau + \beta(y_i - \tau)) - \rho(\tau + \beta(y_i - \tau))^\gamma \right) \\ &+ (n - n_u - n_a) \left(-\alpha(\tau + \beta(\eta - \tau)) - \rho(\tau + \beta(\eta - \tau))^\gamma \right) \end{aligned}$$

where,

$$\sum_{i=1}^n \delta_{1i} = n_u, \quad \sum_{i=1}^n \delta_{2i} = n_a, \quad \sum_{i=1}^n \bar{\delta}_{1i} \bar{\delta}_{2i} = n - n_u - n_a$$

The maximum likelihood estimators $\hat{\alpha}, \hat{\rho}, \hat{\gamma}$ and $\hat{\beta}$ for the parameters α, ρ, γ and β are obtained by solving the system of equations consisting of the first

partial derivatives of the logarithmic maximum likelihood function with respect to α, ρ, γ and β when equated to zero.

The first partial derivatives with respect to α, ρ, γ and β are obtained as follows:

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^n \frac{\delta_{1i}}{\alpha + \rho \gamma y_i^{\gamma-1}} - \sum_{i=1}^n \delta_{1i} y_i + \sum_{i=1}^n \frac{\delta_{2i}}{\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1}} - \sum_{i=1}^n \delta_{2i} (\tau + \beta(y_i - \tau)) - (n - n_u - n_a)(\tau + \beta(\eta - \tau))$$

$$\frac{\partial \ln L}{\partial \rho} = \sum_{i=1}^n \delta_{1i} \frac{\gamma y_i^{\gamma-1}}{\alpha + \rho \gamma y_i^{\gamma-1}} - \sum_{i=1}^n \delta_{1i} y_i^{\gamma} + \sum_{i=1}^n \delta_{2i} \frac{\gamma (\tau + \beta(y_i - \tau))^{\gamma-1}}{\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1}} - \sum_{i=1}^n \delta_{2i} (\tau + \beta(y_i - \tau))^{\gamma} - (n - n_u - n_a)(\tau + \beta(\eta - \tau))^{\gamma}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \gamma} &= \sum_{i=1}^n \delta_{1i} \frac{\rho y_i^{\gamma-1} + \rho \gamma y_i^{\gamma-1} \ln y_i}{\alpha + \rho \gamma y_i^{\gamma-1}} - \sum_{i=1}^n \delta_{1i} \rho y_i^{\gamma} \ln y_i \\ &+ \sum_{i=1}^n \delta_{2i} \frac{\rho (\tau + \beta(y_i - \tau))^{\gamma-1} + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \ln (\tau + \beta(y_i - \tau))}{\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1}} \\ &- \sum_{i=1}^n \delta_{2i} (\rho (\tau + \beta(y_i - \tau))^{\gamma} \ln (\tau + \beta(y_i - \tau))) \\ &- (n - n_u - n_a)(\rho (\tau + \beta(\eta - \tau))^{\gamma} \ln (\tau + \beta(\eta - \tau))) \end{aligned}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta} &= \sum_{i=1}^n \delta_{2i} \frac{\rho \gamma (\gamma - 1) (\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau)}{\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1}} \\ &- \sum_{i=1}^n \delta_{2i} [\alpha (y_i - \tau) + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} (y_i - \tau)] \\ &- (n - n_u - n_a)(\alpha (\eta - \tau) + \rho \gamma (\tau + \beta(\eta - \tau))^{\gamma-1} (\eta - \tau)) \end{aligned}$$

Equating the derivatives to zero we obtain a system of non – linear equations that can't be solved in explicit form. Therefore numerical solution using Mathematica 8 was suggested to obtain the maximum likelihood estimators of the parameters α, ρ, γ and the acceleration factor β .

3.2 Variance – Covariance Matrix

The asymptotic variance-covariance matrix of the estimated parameters $(\hat{\alpha}, \hat{\rho}, \hat{\gamma}, \hat{\beta})$ is obtained by inverting the information matrix consisting of the negatives of expected values of the second order partial derivatives of the logarithmic likelihood function. The elements of the asymptotic variance-covariance matrix can be approximated by Cohen's (1965) result :

$$-E\left[\frac{\partial^2 \ln L(\theta)}{\partial \theta_i \partial \theta_j}\right] \approx -\left.\frac{\partial^2 \ln L(\theta)}{\partial \theta_i \partial \theta_j}\right|_{\theta=\hat{\theta}}$$

Therefore, the elements of the approximate information matrix are given by the second order partial derivatives of the logarithmic likelihood function with respect to α, ρ, γ and β are as follows:

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\sum_{i=1}^n \delta_{1i} \frac{1}{\left(\alpha + \rho \gamma y_i^{r-1}\right)^2} - \sum_{i=1}^n \delta_{2i} \frac{1}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{r-1}\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \rho} = -\sum_{i=1}^n \delta_{1i} \frac{\gamma y_i^{r-1}}{\left(\alpha + \rho \gamma y_i^{r-1}\right)^2} - \sum_{i=1}^n \delta_{2i} \frac{\gamma (\tau + \beta(y_i - \tau))^{r-1}}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{r-1}\right)^2}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha \partial \gamma} &= -\sum_{i=1}^n \delta_{1i} \frac{\rho y_i^{r-1} + \rho \gamma y_i^{r-1} \ln y_i}{\left(\alpha + \rho \gamma y_i^{r-1}\right)^2} \\ &- \sum_{i=1}^n \delta_{2i} \frac{\rho (\tau + \beta(y_i - \tau))^{r-1} + \rho \gamma (\tau + \beta(y_i - \tau))^{r-1} \ln(\tau + \beta(y_i - \tau))}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{r-1}\right)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} &= -\sum_{i=1}^n \delta_{2i} \frac{\rho \gamma (\gamma - 1)(\tau + \beta(y_i - \tau))^{r-2}(y_i - \tau)}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{r-1}\right)^2} \\ &- \sum_{i=1}^n \delta_{2i} (y_i - \tau) - (n - n_u - n_v)(\eta - \tau) \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \ln L}{\partial \rho \partial \alpha} &= -\sum_{i=1}^n \delta_{1i} \frac{\gamma y_i^{r-1}}{(\alpha + \rho \gamma y_i^{r-1})^2} - \sum_{i=1}^n \delta_{2i} \frac{\gamma(\tau + \beta(y_i - \tau))^{r-1}}{(\alpha + \rho \gamma(\tau + \beta(y_i - \tau))^{r-1})^2} \\
 \frac{\partial \ln L}{\partial \rho^2} &= -\sum_{i=1}^n \delta_{1i} \frac{(\gamma y_i^{r-1})^2}{(\alpha + \rho \gamma y_i^{r-1})^2} - \sum_{i=1}^n \delta_{2i} \frac{(\gamma(\tau + \beta(y_i - \tau))^{r-1})^2}{(\alpha + \rho \gamma(\tau + \beta(y_i - \tau))^{r-1})^2} \\
 \frac{\partial \ln L}{\partial \rho \partial \gamma} &= \sum_{i=1}^n \delta_{1i} \frac{(y_i^{r-1} + \gamma y_i^{r-1} \ln y_i)(\alpha + \rho \gamma y_i^{r-1}) - (\gamma y_i^{r-1})(\rho y_i^{r-1} + \rho \gamma y_i^{r-1} \ln y_i)}{(\alpha + \rho \gamma y_i^{r-1})^2} \\
 &\quad - \sum_{i=1}^n \delta_{1i} y_i^r \ln y_i \\
 &\quad + \sum_{i=1}^n \delta_{2i} \frac{L \cdot Z - (\gamma(\tau + \beta(y_i - \tau))^{r-1})(\rho(\tau + \beta(y_i - \tau))^{r-1} + \rho \gamma(\tau + \beta(y_i - \tau))^{r-1} \ln(\tau + \beta(y_i - \tau)))}{(\alpha + \rho \gamma(\tau + \beta(y_i - \tau))^{r-1})^2} \\
 &\quad - \sum_{i=1}^n \delta_{2i} (\tau + \beta(y_i - \tau))^r \ln(\tau + \beta(y_i - \tau)) - (n - n_u - n_a)(\tau + \beta(\eta - \tau))^r \ln(\tau + \beta(\eta - \tau)) \\
 \frac{\partial \ln L}{\partial \rho \partial \beta} &= \\
 &\quad \sum_{i=1}^n \delta_{2i} \frac{(\gamma(\gamma - 1)(\tau + \beta(y_i - \tau))^{r-2} \cdot (y_i - \tau)) \cdot Z - (\gamma(\tau + \beta(y_i - \tau))^{r-1})(\rho \gamma(\gamma - 1)(\tau + \beta(y_i - \tau))^{r-2} \cdot (y_i - \tau))}{(\alpha + \rho \gamma(\tau + \beta(y_i - \tau))^{r-1})^2} \\
 &\quad - \sum_{i=1}^n \delta_{2i} \gamma(\tau + \beta(y_i - \tau))^{r-1} \cdot (y_i - \tau) - (n - n_u - n_a) \gamma(\tau + \beta(\eta - \tau))^{r-1} \cdot (\eta - \tau) \\
 \frac{\partial^2 \ln L}{\partial \gamma \partial \alpha} &= -\sum_{i=1}^n \delta_{1i} \frac{\rho y_i^{r-1} + \rho \gamma y_i^{r-1} \ln y_i}{(\alpha + \rho \gamma y_i^{r-1})^2} - \sum_{i=1}^n \delta_{2i} \frac{F}{(\alpha + \rho \gamma(\tau + \beta(y_i - \tau))^{r-1})^2} \\
 \frac{\partial^2 \ln L}{\partial \gamma \partial \rho} &= \sum_{i=1}^n \delta_{1i} \frac{(y_i^{r-1} + \gamma y_i^{r-1} \ln y_i)(\alpha + \rho \gamma y_i^{r-1}) - (\rho y_i^{r-1} + \rho \gamma y_i^{r-1} \ln y_i)}{(\alpha + \rho \gamma y_i^{r-1})^2} - \sum_{i=1}^n \delta_{1i} y_i^r \ln y_i \\
 &\quad + \sum_{i=1}^n \delta_{2i} \frac{((\tau + \beta(y_i - \tau))^{r-1} + \gamma(\tau + \beta(y_i - \tau))^{r-1} \ln(\tau + \beta(y_i - \tau))) \cdot B - F(\gamma(\tau + \beta(y_i - \tau))^{r-1})}{(\alpha + \rho \gamma(\tau + \beta(y_i - \tau))^{r-1})^2} \\
 &\quad - \sum_{i=1}^n \delta_{2i} (\tau + \beta(y_i - \tau))^r \cdot \ln(\tau + \beta(y_i - \tau)) - (n - n_u - n_a) (\tau + \beta(\eta - \tau))^r \cdot \ln(\tau + \beta(\eta - \tau))
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^2 \ln L}{\partial \gamma^2} = & \sum_{i=1}^n \delta_{1i} \frac{\left[\rho y_i^{\gamma-1} \ln y_i + \rho y_i^{\gamma-1} \ln y_i + \rho \gamma (\ln y_i)^2 y_i^{\gamma-1} \right] \cdot \left[\alpha + \rho \gamma y_i^{\gamma-1} \right] - \left[\rho y_i^{\gamma-1} + \rho \gamma y_i^{\gamma-1} \ln y_i \right] \cdot \left[\rho y_i^{\gamma-1} + \rho \gamma y_i^{\gamma-1} \ln y_i \right]}{\left(\alpha + \rho \gamma y_i^{\gamma-1} \right)^2} \\
 & - \sum_{i=1}^n \delta_{1i} \left(\rho (\ln y_i)^2 y_i^{\gamma} \right) \\
 & + \sum_{i=1}^n \delta_{2i} \frac{H \cdot B - F \left[\rho (\tau + \beta(y_i - \tau))^{\gamma-1} + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \ln(\tau + \beta(y_i - \tau)) \right]}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \right)^2} \\
 & - \sum_{i=1}^n \delta_{2i} \left[\rho (\tau + \beta(y_i - \tau))^{\gamma} (\ln(\tau + \beta(y_i - \tau)))^2 \right] \\
 & - (n - n_u - n_a) \left[\rho (\tau + \beta(\eta - \tau))^{\gamma} \cdot (\ln(\tau + \beta(\eta - \tau)))^2 \right] \\
 \\
 \frac{\partial^2 \ln L}{\partial \gamma \partial \beta} = & \sum_{i=1}^n \delta_{2i} \frac{G \cdot B - F \left[\rho \gamma (\gamma-1) (\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau) \right]}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \right)^2} \\
 & - \sum_{i=1}^n \delta_{2i} \left[\rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \cdot (y_i - \tau) \cdot \ln(\tau + \beta(y_i - \tau)) + \rho (\tau + \beta(y_i - \tau))^{\gamma} \cdot \frac{(y_i - \tau)}{\tau + \beta(y_i - \tau)} \right] \\
 & - (n - n_u - n_a) \left[\rho \gamma (\tau + \beta(\eta - \tau))^{\gamma-1} (\eta - \tau) \cdot \ln(\tau + \beta(\eta - \tau)) + \rho (\tau + \beta(\eta - \tau))^{\gamma} \cdot \frac{(\eta - \tau)}{\tau + \beta(\eta - \tau)} \right] \\
 \\
 \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = & - \sum_{i=1}^n \delta_{2i} \frac{\rho \gamma (\gamma-1) (\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau)}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \right)^2} - \sum_{i=1}^n \delta_{2i} (y_i - \tau) - (n - n_u - n_a) (\eta - \tau) \\
 \\
 \frac{\partial^2 \ln L}{\partial \beta \partial \rho} = & \sum_{i=1}^n \delta_{2i} \frac{\gamma (\gamma-1) (\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau) \cdot T - S \cdot \gamma (\tau + \beta(y_i - \tau))^{\gamma-1}}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \right)^2} \\
 & - \sum_{i=1}^n \delta_{2i} \left[\gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \cdot (y_i - \tau) \right] - (n - n_u - n_a) \left[\gamma (\tau + \beta(\eta - \tau))^{\gamma-1} \cdot (\eta - \tau) \right] \\
 \\
 \frac{\partial^2 \ln L}{\partial \beta \partial \gamma} = & \sum_{i=1}^n \delta_{2i} \frac{E \cdot T - S \cdot \left[\rho (\tau + \beta(y_i - \tau))^{\gamma-1} + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \ln(\tau + \beta(y_i - \tau)) \right]}{\left(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \right)^2} \\
 & - \sum_{i=1}^n \delta_{2i} \left[\rho (\tau + \beta(y_i - \tau))^{\gamma-1} (y_i - \tau) + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} (y_i - \tau) \ln(\tau + \beta(y_i - \tau)) \right] \\
 & - (n - n_u - n_a) \left[\rho (\tau + \beta(\eta - \tau))^{\gamma-1} (\eta - \tau) + \rho \gamma (\tau + \beta(\eta - \tau))^{\gamma-1} \ln(\tau + \beta(\eta - \tau)) \right]
 \end{aligned}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \sum_{i=1}^n \delta_{2i} \frac{[\rho \gamma (\gamma-1)(\gamma-2)(\tau + \beta(y_i - \tau))^{\gamma-3} (y_i - \tau)^2] \cdot T - S \cdot [\rho \gamma (\gamma-1)(\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau)]}{(\alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1})^2} - \sum_{i=1}^n \delta_{2i} [\rho \gamma (\gamma-1)(\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau)^2] - (n - n_u - n_o) [\rho \gamma (\gamma-1)(\tau + \beta(\eta - \tau))^{\gamma-2} (\eta - \tau)^2]$$

Where

$$L = (\tau + \beta(y_i - \tau))^{\gamma-1} + \gamma(\tau + \beta(y_i - \tau))^{\gamma-1} \ln(\tau + \beta(y_i - \tau))$$

$$F = \rho(\tau + \beta(y_i - \tau))^{\gamma-1} + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \ln(\tau + \beta(y_i - \tau))$$

$$B = \alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1}$$

$$H = \rho(\tau + \beta(y_i - \tau))^{\gamma-1} \cdot \ln(\tau + \beta(y_i - \tau)) + \rho(\tau + \beta(y_i - \tau))^{\gamma-1} \cdot \ln(\tau + \beta(y_i - \tau)) + \rho \gamma (\ln(\tau + \beta(y_i - \tau)))^2 (\tau + \beta(y_i - \tau))^{\gamma-1}$$

$$G = \rho(\gamma-1)(\tau + \beta(y_i - \tau))^{\gamma-2} \cdot (y_i - \tau) + \rho \gamma (\gamma-1)(\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau) \ln(\tau + \beta(y_i - \tau)) + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1} \cdot \frac{(y_i - \tau)}{\tau + \beta(y_i - \tau)}$$

$$S = \rho \gamma (\gamma-1)(\tau + \beta(y_i - \tau))^{\gamma-2} \cdot (y_i - \tau)$$

$$T = \alpha + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-1}$$

$$E = \rho(\gamma-1)(\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau) + \rho \gamma (\tau + \beta(y_i - \tau))^{\gamma-2} (y_i - \tau) + \rho \gamma (\gamma-1)(y_i - \tau)(\tau + \beta(y_i - \tau))^{\gamma-2} \cdot \ln(\tau + \beta(y_i - \tau))$$

Since the maximum likelihood estimates of the parameters are being used in finding the variance-covariance matrix according to Cohen's(1965), the parameters α , ρ , γ , and β in the second order partial derivatives of the logarithmic likelihood function shall be replaced by their estimates $\hat{\alpha}$, $\hat{\rho}$, $\hat{\gamma}$, and $\hat{\beta}$

Numerical Illustration

When conducting an estimation experiment it is requisite to study the characteristics of the expressions derived for the estimators. To do so a simulation study is performed. In this paper the simulation study is performed using Mathematica 8. The behavior of the resulted estimators of the acceleration factor and the Modified Weibull parameters has been studied in terms of the Mean Squared Error (MSE) and the Absolute Relative Bias (RABias).

Three different sample sizes were used, that are $n=10$, $n=20$, and $n=50$. Also different combinations of initial values for the parameters are used while the acceleration factor β was set fixed at 1.5. The different combinations are $(\alpha=2, \rho=1.5, \gamma=1.3)$, $(\alpha=2.5, \rho=2.5, \gamma=1.3)$, and $(\alpha=3, \rho=1.5, \gamma=1.3)$. Censoring time under normal conditions τ (also known as the switching time) is set in advance to 0.3 and the test termination time η is set to 1. The Kolmogorov Smirnov goodness of fit test was also conducted to guarantee that the generated random numbers follow the Modified Weibull distribution. Two significance levels were used, that's' 1% and 5%.

The resulted MSE values for the parameters and acceleration factor are displayed in the following tables:

	n = 10					
	(2, 1.5, 1.3, 1.5) *		(2.5, 2.5, 1.3, 1.5) **		(3, 1.5, 1.3, 1.5) **	
	MSE	RABias	MSE	RABias	MSE	RABias
α	75.3634	0.397597	7.949	0.5169044	13.2469	0.586204
ρ	78.3951	1.57422	13.4181	1.2284	13.7918	1.529993
γ	0.2677	0.210127	0.8628	0.004758	0.9015	0.0360077
β	6624.7114	37.34727	16920.4	55.60567	7444.0274	41.1966

* 500 trials, ** 100 trials

n = 20						
(2, 1.5, 1.3, 1.5) *		(2.5, 2.5, 1.3, 1.5) *		(3, 1.5, 1.3, 1.5) *		
	MSE	RABias	MSE	RABias	MSE	RABias
α	8.0922	0.6699345	5.8069	0.8581124	8.5774	0.883583
ρ	4698990.253	71	7.8098	1.04834	4.0104	1.07423
γ	0.9399	0.14698	0.2728	0.175245	0.16498	0.203263
β	3377.3179	31.83205	12033.3565	60.33331	10456.9212	56.05236

Significance level 1%. * 500 trials. ** 100 trials

n = 20						
(2, 1.5, 1.3, 1.5) *		(2.5, 2.5, 1.3, 1.5) *		(3, 1.5, 1.3, 1.5) *		
	MSE	RABias	MSE	RABias	MSE	RABias
α	13.4182	0.88219	6.5848	0.8353152	7.8599	0.93436
ρ	12.3805	0.969861	9.2227	1.08082	2.2496	0.999904
γ	0.4903	0.11943	0.1217	0.24598	0.1327	0.207948
β	3069.7622	29.94447	6452.8583	46.33241	11342.0565	52.645867

Significance level 5%. * 500 trials. ** 100 trials

n = 50						
	(1.5, 1.5, 1.3, 1.5) **		(1.5, 1.5, 1.3, 1.5) *		(2, 1.5, 1.3, 1.5) *	
	MSE	RABias	MSE	RABias	MSE	RABias
α	7.38998	0.489217	6.6578	0.627048	6.7378	0.997803
ρ	9.6181	1.289953	8.4223	1.155173	4.7632	0.849984
γ	0.3059	0.184686	0.8052	0.11134	0.7549	0.126129
β	1237.6308	21.239207	1328.6315	21.97689	1956.5841	28.449627

Significance level 1%.

* 500 trials.

** 100 trials

4. Conclusion

After conducting the numerical study it was noticed that an increase in the values of α and ρ results in a decrease in the MSE values of α and ρ , and an increase in the MSE values of γ and β at a sample size of 10. When the sample size is raised to 20, all MSE values decreased except for the acceleration factor, it scored an increasing value in such a situation. At fixed values of ($\alpha = 3, \rho = 1.5, \gamma = 1.3$), the MSE of the Modified Weibull distribution's parameter decrease but not the acceleration factor. It was generally noticed that the acceleration factor's MSE decreased when sample size increased. At fixed values of ρ and γ but increasing ones of α , the MSE for the acceleration factor increased. Increasing the sample size in the case ($\alpha = 2, \rho = 1.5, \gamma = 1.3$) led to a decrease in the RABias values of the acceleration factor and the two shape parameters but an increase in the value of the scale parameter.

References

- 1) Attia, A. F., Abdel – Ghaly, A. A., and Abdel – Ghani, M. M. (1996). “The Estimation Problem of Partially Accelerated Life Tests for Weibull distribution by Maximum Likelihood Method with Censored Data”, Proceeding of the 31st Annual Conference of Statistics, Computer Science, and Operation Research, ISSR, Cairo University, 128 – 138.
- 2) Cohen, A. Clifford (1965). Maximum Likelihood Estimation in the Weibull Distribution Based On Complete and Censored Samples. *Technometrics*, Vol 7, pp. 579 – 588.
- 3) Goel, P. K. (1971). “Some Estimation Problems is the Study of Tampered Random Variables”. PhD Thesis, Department of statistics, Cranegie – Mellon University, Pittsburgh, Pennsylvania.
- 4) Goel, P. K. and DeGroot, M. H. (1979). “Bayesian Estimation and Optimal Design in Partially Accelerated Life Testing”. *Naval Research Logistics Quarterly*, 16 (2) 223 – 235.
- 5) Ismail, A. A. (2004). “The Test Design and Parameter Estimation of Pareto Lifetime Distribution under Partially Accelerated Life Tests”. Ph.D thesis, Department of Statistics, Faculty of Economics and Political Science, Cairo University, Egypt.
- 6) Nelson, W. (1980). Accelerated Life Testing: Step-Stress Models and Data Analyses, *IEEE Transactions of Reliability*, Vol. R-29, issue 2, 103-108.
- 7) Nassr, S. G. (2008). The Estimation Problems of Life Time Model under Partially Accelerated Life Testing.. M. Sc. Thesis ISSR. Cairo University.
- 8) Sarhan, A. M. and Zaindin, M. (2009). “Modified Weibull Distribution”. *Applied Sciences*, Vol. 11, pp. 123 – 136.

The Empirical Probability Generating Method for Intervened Poisson Distribution Based on Rapid Estimation

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Abstract: In this paper, we introduce the rapid estimation method using the empirical probability generating function, and finding asymptotic covariance matrix for the intervened Poisson distribution. Numerical illustrations are considered to studies the properties of estimators.

Keywords: Empirical probability generating, intervened Poisson distribution, rapid estimation methods, asymptotic covariance matrix.

1- Introduction

The intervened Poisson distribution (IPD) was introduced by Shanmugam (1985) as a replacement for the zero truncated Poisson distribution (ZTPD), in order to model rare event count data, when some intervention processes may alter the mean of the rare event generating process under observation. Scollnik (1995) carried out a Bayesian analysis of these data using the Gibbs sampler and adaptive rejection sampling.

1.1 Empirical Probability Generating Function

Nakamura and PerezAbreu (1993) defined the empirical probability generating function as follows, if $x_1, \dots, x_n$ be random sample from a discrete distribution $F(x)$ over $0, 1, 2, \dots$ with corresponding probabilities $p(x_k)$, $k = 0, 1, 2, \dots$ then the empirical probability generating function as,

$$G_n(t) = (1/n) \sum_{i=1}^n t^{x_i} \quad \text{for } t \in [0, 1]$$

1.2 Properties of Intervened Poisson Distribution

For investigating the properties of IPD, let Y be the number of cholera cases in a household, the event is not observable since the observational apparatus is activated only when $Y > 0$, Y is the random variable of a rare event, it is appropriate to consider the ZTPD for the positive integer- valued Y

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with probability function form as follows

$$\Pr(Y = y) = (e^\theta - 1)^{-1} \theta^y / y!, \quad y = 1, 2, \dots$$

Where $\theta > 0$ is called the incidence parameter. In the example of cholera incidence, health service agencies and others have to use various preventive treatments; these treatments have the effect of changing θ from the incidence to another. Shanmugam (1985) assumed that such effects due to changing θ to $\rho\theta$, such that, $E(Z) = \rho\theta$, where $0 < \rho < \infty$ is an intervention parameter, and Z is the number of cases that occurred after the preventive treatments were applied. Note that Z is a Poisson random variable with mean $\rho\theta$, Y and Z are stochastically independent. He assumed that the observational apparatus has kept a record of the random variable $X = Y + Z$ which is the total number of cholera cases that occurred all together, then the pdf of X is

$$\begin{aligned} \Pr(X = x) &= \sum_{i=1}^{x-1} \Pr(Y = x - i) \Pr(Z = i / Y = x - i) \\ &= \frac{e^{-\rho\theta}}{(e^\theta - 1)} \left[(1 + \rho)^x - \rho^x \right] \theta^x / x! \end{aligned} \quad (1.1)$$

where $x = 1, 2, \dots$, we shall call the pdf in (1.1) an intervened Poisson distribution, the mean and variance of IPD can be represented as

$$\mu = E(X) = \theta \left[\rho + 1 + (e^\theta - 1)^{-1} \right]$$

and

$$\sigma^2 = \text{var}(X) = \mu - e^\theta (\theta / e^\theta - 1)^2,$$

with probability generating function (PGF)

$$G(t) = \frac{e^{\rho\theta t} (e^\theta t - 1)}{e^{\rho\theta} (e^\theta - 1)}$$

the first derivative of PGF is

$$G'(t) = F_x (e^{\rho\theta t} (e^\theta (1 + \rho\theta t) - \rho\theta)) \quad (1.2)$$

where,

$$F = \frac{1}{e^{\rho\theta} (e^\theta - 1)}$$

The second derivative of PGF is

$$G''(t) = F(e^{\rho\theta} (e^\theta (2\theta\rho + t(\theta\rho)^2) - (\theta\rho)^2) \quad (1.3)$$

1.3 Maximum Likelihood Estimation For Intervened Poisson Distribution

The maximum likelihood method can be introduced as following:

let $x_1, x_2, \dots, x_n$ be a random sample of size n from IPD distribution, the likelihood function of IPD is

$$L = \prod_{i=1}^n \frac{e^{-\rho\theta}}{e^\theta - 1} ((1 + \rho)^{x_i} - \rho^{x_i}) \frac{\theta^{x_i}}{x_i!},$$

and the $\ln$ of the likelihood function will be

$$\begin{aligned} \ln L &= \ln \prod_{i=1}^n \frac{e^{-\rho\theta}}{e^\theta - 1} ((1 + \rho)^{x_i} - \rho^{x_i}) \frac{\theta^{x_i}}{x_i!} \\ &= \sum_{i=1}^n \ln \left(\frac{e^{-\rho\theta}}{e^\theta - 1} ((1 + \rho)^{x_i} - \rho^{x_i}) \frac{\theta^{x_i}}{x_i!} \right) \\ &= \sum_{i=1}^n (-\rho\theta - \ln(e^\theta - 1) + \ln((1 + \rho)^{x_i} - \rho^{x_i}) + \ln \theta^{x_i} - \ln x_i!), \end{aligned}$$

and,

$$\ln L = (-n\rho\theta - n \ln(e^\theta - 1) + \sum_{i=1}^n \ln((1 + \rho)^{x_i} - \rho^{x_i}) + \sum_{i=1}^n x_i \ln \theta - \sum_{i=1}^n \ln x_i!) \quad (1.4)$$

Differentiating (1.4) with respect to θ, ρ , and equaling it by zero, then

$$\frac{\partial \ln L}{\partial \theta} = -n\rho - \frac{ne^\theta}{(e^\theta - 1)} + \frac{\sum_{i=1}^n x_i}{\theta},$$

and

$$\frac{\partial \ln L}{\partial \rho} = -n\theta + \sum_{i=1}^n \frac{(x_i (1 + \rho)^{x_i-1} - x_i \rho^{x_i-1})}{((1 + \rho)^{x_i} - \rho^{x_i})}$$

Putting $\frac{\partial \ln L}{\partial \hat{\rho}} = 0$, we get

$$-n\hat{\rho} - \frac{ne^{\hat{\theta}}}{(e^{\hat{\theta}} - 1)} + \frac{\sum_{i=1}^n x_i}{\hat{\theta}} = 0$$

$$-n\hat{\theta} + \sum_{i=1}^n \frac{(x_i (1 + \hat{\rho})^{x_i-1} - x_i \hat{\rho}^{x_i-1})}{((1 + \hat{\rho})^{x_i} - \hat{\rho}^{x_i})} = 0$$

It is noted that, there is no explicit solution for $\hat{\theta}, \hat{\rho}$, so computer facilities and numerical methods are needed to obtain these estimators.

1.4 Rapid estimation for discrete distributions

Kemp and Kemp (1988) suggested estimators based on the EPGF. The methods involve solving estimating equations obtained by equating the EPGF and PGF at a fixed finite set of values, then

$$EPGF(t_i) = G(t_i), \quad i = 1, 2. \quad (1.5)$$

Additionally, they explain how to choose t_1 and t_2 , they suggested that this choice was restricted as $-1 \leq t_i \leq 1$, such that the following cases obtained

i- If $t_2 \rightarrow t_1$ then the estimating equations become

$$\begin{aligned} EPGF(t_1) &= G(t_1), \\ EPGF'(t_1) &= G'(t_1), \end{aligned} \quad (1.6)$$

and if $t_1 \rightarrow 1$, implies that $EPG(t_1) = G(t_1)$ which can be written as

$$EPGF'(1) = G'(1), \quad (1.7)$$

then, the first moment equation can be obtained as

$$\bar{X} = \mu(\hat{\lambda}, \hat{\theta}) \quad (1.8)$$

which is one of the ML equations by a power series distribution, when both t_1 and t_2 tend to unity, the equations tend to

$$\begin{aligned} EPGF'(1) &= G'(1) \\ EPGF''(1) &= G''(1), \end{aligned} \quad (1.9)$$

which are the estimation equations for the method of moments.

ii- When $t_1 \rightarrow 1$ and $t_2 = 0$, (1.5) tends to (1.7), and also from (1.5) getting the next equation

$$f_{\cdot} = p_{\cdot}(\lambda^*, \theta^*) \quad (1.10)$$

which are the equations for the mean and zero frequency method.

iii- When $t_1 \rightarrow 1$ and $t_2 \rightarrow -1$, (1.5) tends (1.7) and from (1.5) also getting the next equation

$$\sum f_{2x} = \sum p_{2x}(\lambda^*, \theta^*) \quad (1.11)$$

which are the equations for the method of even-point.

Moreover, the equations estimating can be formed by equating corresponding both EPGF and PGF; the derivative, $\frac{dG(t)}{dt}$, and the logarithmic derivative, $\frac{d(\log G(t))}{dt}$, give

$$EPGF'(t_i) = G'(t_i), \quad i = 1, 2 \quad (1.12)$$

and

$$\frac{EPGF'(t_i)}{EPGF(t_i)} = \frac{G'(t_i)}{G(t_i)}, \quad i = 1, 2 \quad (1.13)$$

Respectively, when $t_1 = 1$ and $t_2 \rightarrow 1$ both (1.12) and (1.13) limit to the method of moments, putting $t_1 \rightarrow 1$ and $t_2 \rightarrow 0$ in (1.13) gives the method of mean and ratio of first two frequencies. This dose not exhaust the possibilities, useful new procedures can arise from the use of values other than -1,0,1 for t_1 and/ or t_2 , and also from using mixed pairs of equations, (1.5) for t_1 and (1.13) for t_2 .

2- Rapid Method for the Intervened Poisson Distribution

Accordingly, we use the method of Kemp and Kemp(1988) for the IPD to solve estimating equations obtained by equating the EPGF and PGF, we obtain the methods of estimation namely: the method of moment, the method of mean and zero frequency, the method of mean ratio of first two frequencies, and the method of even point appear when we choose different value of t_1 and t_2 such that $-1 \leq t_i \leq 1, i = 1, 2$, we apply these methods to IPD, when $t_1, t_2 \rightarrow 1$, by using (1.9), then the moment method form is

$$\begin{aligned}\bar{x} &= \mu \\ \bar{x} &= \tilde{\theta}\tilde{\rho} + \tilde{\theta} + \tilde{\theta}(e^{\tilde{\theta}} - 1)^{-1} \\ \tilde{\rho} &= \frac{\bar{x}}{\tilde{\theta}} - (1 + (e^{\tilde{\theta}} - 1)^{-1}),\end{aligned}\tag{2.1}$$

and,

$$\begin{aligned}\sigma^2 &= s^2 \\ s^2 &= \bar{x} - e^{\tilde{\theta}} \left(\frac{\tilde{\theta}}{e^{\tilde{\theta}} - 1} \right)^2 \\ \bar{x} - s^2 &= e^{\tilde{\theta}} \left(\frac{\tilde{\theta}}{e^{\tilde{\theta}} - 1} \right)^2,\end{aligned}\tag{2.2}$$

substituting for $t_1 = 1$, $t_2 = 0$, then we get $\bar{x} = \mu$, $f_1 = p_1$, respectively. This is called the second rapid estimation method; this is illustrated in detail as follows

$$\bar{x} = \mu$$

then

$$\dot{\rho} = \frac{\bar{x}}{\dot{\theta}} - (1 + (e^{\dot{\theta}} - 1)^{-1}),\tag{2.3}$$

and,

$$f_1 = p_1$$

from the proprieties of PGF, we can obtain

$$p_1 = G'(0) = \frac{e^{\theta} - \rho\theta}{e^{\rho\theta}(e^{\theta} - 1)}$$

then

$$f_1 = e^{\theta^*} - \dot{\rho}\dot{\theta} / e^{\rho^*\theta^*} (e^{\theta^*} - 1)\tag{2.4}$$

The third rapid estimation method, which is called even point method, it can be performed by setting $t_1 \rightarrow 1$, $t_2 = -1$, we get $\sum f_{2x} = \sum p_{2x}$, this is illustrated in detail as follows

$$\begin{aligned}\bar{x} &= \mu \\ \rho &= \frac{\bar{x}}{\theta} - (1 + (e^{\theta''} - 1)^{-1}),\end{aligned}\tag{2.5}$$

and

$$\sum f_{2x} = \sum p_{2x}\tag{2.6}$$

using the proprieties of PGF, we can get

$$\sum p_{2x} = \frac{1}{2}[G(-1)+1] \quad (2.7)$$

where

$$G(-1) = \frac{e^{-\rho\theta}(-e^\theta - 1)}{e^{\rho\theta}(e^\theta - 1)} \quad (2.8)$$

substituting $G(-1)$ in (2.8) by (2.7), we get

$$2\sum_x f_{2x} - 1 = -e^{-\rho\theta} (e^\theta + 1) / e^{\rho\theta} (e^\theta - 1) \quad (2.9)$$

the fourth Rapid estimation method ,can be obtained by setting $t_1=0$, $t_2 = -1$

$$f_1 = e^{\theta} - \rho \theta / e^{\rho\theta} (e^\theta - 1), \quad (2.10)$$

and,

$$\sum_x x f_x t^{x-1} = \sum_x x p_x t^{x-1} \quad (2.11)$$

$$\sum_x x p_x (t)^{x-1} = G'(t) \quad (2.12)$$

Setting $t = -1$ in (2.6), substituting in (2.11) and (2.12), respectively, we get

$$\sum_x (-1)^{x-1} x f_x = \frac{1}{e^{\rho\theta} (e^\theta - 1)} (e^{\theta} - \rho \theta - e^{\rho\theta} (e^\theta - 1)) \quad (2.13)$$

3-Asymptotic Covariance Matrix

Dowling and Nakamura (1997). Introduced a compact form for the asymptotic covariance matrix as follows

$$\Sigma(t) = A^{-1}(t) B(t) (A^{-1}(t))^T \quad (3.1)$$

where,

$$A(t) = \begin{pmatrix} -\frac{\partial G'(t_1, (\theta, \rho))}{\partial \theta} & -\frac{\partial G'(t_1, (\theta, \rho))}{\partial \rho} \\ -\frac{\partial G'(t_2, (\theta, \rho))}{\partial \theta} & -\frac{\partial G'(t_2, (\theta, \rho))}{\partial \rho} \end{pmatrix} \quad (3.2)$$

and,

$$B(t) = t t^T G''(t t^T, (\theta, \rho)) + G'(t t^T, (\theta, \rho)) - G'(t, (\theta, \rho)) G'(t^T, (\theta, \rho))$$

where,

$$t t^T = \begin{pmatrix} t_1^2 & t_1 t_2 \\ t_2 t_1 & t_2^2 \end{pmatrix}, \quad (3.3)$$

According to them, we expand this compact form of the asymptotic covariance matrix by applying this from to IPD, and we try to obtain illustrative form for the asymptotic covariance matrix by differentiating PGF as follows

$$G'(t^T, (\theta, \rho)) = \begin{pmatrix} G'(t_1^2) & G'(t_1 t_2) \\ G'(t_2 t_1) & G'(t_2^2) \end{pmatrix}$$

and,

$$G'(t) = \begin{pmatrix} G'(t_1) \\ G'(t_2) \end{pmatrix}$$

form(1.2),(1.3) we can write above equation as,

$$G'(t^T, (\theta, \rho)) = \begin{pmatrix} F(e^{\rho\theta_1^2} (e^\theta (1 + \rho\theta_1^2) - \rho\theta)) & F(e^{\rho\theta_1 t_2} (e^\theta (1 + \rho\theta_1 t_2) - \rho\theta)) \\ F(e^{\rho\theta_2 t_1} (e^\theta (1 + \rho\theta_2 t_1) - \rho\theta)) & F(e^{\rho\theta_2^2} (e^\theta (1 + \rho\theta_2^2) - \rho\theta)) \end{pmatrix}, \quad (3.4)$$

$$G'(t, (\theta, \rho)) = \begin{pmatrix} F(e^{\rho\theta_1} (e^\theta (1 + \rho\theta_1) - \rho\theta)) \\ F(e^{\rho\theta_2} (e^\theta (1 + \rho\theta_2) - \rho\theta)) \end{pmatrix} \quad (3.5)$$

$$G'(t, (\theta, \rho)) G'(t^T, (\theta, \rho)) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where,

$$\begin{aligned} g_{11} &= F^2 [e^{\rho\theta_1} (e^\theta (1 + \rho\theta_1) - \rho\theta)]^2 \\ g_{12} &= F^2 [e^{\rho\theta_1} (e^\theta (1 + \rho\theta_1) - \rho\theta)] [e^{\rho\theta_2} (e^\theta (1 + \rho\theta_2) - \rho\theta)] \\ g_{21} &= F^2 [e^{\rho\theta_1} (e^\theta (1 + \rho\theta_1) - \rho\theta)] [e^{\rho\theta_2} (e^\theta (1 + \rho\theta_2) - \rho\theta)] \\ g_{22} &= F^2 [e^{\rho\theta_2} (e^\theta (1 + \rho\theta_2) - \rho\theta)]^2 \end{aligned}$$

and,

$$G''(t) = \begin{pmatrix} F(e^{\rho\theta_1^2} (e^\theta (2\theta\rho + t_1^2 (\theta\rho)^2) - (\theta\rho)^2)) & F(e^{\rho\theta_1 t_2} (e^\theta (2\theta\rho + t_1 t_2 (\theta\rho)^2) - (\theta\rho)^2)) \\ F(e^{\rho\theta_2 t_1} (e^\theta (2\theta\rho + t_2 t_1 (\theta\rho)^2) - (\theta\rho)^2)) & F(e^{\rho\theta_2^2} (e^\theta (2\theta\rho + t_2^2 (\theta\rho)^2) - (\theta\rho)^2)) \end{pmatrix},$$

from above equation and (3.3), (3.4), (3.5), we obtain

$$B(t) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad (3.6)$$

where,

$$b_{11} = t_1^2 F(e^{\rho\theta_1} (e^\theta (2\theta\rho + t_1^2 (\theta\rho)^2) - (\theta\rho)^2) + t_1 t_2 F(e^{\rho\theta_1 t_1} (e^\theta (2\theta\rho + t_2 t_1 (\theta\rho)^2) - (\theta\rho)^2) + F(e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_1) - \rho\theta)) - F^2 [e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_1) - \rho\theta)]^2,$$

$$b_{12} = t_1^2 F(e^{\rho\theta_1 t_2} (e^\theta (2\theta\rho + t_1 t_2 (\theta\rho)^2) - (\theta\rho)^2) + t_1 t_2 F(e^{\rho\theta_1} (e^\theta (2\theta\rho + t_2^2 (\theta\rho)^2) - (\theta\rho)^2) + F(e^{\rho\theta_1 t_2} (e^\theta (1 + \rho\theta t_2) - \rho\theta)) - F^2 [e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_1) - \rho\theta)] [e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_2) - \rho\theta)],$$

$$b_{21} = t_2 t_2 F(e^{\rho\theta_1} (e^\theta (2\theta\rho + t_1^2 (\theta\rho)^2) - (\theta\rho)^2) + t_2^2 F(e^{\rho\theta_1 t_1} (e^\theta (2\theta\rho + t_2 t_1 (\theta\rho)^2) - (\theta\rho)^2) + F(e^{\rho\theta_1 t_1} (e^\theta (1 + \rho\theta t_2) - \rho\theta)) - F^2 [e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_1) - \rho\theta)] [e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_2) - \rho\theta)],$$

and,

$$b_{22} = t_1 t_2 F(e^{\rho\theta_1 t_2} (e^\theta (2\theta\rho + t_1 t_2 (\theta\rho)^2) - (\theta\rho)^2) + t_1^2 F(e^{\rho\theta_1} (e^\theta (2\theta\rho + t_2^2 (\theta\rho)^2) - (\theta\rho)^2) + F(e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_2) - \rho\theta)) - F^2 [e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_2) - \rho\theta)]^2.$$

from(1.2),get that

$$\frac{\partial G'(t)}{\partial \theta} = F^2 [(e^{\theta\rho} (e^\theta - 1))] [e^{\theta\rho} (e^\theta (\rho t (2 + \rho\theta t + \theta) + 1) - (\rho + \rho^2 t \theta))] - [e^{\rho\theta} (e^\theta (1 + \rho\theta t) - \rho\theta)] [(1 + \rho) e^{(1+\rho)\theta} - \rho e^{\theta\rho}]$$

and,

$$\frac{\partial G'(t)}{\partial \rho} = F^2 [(e^{\theta\rho} (e^\theta - 1))] [e^{\theta\rho} (e^\theta (2\theta t + \rho(\theta t)^2) - (\theta + \theta^2 \rho t))] - [e^{\rho\theta} (e^\theta (1 + \rho\theta t) - \rho\theta)] [\theta e^{\rho\theta+\theta} - \theta e^{\theta\rho}]$$

than,

$$A(t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad (3.7)$$

where,

$$a_{11} = -F^2 [(e^{\theta\rho} (e^\theta - 1))] [e^{\theta\rho_1} (e^\theta (\rho t_1 (2 + \rho\theta t_1 + \theta) + 1) - (\rho + \rho^2 t_1 \theta))] - [e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_1) - \rho\theta)] [(1 + \rho) e^{(1+\rho)\theta} - \rho e^{\theta\rho}]$$

$$a_{12} = -F^2 [(e^{\theta\rho} (e^\theta - 1))] [e^{\theta\rho_1} (e^\theta (2\theta t_1 + \rho(\theta t_1)^2) - (\theta + \theta^2 \rho t_1))] - [e^{\rho\theta_1} (e^\theta (1 + \rho\theta t_1) - \rho\theta)] [\theta e^{\rho\theta+\theta} - \theta e^{\theta\rho}]$$

$$a_{21} = -F^2 \left[(e^{\theta\rho}(e^\theta - 1)) \left[e^{\theta\rho t_2} (e^\theta (\rho t_2 (2 + \rho\theta t_2 + \theta) + 1) - (\rho + \rho^2 t_2 \theta)) \right] \right. \\ \left. - \left[e^{\rho\theta t_2} (e^\theta (1 + \rho\theta t_2) - \rho\theta) \right] \left[(1 + \rho)e^{(1+\rho)\theta} - \rho e^{\theta\rho} \right] \right]$$

and,

$$a_{22} = -F^2 \left[(e^{\theta\rho}(e^\theta - 1)) \left[e^{\theta\rho t_2} (e^\theta (2\theta t_2 + \rho(\theta t_2)^2) - (\theta + \theta^2 \rho t_2)) \right] \right. \\ \left. - \left[e^{\rho\theta t_2} (e^\theta (1 + \rho\theta t_2) - \rho\theta) \right] \left[\theta e^{\rho\theta + \theta} - \theta e^{\theta\rho} \right] \right]$$

from (3.6), and (3.7) we obtain the asymptotic covariance matrix for IDP by using EPGF.

4- Simulation Study

In this thesis, we perform and obtain the performance of the unknown parameters of the two parameters intervened Poisson distribution, using the method of the first rapid method (ME), the second rapid method, the third rapid method, the fourth rapid method, with a comparative study between the previous methods of estimation.

The simulation study is performed depending on different sample sizes, particular $n=10, 20, 30$, and 50 . All the results are depending on 1000 replications and with different parameters values of the θ, ρ , and calculating the mean and median values of the parameters, the comparison bases on the criteria of mean square error as a measure of deviation for mean, and the median absolute deviation as a measure of deviation for median.

It is observed during the simulation, that there are outliers among the estimated parameters values, so it is noted that estimated values using first rapid method, the second rapid method, and fourth rapid method have extreme for both θ and ρ , additionally, the MSE for these methods denotes the high variability, but it is noted that both of third rapid method and ML estimation are closed to initials values for both θ and ρ , moreover they are more efficient among other methods. Accordingly, we can not bound the ML estimation for the rapid estimation methods.

For avoiding the outlier effect on the comparison among the different estimation methods for parameters, we have to depend on the median as a

robust or reliable estimation of center tendency for the estimated parameters, and MAD as robust measure for estimates values deviation.

Table (1) contains, the median values of the first rapid method (ME), the second rapid method, the third rapid method, the fourth rapid method, and ML for $\theta = 1$, $\rho = 0.9$, along with the MAD's between brackets,

Table(1) The Simulation Results Based on The Median (N=10, N=20)

Methods	N=10		N=20	
	$\hat{\theta}$	$\hat{\rho}$	$\hat{\theta}$	$\hat{\rho}$
The first method(ME)	0.938 (1.015)	0.197 (1.015)	0.9 (0.86)	0.267 (1.058)
The second method	0.018 (0.014)	55.96 (53.908)	0.16 (0.154)	6.164 (4.36)
The third method	0.804 (0.064)	1.027 (0.23)	0.767 (0.051)	1.0301 (0.141)
The fourth method	1.216 (0.823)	0.922 (1.21)	1.5 (1.27)	0.741 (0.739)
ML	0.975 (0.0049)	0.351 (0.00042)	0.926 (0.043)	0.353 (0.000334)

It is observed that, at in $n = 10$, for θ , the median values of the first rapid method (ME), the third rapid method, fourth rapid methods, and the ML estimates are closed to the initial values, the ML estimates lie between the first and the fourth rapid method estimates. According to MAD for error, it is noted that ML estimation is the most efficient among the estimation methods, but the third rapid method is the most efficient estimation among the rapid estimation methods. For ρ , the third and the fourth rapid methods are the closed values to the initial. the ML estimates lie between the first and the fourth estimates. According to MAD for error, it is noted that the third method and ML are more efficient as estimations for θ and ρ .

At in $n = 20$, for θ , the median values of the first rapid method(ME), the third rapid method, and the ML estimates are closed to the initial values.

The ML estimates lie between the first and the fourth estimates. According to MAD for error, it is noted that ML estimation is the most efficient among the estimation methods, and the third rapid method is the most efficient among the rapid estimation methods. For ρ , the third rapid method, and fourth rapid methods are closed to the initial values, the ML estimates lie between the first and the fourth rapid method estimates. According to MAD for error, it is noted that the third method and ML are more efficient as estimations for θ and ρ .

Table (2) contains, the median values of the first rapid method(ME), the second rapid method, the third rapid method, the fourth rapid method, and ML, for $\theta = 1$, $\rho = 0.9$, along with the MAD's between brackets

Table(2) The Simulation Results Based on The Median (N=30, N=50)

Methods	N=30		N=50	
	$\hat{\theta}$	$\hat{\rho}$	$\hat{\theta}$	$\hat{\rho}$
The first method(ME)	0.9 (0.863)	0.351 (1.087)	0.905 (0.865)	0.534 (1.216)
The second method	0.208 (0.177)	4.66 (2.608)	0.241 (0.135)	3.896 (1.54)
The third method	0.765 (0.041)	1.030 (0.141)	0.77 (0.041)	1.032 (0.093)
The fourth method	2.487 (2.069)	0.368 (0.351)	2.775 (2.296)	0.306 (0.292)
ML	0.8265 (0.015)	0.3650 (0.057)	0.87447 (0.0064)	0.432 (0.0037)

It is observed that, at $n = 30$ and $n = 50$, for θ , the median values the first rapid method(ME), the third rapid method, and the ML estimates are closed to the initial value of θ , the ML estimates lie between the first and the third sets of estimates, taking in consideration that the first rapid method is the closest. According to MAD for error, the ML and the third rapid estimation are more efficient than other methods. For ρ , the third rapid method is the closest

value, the ML estimates lie between the first and the fourth rapid method estimates. According to MAD for error, the ML and the third rapid estimation are more efficient than other methods.

Conclusion:

According to the simulation study results, the first rapid method, third rapid method, and the fourth rapid methods bound ML estimation with different combination at different sample sizes basing on median. The third rapid method is the closest to the initial value for both ρ and θ whether we base on median or mean, and it is the most efficient among the rapid estimation methods for both ρ and θ whether we base on MSE or MAD . On the other hand, the second rapid method is the worst method among the rapid estimation methods.

References

- 1- Dowling, M. and Nakamura, M.(1997). Estimating Parameter For Discrete Distribution Via The Empirical Probability Generating Function, Communications in Statistics, Simulation and computation,26,301-313
- 2- Kemp, C. and Kemp, A. (1988). Rapid Estimation For Discrete Distribution. Statistician, 37,243-255.
- 3- Scollnik, D. (1995). Bayesian Analysis of An Intervened Poisson Distribution, Communications in Statistics, Theory and Methods, 24, 735-754.
- 4- Shanmugam, R.(1985). An Intervened Poisson distribution and its Medical Application, Biometrics, 41, 1025-1029.

On The Kumaraswamy- Gumbel Minimum Distribution

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Abstract:

In this paper, the Kumaraswamy – Gumbel minimum (kw-Gu) distribution as a special model from the class of Kumaraswamy Generalized (kw-G) distributions is considered. Some properties of the density function (pdf), as the cumulative distribution function (cdf), moments, quantiles, the median, the mode, the mean deviation, the survival function, and hazard function, are obtained. Also, the entropy, order statistics, L-moments and parameters estimation based on maximum likelihood are obtained. Numerical illustration is used to study the properties of the parameters.

Keywords: Kumaraswamy generalized distributions, moments, order statistics, L-moments, entropy, maximum likelihood estimation.

1. Introduction :

Cordeiroa and Castro (2010) presented the class of Kw-G distributions where the pdf of the class of kw-G distributions was presented as follows:

$$f(x) = ab g(x) G(x)^{a-1} \left\{ 1 - G(x)^a \right\}^{b-1}; a, b > 0, -\infty < x < \infty \quad (1)$$

and the cdf is:

$$F(x) = 1 - \left\{ 1 - G(x)^a \right\}^b; a, b > 0, -\infty < x < \infty \quad (2)$$

Then, if a is real non integer, an expansion for density function of the class of kw-G distributions will be

$$f(x) = g(x) \sum_{i,j=0}^{\infty} \sum_{k=0}^j w_{i,j,k} G(x)^k, \quad (3)$$

where,

$$w_{i,j,k}(a,b) = (-1)^{i+j+k} ab \binom{a(i+1)-1}{j} \binom{b-1}{i} \binom{j}{k}$$

A simplified expansion for $F(x)^r$ was used by Cordeiroa and Castro (2010), as follows:

$$F(x)^r = \sum_{q=0}^{\infty} d_{q,r}(a,b) G(x)^q, \quad (4)$$

where,

$$d_{q,r}(a,b) = \sum_{w=0}^r \binom{r}{w} (-1)^w \sum_{m=0}^{\infty} \sum_{p=q}^{\infty} (-1)^{m+p+q} \binom{wb}{m} \binom{ma}{p} \binom{p}{q}$$

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For integer $b > 0$, the index i stops at $b - 1$ and for integer $a > 0$ the index j stops at $a(i + 1) - 1$.

2. The Kumaraswamy- Gumbel Minimum Distribution

The pdf and cdf of the Gumbel minimum (minimum extreme events) distribution with location parameter $\mu > 0$ and scale parameter $\sigma > 0$, respectively are :

$$g(x) = \frac{1}{\sigma} \exp \left\{ \frac{x - \mu}{\sigma} - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} ; x, \mu \in R, \sigma > 0 \quad (5)$$

and,

$$G(x) = 1 - \exp \left\{ - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} ; x, \mu \in R, \sigma > 0 \quad (6)$$

Substituting from equation (5) and (6) into equation (1) yields the probability density function of the kw-Gu distribution as:

$$f(x) = ab \frac{1}{\sigma} \exp \left\{ \frac{x - \mu}{\sigma} - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} \left[1 - \exp \left\{ - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} \right]^{a-1} \left\{ 1 - \left[1 - \exp \left\{ - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} \right]^a \right\}^{b-1} ; a, b, \sigma > 0, -\infty < x, \mu < \infty \quad (7)$$

Substituting from equation (6) into equation (2), yields cumulative distribution function of the kw-Gu distribution .

$$F(x) = 1 - \left\{ 1 - \left[1 - \exp \left\{ - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} \right]^a \right\}^b ; a, b, > 0, -\infty < x < \infty \quad (8)$$

It is noted that, when $a=1$ they give the pdf and cdf of the exponentiated minimum Gumbel distribution. For $a=1$ and $b=1$ equation (7) and (8) give the pdf and cdf of the ordinary minimum Gumbel distribution.

General Expansion for the kw-Gu Distribution Density Function:

A general expansion for the kw-Gu distribution density function can be yielded by substituting from equation (5) and (6) into equation (3) as follows:

$$f(x) = \frac{1}{\sigma} \sum_{i,j=0}^{\infty} w_{i,j} \exp \left\{ \frac{x - \mu}{\sigma} - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} \left[\exp \left\{ - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} \right]^j \quad (9)$$

where,

$$w_{i,j}(a,b) = (-1)^{i+j} ab \binom{a(i+1)-1}{j} \binom{b-1}{i}, \quad \sum_{i,j=0}^{\infty} \frac{1}{j+1} w_{i,j} = 1$$

For integer $b > 0$, the index i stops at $b-1$ and for integer $a > 0$ the index j stops at $a(i+1)-1$.

3. Some properties for the kw-Gu distribution

3.1 Moments

In this section some properties for the kw-Gu distribution will be calculated as follows:

Moments of the kw-Gu distribution can be given simply from (9) as follows:

$$\mu^r = \int_{-\infty}^{\infty} x^r \cdot \frac{1}{\sigma} \sum_{i,j=0}^{\infty} w_{i,j} \left[\exp \left\{ -\exp \left(\frac{x-\mu}{\sigma} \right) \right\} \right]^j \exp \left\{ \frac{x-\mu}{\sigma} - \exp \left(\frac{x-\mu}{\sigma} \right) \right\} dx$$

Then,

$$\mu^r = \sum_{i,j=0}^{\infty} w_{i,j} I \quad (10)$$

where,

$$I = \frac{1}{\sigma} \int_{-\infty}^{\infty} x^r e^{-j} e^{\frac{x-\mu}{\sigma}} e^{\frac{x-\mu}{\sigma}} e^{-e^{\frac{x-\mu}{\sigma}}} dx$$

Using the following substitutes

$$u = e^{\frac{x-\mu}{\sigma}}, \quad du = e^{\frac{x-\mu}{\sigma}} \frac{1}{\sigma} dx, \quad x = \mu + \sigma \ln u$$

yields,

$$I = \int_0^{\infty} (\mu + \sigma \ln u)^r e^{-(j+1)u} du$$

Using binomial expansion yields:

$$I = \sum_{m=0}^r (-\sigma)^m \binom{r}{m} \mu^{r-m} I_{(m,j)} \quad (11)$$

where
$$I_{(m,j)} = \int_0^{\infty} (\ln u)^m \cdot e^{-(j+1)u} du$$

Now, $I(m, j)$ can be calculated by using the following fact, as it was presented in Prudnikov, et al.(1986), Nadarajah and S. Kotz .(2004) :

$$I(m, j) = \left(\frac{\partial}{\partial \alpha} \right)^m \left[(1+j)^{-\alpha} \Gamma(\alpha) \right] ; \alpha = 1 \quad (12)$$

where $m = 0, 1, 2, \dots$ and $(1+j) > 0$.

Then from equations (12) and (11) into equation (10) yields,

$$\mu^r = \sum_{i,j=0}^{\infty} w_{i,j} \cdot \sum_{m=0}^r (-\sigma)^m \binom{r}{m} \mu^{r-m} \left(\frac{\partial}{\partial \alpha} \right)^m \left[(1+j)^{-\alpha} \Gamma(\alpha) \right] ; \alpha = 1 \quad (13)$$

and it can be summarized from equation (12) as :

$$\left. \begin{aligned} I_{(0,j)} &= (1+j)^{-1} \\ I_{(1,j)} &= (1+j)^{-1} [-\ln(1+j)] - (1+j)^{-1} \gamma \\ I_{(2,j)} &= \frac{[\ln(1+j)]^2}{(1+j)} + \frac{2\gamma \ln(1+j)}{(1+j)} + \frac{\gamma^2}{(1+j)} + \frac{\pi^2}{(1+j)6} \end{aligned} \right\} \quad (14)$$

where, γ is Euler constant

From equation (13) and $r=1$ yields that:

$$E(X) = \sum_{i,j=0}^{\infty} w_{i,j} \left[\left(\mu I_{(0,j)} + \sigma I_{(1,j)} \right) \right]$$

and substituting from (14) into the last equation, yields:

$$E(X) = \sum_{i,j=0}^{\infty} w_{i,j} \left[\frac{\mu}{j+1} - \sigma \left(\frac{\gamma}{j+1} + \frac{\ln(j+1)}{j+1} \right) \right]$$

Similarly,

$$\begin{aligned} E(X^2) = \sum_{i,j=0}^{\infty} w_{i,j} & \left[\sigma^2 \left(\frac{\gamma^2}{j+1} + \frac{[\ln(j+1)]^2}{j+1} + \frac{\pi^2}{6(j+1)} + \frac{2\gamma \ln(j+1)}{j+1} \right) + \frac{\mu^2}{j+1} \right. \\ & \left. - 2\mu\sigma \left(\frac{\gamma}{j+1} + \frac{\ln(j+1)}{j+1} \right) \right] \end{aligned}$$

where $w_{i,j} = (-1)^{i+j} \binom{a(i+1)-1}{j} \binom{b-1}{i}$

3.2 Quantile function and The Median

Starting with the well known definition of the 100 q-th quantile, it is clear that

$$q = P(X \leq x_q) = F(x_q); \quad x_q > 0, 0 < q < 1$$

Quantiles of the kw-Gu distribution can be yielded by equating the cdf with q :

$$q = 1 - \left\{ 1 - \left(1 - \exp \left\{ - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} \right)^a \right\}^b; \quad a, b, \sigma > 0, -\infty < x, \mu < \infty$$

$$\text{Then, } x_q = \mu + \sigma \ln \left[-\ln \left(1 - \sqrt[b]{1 - \sqrt[a]{1 - q}} \right) \right]$$

Obviously, by substituting q by 1/2 (the second quartile) the median will be got. So, the median is

$$M = \mu + \sigma \ln \left[-\ln \left(1 - \sqrt[b]{1 - \sqrt[a]{\frac{1}{2}}} \right) \right]$$

3.3 The Mode

The mode of The kw-Gu distribution can be got by differentiating the normal logarithm of equation (7) with respect to x as follows :

$$\begin{aligned} f'(x) = [f(x)]^2 & \left[\left(\frac{1}{\sigma} - \frac{u}{\sigma} \right) [1 - e^{-u}]^{2-2a} \left[1 - [1 - e^{-u}]^a \right]^{2-2b} \frac{\sigma^2}{a^2 b^2} \right. \\ & - (b-1) u e^{-u} [1 - e^{-u}]^{1-a} \left[1 - [1 - e^{-u}]^a \right]^{1-2b} \frac{\sigma}{a b^2} \\ & \left. + (a-1) u e^{-u} [1 - e^{-u}]^{1-2a} \left[1 - [1 - e^{-u}]^a \right]^{2-2b} \frac{\sigma}{a^2 b^2} \right] \end{aligned}$$

$$\text{where, } u = \exp \left(\frac{x - \mu}{\sigma} \right)$$

Then equating it with zero .

$$\begin{aligned} & \left[\left(\frac{1}{\sigma} - \frac{u}{\sigma} \right) [1 - e^{-u}]^{2-2a} \left[1 - [1 - e^{-u}]^a \right]^{2-2b} \frac{\sigma}{a} \right. \\ & - (b-1) u e^{-u} [1 - e^{-u}]^{1-a} \left[1 - [1 - e^{-u}]^a \right]^{1-2b} \\ & \left. + (a-1) u e^{-u} [1 - e^{-u}]^{1-2a} \left[1 - [1 - e^{-u}]^a \right]^{2-2b} \frac{1}{a} \right] = 0 \end{aligned}$$

But the last equation is a nonlinear equation and does not have an analytic solution with respect to x , Therefore, we have to solve it numerically.

If x_0 is a zero of $f(x)$, then $f''(x_0) < 0$

Now,

$$f''(x) = B'1 - B'2 + B'3,$$

where,

$$B1 = \left(\frac{1}{\sigma} - \frac{u}{\sigma} \right) \left[1 - e^{-u} \right]^{2-2a} \left[1 - \left[1 - e^{-u} \right]^a \right]^{2-2b} \frac{\sigma}{a}$$

$$B2 = (b-1) u e^{-u} \left[1 - e^{-u} \right]^{1-a} \left[1 - \left[1 - e^{-u} \right]^a \right]^{1-2b}$$

$$B3 = (a-1) u e^{-u} \left[1 - e^{-u} \right]^{1-2a} \left[1 - \left[1 - e^{-u} \right]^a \right]^{2-2b} \frac{1}{a}$$

Then,

$$\begin{aligned} B'1 &= (2b-2) e^{-u} u \left(\frac{1}{\sigma} - \frac{u}{\sigma} \right) \left[1 - \left(1 - e^{-u} \right)^a \right]^{1-2b} \left[1 - e^{-u} \right]^{1-a} \\ &\quad - \frac{1}{a\sigma} u \left[1 - \left(1 - e^{-u} \right)^a \right]^{2-2b} \left[1 - e^{-u} \right]^{2-2a} \\ &\quad - \frac{1}{a} (2a-2) e^{-u} u \left(\frac{1}{\sigma} - \frac{u}{\sigma} \right) \left[1 - \left(1 - e^{-u} \right)^a \right]^{2-2b} \left[1 - e^{-u} \right]^{1-2a} \\ B'2 &= \frac{1}{\sigma} (b-1) e^{-u} u \left[1 - \left(1 - e^{-u} \right)^a \right]^{1-2b} \left[1 - e^{-u} \right]^{1-a} \\ &\quad - \frac{1}{\sigma} (b-1) e^{-u} u^2 \left[1 - \left(1 - e^{-u} \right)^a \right]^{1-2b} \left[1 - e^{-u} \right]^{1-a} \\ &\quad + \frac{a e^{-2u} u^2 (b-1)(2b-1)}{\sigma \left[1 - \left(1 - e^{-u} \right)^a \right]^{2b}} - \frac{(a-1)(b-1) e^{-2u} u^2 \left[1 - \left(1 - e^{-u} \right)^a \right]^{1-2b}}{\sigma \left[1 - e^{-u} \right]^a} \end{aligned}$$

$$\begin{aligned}
 B'3 = & \frac{1}{a\sigma} (a-1) e^{-u} u \left[1 - \left(1 - e^{-u} \right)^a \right]^{2-2b} \left[1 - e^{-u} \right]^{1-2a} \\
 & - \frac{1}{a\sigma} (a-1) e^{-u} u^2 \left[1 - \left(1 - e^{-u} \right)^a \right]^{2-2b} \left[1 - e^{-u} \right]^{1-2a} \\
 & - \frac{(a-1)(2a-1) e^{-2u} u^2 \left[1 - \left(1 - e^{-u} \right)^a \right]^{2-2b}}{a\sigma \left[1 - e^{-u} \right]^{2a}} \\
 & + \frac{1}{\sigma} (a-1)(2b-2) e^{-u} u^2 \left[1 - \left(1 - e^{-u} \right)^a \right]^{1-2b} \left[1 - e^{-u} \right]^{-a}
 \end{aligned}$$

3.4 The Mean Deviation

Basically, the mean deviation is a measure for the amount of scatter in X, Generally, the mean deviation about the mean and about the median is expressed respectively by

$$\delta_1(x) = \int_{-\infty}^{\infty} |x - \mu| f(x) dx \quad \text{and} \quad \delta_2(x) = \int_{-\infty}^{\infty} |x - M| f(x) dx,$$

where μ is the mean and M is the median.

These measures can be expressed easily by:

$$\delta_1(X) = 2\mu F(\mu) - 2T(\mu) \quad \text{and} \quad \delta_2(X) = \mu - 2T(M)$$

$$\text{where } T(q) = \int_{-\infty}^q x f(x) dx$$

from equation (9) the last equation will be:

$$T(q) = \frac{1}{\sigma} \sum_{i,j=0}^{\infty} w_{i,j} I(q)$$

where,

$$I(q) = \int_{-\infty}^q x \left[\exp \left\{ - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} \right]^j \exp \left\{ \frac{x - \mu}{\sigma} - \exp \left(\frac{x - \mu}{\sigma} \right) \right\} dx$$

Using the following substitutes into the last equation :

$$u = e^{\frac{x - \mu}{\sigma}}, \quad du = e^{\frac{x - \mu}{\sigma}} \frac{1}{\sigma} dx, \quad x = \mu + \sigma \ln u$$

then,

$$I(q) = \sigma \mu \int_0^{\exp\{(q-\mu)/\sigma\}} \exp\{-u(j+1)\} du + \sigma^2 \int_0^{\exp\{(q-\mu)/\sigma\}} (\ln u) \exp\{-u(j+1)\} du$$

Calculating the first integration yields:

$$\sigma \mu \int_0^{\exp\{(q-\mu)/\sigma\}} \exp\{-u(j+1)\} du = \frac{\sigma \mu}{(j+1)} \left[1 - \exp\left\{-(j+1) \exp\left(\frac{q-\mu}{\sigma}\right)\right\} \right]$$

However, the second integration

$$\sigma^2 \int_0^{\exp\{(q-\mu)/\sigma\}} (\ln u) \exp\{-u(j+1)\} du$$

needs to a numerical integration. Finally,

$$T(q) = \frac{1}{\sigma} \sum_{i,j=0}^{\infty} w_{i,j} \left\{ \frac{\sigma \mu}{(j+1)} \left[1 - \exp\left\{-(j+1) \exp\left(\frac{q-\mu}{\sigma}\right)\right\} \right] + \sigma^2 \int_0^{\exp\{(q-\mu)/\sigma\}} (\ln u) \exp\{-u(j+1)\} du \right\}$$

Clearly, $F(\mu)$ are easily calculated from equation (8) and from $T(q)$ the measures $\delta_1(X)$ and $\delta_2(X)$ can be easily calculated.

4. Order Statistics

A simple random sample $X_1, X_2, \dots, X_n$ given from the kw-Gu distribution where X 's are i.i.d random variables, Arnold et al.(1992), The density $f_{u:n}(x_u)$ of the u -th order statistic, for $u = 1, 2, \dots, n$ is given by:

$$f_{u:n}(x_u) = \frac{f(x_u)}{\beta(u, n-u+1)} F(x_u)^{u-1} \{1 - F(x_u)\}^{n-u}$$

Using binomial expansion, yields:

$$f_{u:n}(x_u) = \frac{f(x_u)}{\beta(u, n-u+1)} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v} F(x_u)^{u+v-1} \quad (15)$$

from (6):

$$F(x_u)^{u+v-1} = \sum_{q=0}^{\infty} d_{q, u+v-1}(a, b) G(x_u)^q$$

where,

$$d_{q, u+v-1}(a, b) = \sum_{w=0}^{u+v-1} \binom{u+v-1}{w} (-1)^w \sum_{m=0}^{\infty} \sum_{p=q}^{\infty} (-1)^{m+p+q} \binom{wb}{m} \binom{ma}{p} \binom{p}{q}$$

Substitution from (6) in to last equation, yields:

$$F(x_u)^{u+v-1} = \sum_{q=0}^{\infty} d_{q,u+v-1}(a,b) \left[1 - \exp \left\{ -\exp \left(\frac{x_u - \mu}{\sigma} \right) \right\} \right]^q$$

Then,

$$F(x_u)^{u+v-1} = \sum_{s=0}^q H_{q,s}(a,b) \left[\exp \left\{ -\exp \left(\frac{x_u - \mu}{\sigma} \right) \right\} \right]^s \quad (16)$$

where,

$$H_{q,s} = (-1)^s \binom{q}{s} \sum_{q=0}^{\infty} d_{q,u+v-1}(a,b)$$

Substituting from (9) and (16) into (15),yields:

$$f_{u:n}(x_u) = \frac{\frac{1}{\sigma} \sum_{i,j=0}^{\infty} w_{i,j} \left[\exp \left\{ -\exp \left(\frac{x_u - \mu}{\sigma} \right) \right\} \right]^j}{\beta(u, n-u+1)} \exp \left\{ \frac{x_u - \mu}{\sigma} - \exp \left(\frac{x_u - \mu}{\sigma} \right) \right\}$$

$$\cdot \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v} \sum_{s=0}^q H_{(q,s)} \left[\exp \left\{ -\exp \left(\frac{x_u - \mu}{\sigma} \right) \right\} \right]^s$$

Then,

$$f_{u:n}(x_u) = \frac{\frac{1}{\sigma} \sum_{i,j=0}^{\infty} w_{i,j} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v} \sum_{s=0}^q H_{(q,s)}}{\beta(u, n-u+1)} \quad (17)$$

$$\cdot \left[\exp \left\{ -\exp \left(\frac{x_u - \mu}{\sigma} \right) \right\} \right]^{s+j} \exp \left\{ \frac{x_u - \mu}{\sigma} - \exp \left(\frac{x_u - \mu}{\sigma} \right) \right\}$$

where,

$$w_{i,j}(a,b) = (-1)^{i+j} ab \binom{a(i+1)-1}{j} \binom{b-1}{i}, H_{q,s} = (-1)^s \binom{q}{s} \sum_{q=0}^{\infty} d_{q,u+v-1}(a,b)$$

Moments of Order Statistics

Basically, moments of order statistics, Arnold et al.(1988)., is defined by :

$$E(X_{u:n}^r) = \int_{-\infty}^{\infty} x_u^r f_{u:n}(x_u) dx_u$$

Substituting from equation (17) into the last equation yields:

$$E(X_{u:n}^r) = \frac{1}{\sigma} \frac{1}{\beta(u, n-u+1)} \sum_{i,j=0}^{\infty} w_{i,j} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v} \sum_{s=0}^q H_{(q,s)}$$

$$\int_{-\infty}^{\infty} x_u^r \left[\exp \left\{ -\exp \left(\frac{x_u - \mu}{\sigma} \right) \right\} \right]^{s+j} \exp \left\{ \frac{x_u - \mu}{\sigma} - \exp \left(\frac{x_u - \mu}{\sigma} \right) \right\} dx_u$$

Using the following substitutes into the last equation:

$$u = e^{\frac{x-\mu}{\sigma}}, \quad du = e^{\frac{x-\mu}{\sigma}} \frac{1}{\sigma} dx, \quad x = \mu + \sigma \ln u$$

yields,

$$E(X_{u:n}^r) = \frac{\sum_{i,j=0}^{\infty} w_{i,j} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v}}{\beta(u, n-u+1)} \sum_{s=0}^q H_{(q,s)} I \quad (18)$$

where,

$$I = \int_0^{\infty} (\mu + \sigma \ln u)^r \exp \{ -(j+s+1)u \} du$$

Using binomial expansion for $(\mu + \sigma \ln u)^r$, yields:

$$I = \sum_{m=0}^r (\sigma)^m \binom{r}{m} \mu^{r-m} I(m, j+s) \quad (19)$$

where,

$$I(m, j+s) = \int_0^{\infty} (\ln u)^m \exp \{ -(j+s+1)u \} du$$

$I(m, j+s)$ can be calculated by using the following fact as it was presented in

Prudnikov et al.(1986), Nadarajah, S. and Kotz, S.(2004)

$$I(m, j+s) = \left(\frac{\partial}{\partial \alpha} \right)^m \left[(1+j+s)^{-\alpha} \Gamma(\alpha) \right]; \alpha = 1 \quad (20)$$

where, $m = 0, 1, 2, \dots$, and $(1+j+s) > 0$.

Combining equation (18), (19), and (20) yields:

$$E(X_{u:n}^r) = \frac{\sum_{i,j=0}^{\infty} w_{i,j} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v}}{\beta(u, n-u+1)} \sum_{s=0}^q H_{(q,s)} \sum_{m=0}^r (\sigma)^m \binom{r}{m} \mu^{r-m} \left(\frac{\partial}{\partial \alpha} \right)^m \left[(j+s+1)^{-\alpha} \Gamma(\alpha) \right]$$

On the other hand,

$$\left. \begin{aligned} I_{(0,j+s)} &= (1+j+s)^{-1} \\ I_{(1,j+s)} &= -(1+j+s)^{-1} [\ln(1+j+s)] - (1+j+s)^{-1} \gamma \\ I_{(2,j+s)} &= \frac{[\ln(1+j+s)]^2}{(1+j+s)} + \frac{2\gamma \ln(1+j+s)}{(1+j+s)} + \frac{\gamma^2}{(1+j+s)} + \frac{\pi^2}{(1+j+s)6} \end{aligned} \right\} \quad (21)$$

Then at $r=1$ yields,

$$E(X_{u:n}) = \frac{\sum_{i,j=0}^{\infty} w_{i,j} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v}}{\beta(u, n-u+1)} \sum_{s=0}^q H_{(q,s)} \left[\left(\mu I_{(0,j+s)} + \sigma I_{(1,j+s)} \right) \right]$$

Substituting from (21) in to the last equation, yields:

$$E(X_{u:n}) = \frac{\sum_{i,j=0}^{\infty} w_{i,j} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v}}{\beta(u, n-u+1)} \sum_{s=0}^q H_{(q,s)} \left[-\sigma \left(\frac{\gamma}{j+s+1} + \frac{\ln(j+s+1)}{j+s+1} \right) + \frac{\mu}{j+s+1} \right]$$

Similarly,

$$\begin{aligned} E(X_{u:n}^2) &= \frac{\sum_{i,j=0}^{\infty} w_{i,j} \sum_{v=0}^{n-u} (-1)^v \binom{n-u}{v}}{\beta(u, n-u+1)} \sum_{s=0}^q H_{(q,s)} \\ &\quad \left[\sigma^2 \left(\frac{\gamma^2}{j+s+1} + \frac{[\ln(j+s+1)]^2}{j+s+1} + \frac{\pi^2}{6(j+s+1)} + \frac{2\gamma \ln(j+s+1)}{j+s+1} \right) \right. \\ &\quad \left. + \frac{\mu^2}{j+s+1} - 2\mu \sigma \left(\frac{\gamma}{j+s+1} + \frac{\ln(j+s+1)}{j+s+1} \right) \right] \end{aligned}$$

where, γ is euler constant.

5. L-Moments

L-moments are linear functions of expected order statistics, and L-moments is defined as it was presented in Hosking (1990), Abd el- Sattar (2009), by :

$$\lambda_{\alpha+1} = (\alpha+1)^{-1} \sum_{t=0}^{\alpha} (-1)^t \binom{\alpha}{t} E(x_{\alpha+1-t:\alpha+1}), \alpha = 0, 1, \dots$$

The first four L-moments are

$$\begin{aligned} \lambda_1 &= E(X_{1:1}), \quad \lambda_2 = \frac{1}{2} E(X_{2:2} - 2X_{1:2}), \quad \lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) \\ \lambda_4 &= \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) \end{aligned}$$

Therefore,

$$\lambda_1 = \sum_{i,j=0}^{\infty} w_{i,j} \sum_{s=0}^q H_{(q,s)} \left[-\sigma \left(\frac{\gamma}{j+s+1} + \frac{\ln(j+s+1)}{j+s+1} \right) + \frac{\mu}{j+s+1} \right]$$

where,

$$w_{i,j}(a,b) = (-1)^{i+j} a b \binom{a(i+1)-1}{j} \binom{b-1}{i}, \quad H_{q,s} = (-1)^s \binom{q}{s} \sum_{q=0}^{\infty} d_{q,1}(a,b)$$

$$d_{q,1}(a,b) = \sum_{w=0}^1 \binom{1}{w} (-1)^w \sum_{m=0}^{\infty} \sum_{p=q}^{\infty} (-1)^{m+p+q} \binom{wb}{m} \binom{ma}{p} \binom{p}{q}$$

Hence, λ_1, λ_2 and λ_n can be obtained similarly.

6. The Survival and Hazard Function

Generally, the survival function of the kw-Gu distribution is:

$$S(x) = \left\{ 1 - \left\{ 1 - \exp \left\{ - \exp \left(\frac{x-\mu}{\sigma} \right) \right\} \right\}^a \right\}^b$$

Hazard function is expressed by

$$H(x) = \frac{ab \frac{1}{\sigma} \left\{ 1 - \exp \left\{ - \exp \left(\frac{x-\mu}{\sigma} \right) \right\} \right\}^{a-1} \exp \left\{ \frac{x-\mu}{\sigma} - \exp \left(\frac{x-\mu}{\sigma} \right) \right\}}{\left[1 - \left\{ 1 - \exp \left\{ - \exp \left(\frac{x-\mu}{\sigma} \right) \right\} \right\}^a \right]}$$

7. Rényi Entropy

The entropy of a random variable is a measure of variation of the uncertainty, The entropy has been used in various situations in science and engineering, the Rényi entropy is defined by:

$$e_R(\rho) = \frac{1}{1-\rho} \log \left[\int_{-\infty}^{\infty} f(x)^\rho dx \right]$$

Substituting from equation (7) into the last equation, yields:

$$e_R(\rho) = \frac{1}{1-\rho} \log \left[\int_{-\infty}^{\infty} \left(\frac{a-b}{\sigma} \right)^\rho \left[\exp \left\{ \frac{x-\mu}{\sigma} - \exp \left(\frac{x-\mu}{\sigma} \right) \right\} \right]^\rho \left[1 - \exp \left\{ -\exp \left(\frac{x-\mu}{\sigma} \right) \right\} \right]^{\rho(a-1)} \right. \\ \left. \left\{ 1 - \left[1 - \exp \left\{ -\exp \left(\frac{x-\mu}{\sigma} \right) \right\} \right]^a \right\}^{\rho(b-1)} dx \right]$$

If a and b are real non integer, and using binomial expansions, yields:

$$e_R(\rho) = \frac{1}{1-\rho} \log \left[\sum_{i,j=0}^{\infty} n_{i,j} \int_{-\infty}^{\infty} \frac{1}{\sigma^{\rho-1}} \left[\exp \left(\frac{x-\mu}{\sigma} \right) \right]^{\rho-1} \right. \\ \left. \frac{1}{\sigma} \left[\exp \left(\frac{x-\mu}{\sigma} \right) \right] \frac{1}{\sigma^\rho} \exp \left\{ -(j+\rho) \exp \left(\frac{x-\mu}{\sigma} \right) \right\} dx \right]$$

where,

$$n_{i,j} = (ab)^\rho \binom{\rho(a-1)+ai}{j} \binom{\rho(b-1)}{i} (-1)^{i+j}$$

Using the following substitutes into the last equation:

$$u = e^{-\frac{x-\mu}{\sigma}}, \quad du = e^{-\frac{x-\mu}{\sigma}} \frac{1}{\sigma} dx$$

yields,

$$e_R(\rho) = \frac{1}{1-\rho} \log \left[\sum_{i,j=0}^{\infty} n_{i,j} \int_0^1 \frac{1}{(j+\rho)^{\rho-1} \sigma^{2\rho-1}} [(j+\rho)u]^{\rho-1} \exp \{ -(j+\rho)u \} du \right]$$

Finally,

$$e_R(\rho) = \frac{1}{1-\rho} \log \left[\sum_{i,j=0}^{\infty} n_{i,j} \frac{1}{(j+\rho)^{\rho-1} \sigma^{2\rho-1}} \Gamma(\rho) \right]$$

where , $n_{i,j} = (ab)^{\rho} \binom{\rho(a-1)+ai}{j} \binom{\rho(b-1)}{i} (-1)^{i+j}$

8. parameters Estimation for The kw-Gu Distribution

The purpose of the present section is to estimate the kw-Gu distribution parameters based on maximum likelihood method.

Given the iid random variables $X_1, X_2, \dots, X_n$ from the kw-Gu (a, b, μ, σ) distribution

Hence force, let θ is the parameters vector

Suppose that $u_i = e^{-\frac{x_i - \mu}{\sigma}}$, $v_i = \frac{x_i - \mu}{\sigma}$,

then the likelihood function will be:

$$L(\theta; x_i) = \left[\frac{ab}{\sigma} \right]^n e^{\sum_{i=1}^n (v_i - u_i)} \prod_{i=1}^n \left[1 - e^{-u_i} \right]^{a-1} \prod_{i=1}^n \left[1 - \left(1 - e^{-u_i} \right)^a \right]^{b-1}$$

Hence, the log likelihood function will be:

$$\ell(\theta; x) = n \log \frac{ab}{\sigma} + \sum_{i=1}^n (v_i - u_i) + (a-1) \sum_{i=1}^n \log \left[1 - e^{-u_i} \right] + (b-1) \sum_{i=1}^n \log \left[1 - \left(1 - e^{-u_i} \right)^a \right]$$

Differentiating $\ell(\theta; x)$ with respect to a yields:

$$\frac{\partial \ell(\theta; x)}{\partial a} = \frac{n}{a} + \sum_{i=1}^n \log \left\{ 1 - e^{-u_i} \right\} \left\{ 1 - \sum_{i=1}^n \frac{(b-1)}{\left[1 - e^{-u_i} \right]^{-a} - 1} \right\} \quad (22)$$

Differentiating $\ell(\theta; x)$ with respect to b yields:

$$\frac{\partial \ell(\theta; x)}{\partial b} = \frac{n}{b} + \sum_{i=1}^n \log \left\{ 1 - \left[1 - e^{-u_i} \right]^a \right\} \quad (23)$$

Differentiating $\ell(\theta; x)$ with respect to μ yields:

$$\frac{\partial \ell(\theta; x)}{\partial \mu} = \sum_{i=1}^n \left\{ \left[-\frac{1}{\sigma} + \frac{1}{\sigma} u_i \right] - \frac{1}{\sigma} \frac{u_i e^{-u_i}}{1 - e^{-u_i}} \left[(a-1) - \frac{a(b-1)}{\left[1 - e^{-u_i} \right]^{-a} - 1} \right] \right\} \quad (24)$$

Differentiating $\ell(\theta; x)$ with respect to σ yields:

$$\frac{\partial \ell(a, b, \mu, \sigma; x)}{\partial \sigma} = \sum_{i=1}^n \left\{ \left\{ -\frac{1}{\sigma} - \frac{v_i}{\sigma} [1 - u_i] \right\} - \frac{1}{\sigma} \frac{v_i u_i e^{-u_i}}{1 - e^{-u_i}} \left[(a-1) - \frac{a(b-1)}{\left[1 - e^{-u_i} \right]^{-a} - 1} \right] \right\} \quad (25)$$

The Variance Covariance Matrix

$$\text{Let } A = \exp\left(\frac{x_i - \mu}{\sigma}\right), \text{ and } B = \exp\left\{-\exp\left(\frac{x_i - \mu}{\sigma}\right)\right\}$$

Then the second partial derivatives of ℓ is :

$$\begin{aligned} \frac{\partial^2 \ell}{\partial a^2} &= 2(b-1) \sum_{i=1}^n \ln(1-B) \sum_{i=1}^n \frac{(1-B)^a \ln(1-B)}{(1-B)^a - 1} - \frac{n}{a^2} \\ &+ \sum_{i=1}^n \left[\frac{(1-B)^a \ln(1-B)^2}{(1-B)^a - 1} - \frac{(1-B)^{2a} \ln(1-B)^2}{\left[(1-B)^a - 1\right]^2} \right] (a-1)(b-1) \sum_{i=1}^n \ln(1-b) \end{aligned}$$

$$\frac{\partial^2 \ell}{\partial b^2} = -\frac{n}{b^2}$$

$$\frac{\partial^2 \ell}{\partial a \partial b} = \sum_{i=1}^n \ln[1 - (1-B)^a] \sum_{i=1}^n \ln(1-B) + (a-1) \sum_{i=1}^n \ln(1-B) \sum_{i=1}^n \frac{(1-B)^a \ln(1-B)^a}{(1-B)^a - 1}$$

$$\frac{\partial^2 \ell}{\partial a \partial \sigma} = -\frac{1}{\sigma^2} \sum_{i=1}^n \frac{B A(\mu - x_i)}{(B-1)} + \frac{(b-1)}{\sigma^2} \sum_{i=1}^n \frac{B A(\mu - x_i)}{\left[\frac{1}{(1-B)^a} - 1\right] (B-1)} + \frac{a(b-1)}{\sigma^2} \sum_{i=1}^n \frac{B A(\mu - x_i) \ln(1-B)}{\left[\frac{1}{(1-B)^a} - 1\right] (B-1) (1-B)^a}$$

$$\frac{\partial^2 \ell}{\partial b \partial \sigma} = \frac{a}{\sigma^2} \sum_{i=1}^n \frac{B A(\mu - x_i)}{\left[\frac{1}{(1-B)^a} - 1\right] (B-1)}$$

$$\frac{\partial^2 \ell}{\partial a \partial \mu} = \frac{1}{\sigma} \sum_{i=1}^n \frac{B A}{(B-1)} - \frac{(b-1)}{\sigma} \sum_{i=1}^n \frac{B A}{(B-1) \left[\frac{1}{(1-B)^a} - 1\right]} - \frac{a(b-1)}{\sigma} \sum_{i=1}^n \frac{B A \ln(1-B)}{(B-1) \left[\frac{1}{(1-B)^a} - 1\right]^2 (1-B)^a}$$

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \mu \partial \sigma} &= \frac{n}{\sigma^2} - \frac{1}{\sigma^2} \sum_{i=1}^n A + \frac{1}{\sigma^3} \sum_{i=1}^n A(\mu - x_i) + \frac{(a-1)}{\sigma^3} \sum_{i=1}^n \frac{(\mu - x_i) A B}{(B-1)} - \frac{(a-1)}{\sigma^2} \sum_{i=1}^n \frac{A B}{(B-1)} \\
&\quad - \frac{(a-1)}{\sigma^3} \sum_{i=1}^n \frac{(\mu - x_i) A^2 B}{(B-1)} + \frac{(a-1)}{\sigma^3} \sum_{i=1}^n \frac{(\mu - x_i) A^2 B^2}{(B-1)^2} + \frac{a(b-1)}{\sigma^2} \sum_{i=1}^n \frac{A B}{(B-1) \left[\frac{1}{(1-B)^a} - 1 \right]} \\
&\quad - \frac{a(b-1)}{\sigma^3} \sum_{i=1}^n \frac{(\mu - x_i) A B}{(B-1) \left[\frac{1}{(1-B)^a} - 1 \right]} + \frac{a(b-1)}{\sigma^3} \sum_{i=1}^n \frac{(\mu - x_i) A^2 B}{(B-1) \left[\frac{1}{(1-B)^a} - 1 \right]} \\
&\quad - \frac{a(b-1)}{\sigma^3} \sum_{i=1}^n \frac{(\mu - x_i) A^2 B^2}{(B-1)^2 \left[\frac{1}{(1-B)^a} - 1 \right]} - \frac{a^2(b-1)}{\sigma^3} \sum_{i=1}^n \frac{(\mu - x_i) A^2 B^2}{(B-1)(1-B)^{a+1} \left[\frac{1}{(1-B)^a} - 1 \right]^2}
\end{aligned}$$

$$\frac{\partial^2 \ell}{\partial b \partial \mu} = -\frac{a}{\sigma} \sum_{i=1}^n \frac{B A}{(B-1) \left[\frac{1}{(1-B)^a} - 1 \right]}$$

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \mu^2} &= -\frac{1}{\sigma^2} \sum_{i=1}^n A + \frac{(a-1)}{\sigma^2} \sum_{i=1}^n \frac{B A^2}{(B-1)} - \frac{(a-1)}{\sigma^2} \sum_{i=1}^n \frac{B A}{(B-1)} - \frac{(a-1)}{\sigma^2} \sum_{i=1}^n \frac{B^2 A^2}{(B-1)^2} \\
&\quad - \frac{a(b-1)}{\sigma} \sum_{i=1}^n \frac{B A}{(B-1) \left[\frac{1}{(1-B)^a} - 1 \right]}
\end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \sigma^2} &= \frac{n}{\sigma^2} + \frac{2}{\sigma^3} \sum_{i=1}^n (\mu - x_i)(A-1) - \frac{1}{\sigma^4} \sum_{i=1}^n (\mu - x_i)^2 A + \frac{(a-1)}{\sigma^4} \sum_{i=1}^n \frac{A^2 B (\mu - x_i)^2}{(B-1)} \\ &\quad - \frac{(a-1)}{\sigma^4} \sum_{i=1}^n \frac{AB (\mu - x_i)^2}{(B-1)} - \frac{(a-1)}{\sigma^4} \sum_{i=1}^n \frac{A^2 B^2 (\mu - x_i)^2}{(B-1)^2} + \frac{2(a-1)}{\sigma^3} \sum_{i=1}^n \frac{AB (\mu - x_i)}{(B-1)} \\ &\quad + \frac{a(b-1)}{\sigma^4} \sum_{i=1}^n \frac{AB (\mu - x_i)^2}{(B-1) \left[\frac{1}{(1-B)^a} - 1 \right]} - \frac{2a(b-1)}{\sigma^3} \sum_{i=1}^n \frac{AB (\mu - x_i)}{(B-1) \left[\frac{1}{(1-B)^a} - 1 \right]} \\ &\quad - \frac{a(b-1)}{\sigma^4} \sum_{i=1}^n \frac{A^2 B (\mu - x_i)^2}{(B-1) \left[\frac{1}{(1-B)^a} - 1 \right]} + \frac{a(b-1)}{\sigma^4} \sum_{i=1}^n \frac{A^2 B^2 (\mu - x_i)^2}{(B-1)^2 \left[\frac{1}{(1-B)^a} - 1 \right]} \\ &\quad + \frac{a^2(b-1)}{\sigma^4} \sum_{i=1}^n \frac{A^2 B^2 (\mu - x_i)^2}{(B-1)(1-B)^{a+1} \left[\frac{1}{(1-B)^a} - 1 \right]^2} \end{aligned}$$

Let θ is the vector of the unknown parameters (a, b, μ, σ) , The element of the 4×4 information matrix $I(a, b, \mu, \sigma)$ can be approximated by:

$$I_{ij}(\hat{\theta}) = - \frac{\partial^2 \ell(\theta)}{\partial \theta_i \partial \theta_j} \Big|_{\theta=\hat{\theta}},$$

and $\Gamma^1(a, b, \mu, \sigma)$ is the variance covariance matrix of the unknown parameters

The asymptotic distributions of the MLE parameters

$$\sqrt{n}(\hat{\theta}_i - \theta_i) \approx N_4(0, I^{-1}(\hat{\theta}_i)), \quad i = 1, \dots, 4$$

and, the approximation $100(1 - \alpha)\%$ confidence intervals for the unknown parameters based on the asymptotic distribution of the Kw-Gu (a, b, μ, σ) are determined, respectively, as

$$\hat{\theta}_i \pm Z_{\frac{\alpha}{2}} \sqrt{I^{-1}(\hat{\theta}_i)}, \quad i = 1, \dots, 4$$

where $z_{\frac{\alpha}{2}}$ is the upper $(\frac{\alpha}{2})^{\text{th}}$ percentile of a standard normal distribution

9. Numerical Illustration:

A numerical example will be given to illustrate the new results of the kw-Gu distribution. This example is about MLEs of parameters of the Kw-Gu distribution. The algorithm for obtaining the parameters estimates will be described in the following steps:

Step (1): Generate a random sample of size n as follows: $u_1, u_2, \dots, u_n$, by using the uniform distribution $(0,1)$

Step (2): transform the uniform random numbers to random numbers of the Kw-Gu distribution by using the following quantile function for the kw-Gu distribution:

$$x_u = \mu + \sigma \ln \left[-\ln \left(1 - \sqrt[a]{1 - b\sqrt{1-u}} \right) \right], 0 < u < 1$$

Step (3): Solve the equations (22-25) by iteration to get the maximum likelihood estimators.

Step (4): Calculating variance covariance matrix of the Kw-Gu distribution and confidence intervals

Example

In this example 60 random numbers were generated by the Mathcad package which calculated MLEs and variance covariance matrix of the Kw-Gu distribution from initial values: $a = 2$, $b = 3$, $\mu = 10$, $\sigma = 2$ as follows

$$\begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{\mu} \\ \hat{\sigma} \end{bmatrix} = \begin{bmatrix} 1.754 \\ 10.865 \\ 12.076 \\ 2.115 \end{bmatrix}, \quad \text{var} = \begin{bmatrix} 10.819 & 248.819 & 24.764 & 12.229 \\ 248.819 & 8.045 \times 10^3 & 883.662 & 302.53 \\ 24.764 & 883.662 & 99.246 & 30.916 \\ 12.229 & 302.53 & 30.916 & 14.06 \end{bmatrix}$$

Confidence intervals			
Significance Level	Parameter Estimation	Upper	Lower
$\alpha = 0.05$	a	8.201	-4.693
$\alpha = 0.05$	b	186.664	-164.933
$\alpha = 0.05$	μ	31.602	-7.449
$\alpha = 0.05$	σ	9.465	-5.235

10. Conclusion

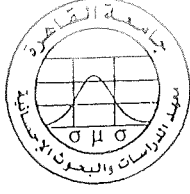
A new distribution, so-called the Kumaraswamy Gumbel Minimum (Kw-Gu) distribution, is calculated and studied some of its general mathematical and statistical properties.

The Kw-Gu distribution generalizes some distributions in the literature and it is an important model for analysis of extreme value data, because of the wide usage of the gumbel minimum distribution as a type one extreme value distribution. The new model contains (as special sub-models) : the exponentiated minimum Gumbel and the Gumbel minimum distributions.

Finally, the Kw-Gu distribution provides a rather flexible mechanism for fitting a wide spectrum of minimum extreme value world data sets.

Reference

- [1] Abd El-Sattar, N. (2009). On the analysis and the estimation using different types of moment. M.sc. thesis, I.S.S.R., Cairo University.
- [2] Arnold, C. B., Balakrishnan, N. and Nagaraja, H. N. (1992). *A first course in order statistics*. John Wiley & Sons, Inc. New York .
- [3] Cordeiroa, G. M. and Castro, M. (2010). A new family of generalized distributions. *Journal of Statistical Computation & Simulation* 1563-5163 ,
- [4] Hosking , J. R. M. (1990). L moments: analysis and estimation of distributions using linear combinations of order statistic, *J. R. Statist. Soc. B*, 52 No.1, 105-124.
- [5] Nadarajah, S. and Kotz , S. (2004) .The beta Gumbel distribution ,*Mathematical Problems in Engineering* 10 323-332 .
- [6] Prudnikov , A. P. , Brychkov, Yu. A. and Marichev, O. I. (1986). *Integrals and Series*, Vol. 3, Gordon and Breach Science Publishers, Amsterdam .



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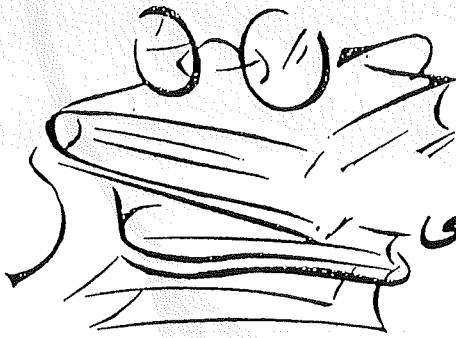


معهد الدراسات والبحوث الإحصائية

المؤتمر السنوى السادس والأربعون

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