جامعة القاهرة
معهد الدراسات و البحوث الإحصائية

المؤتمر السنوى الثالث والأربعون
للإحصاء وعلوم الحاسب
وبحوث العمليات

الإحصاء الرياضى

ينظمه معهد الدراسات والبحوث الإحصائية
جامعة القاهرة
٢٢-٢٥ ديسمبر ٢٠٠٨

بسم الله الرحمن الرحيم

كلمة المؤتمر السنوى الثالث والأربعين

"الإحصاء وعلوم الحاسب وبحوث العمليات"

٢٢-٢٥ ديسمبر ٢٠٠٨

أساتذتى وزملائى الاعـــزاء

السيدات والسادة الحضور الكريم

تعقد فعاليات أنشطة مؤتمرنا السنوي الثالث والأربعين

فى الإحصاء وعلوم الحاسب وبحوث العمليات

تحت شعار

لا تنمية بدون إحصاء

خلصت خبرات السنوات الماضية عن حقيقة دامغة مفادها ان أى خطه أو برنامج أو مشروع لا يمكن انجازه فى غياب قواعد البيانات وبالتالى الإحصاء كأداة لأتخاذ القرارات. وفى هذا السياق فان اعتزازنا بهذا الشعار ينبع من أهمية الدور الذى يقوم به معهدنا العريق فى المشاركة فـــى تحقيق اهداف التنمية الاقتصادية والاجتماعية والتى تتصدر اولويات اهتمام القيادة السياسية إن لم يكن شاغلها الاول.

وفى البداية اتوجه بالشكر والتقدير لحضوركم الكريم كل باسمه ويمتد هذا الشكر والتقدير إلـــى اساتذتنا الاجلاء. وأخص بالشكر والتقدير الى الأستاذة الدكتورة الهام شكرى عميـــد معهـــدنا السابق والتى وضعت كثير من المعايير ارست بها قواعد العمل التنظيمى للمعهد. كـــذلك كـــل

الشكر والتقدير الى الأستاذة الدكتورة أمانى موسى والتى تحملت عبء امانة المؤتمر السنوى لهذا العام بكل تفانى واقتدار. والتى حالت ظروفها الصحية حضورها وأنتهز هذه المناسبة بالدعاء لله بأن يمن عليها نعمة الشفاء. واشكر فريق العمل لامانة المؤتمر الذى عمل فى صمت واستمر فى الاداء حتى تحقق حلمنا بل عيدنا السنوى فى أظهار اعمال المؤتمر فى أبهى صورة.

الساده الحضور الكريم

لعل من حسن الطالع ان يواكب مؤتمرنا هذا فى ان يدرج معهدنا فى مشروع الجودة للتعليم ويخطو خطواته الاولى والذى نرى فيه املا كبيرا فى ان يحقق معهدنا رؤويته ورسالته فى أقصر وقت ممكن. فقد انبثقت رؤية المعهد من فكر اساتذة فى :

" التميز فى الأداء الاكاديمي والمهنى لتحقيق الريادة فى المجالات الاحصائية وعلوم الحاسب ونظم المعلومات وبحوث العمليات ليكون احدى المنارات العلمية والبحثية للارتقاء بمستوى أداء الخريج طبقا للمعايير القومية والعالمية".

وبالتالى تملى علينا تلك الريادة من خلال انفراد معهد الإحصاء بمنح درجة الدبلوم والماجستير والدكتوراة فى الإحصاء وعلوم الحاسب وبحوث العمليات أهمية مواكبة العصر من تكنولوجيا المعلومات وطرق التعليم والتعلم والتى تضفى الى علوم الإحصاء وجه جديد.

ولعل من المناسب ذكره فى ذلك المقام استضافة معهد الإحصاء ورشة العمل عن التعليم الإلكترونى بالتعاون مع وحدة ضمان الجودة بجامعة القاهرة فى أطار مؤتمرنا السنوى.

أساتذتى وزملائى الأعزاء:

تتعد انشطة مؤتمرنا السنوى لهذا العام ليشمل جلسات علمية لتشمل ١٢ جلسة علمية تحتوى على (مايقرب من خمسون بحثا) فى الإحصاء السكانى والتطبيقى والرياضى وعلوم الحاسب والمعلومات وبحوث العمليات.

ويتفاعل المؤتمر بكثير من القضايا والهموم التى يعانى منها مجتمعنا المصرى من خلال ٧ ندوات علمية تتركز حول :

- التعديلات الأدارية لمحافظات مصر ونتائجها السكانية وتتم من خلال شراكة فاعلة من جمعية الديموجرافيين المصريين والجمعية الجغرافية المصرية.
- العشوائيات فى محافظات مصر: الوضع القائم والاسلوب الامثل للتعامل. وتنظمها مركز المعلومات واتخاذ القرار بمجلس الوزراء.
- تحديدالألف قرية الأكثر فقرا - وينظمها مع الجهاز المركزى للتعبئة العامة والإحصاء.
- امراض العيون من المنظور الديموجرافى - وتنظم بالتعاون مع مؤسسة النور الخيرية (مغربى).
- تداعيات الأزمة المالية العالمية على التجارة الخارجية المصرية.

ويتواصل المؤتمر بمحاضرتين عامتين من خيرة علماء مصر بالخارج بل سفرائها من خلال مشروع نقل المعرفة والخبرة اكاديمية البحث العلمى والتى نتشرف بهما وهما:

- الاستاذ الدكتور سمير فريد للحديث عن:

"النمو السكانى وقدرات وأمن الأمم"

- الاستاذ الدكتور سعيد المهدى للحديث عن:

"Active Learning Use of Technology in Teaching Intro Statistics"

السادة الحضور الكريم

بدأت كلمتى بالشكر الواجب .. وفى النهاية تملى على الامانة العلمية لتقديم خالص الشكر والتقدير للجهات والهيئات الداعمة لمؤتمرنا السنوى لهذا العام وعلى رأسها سعادة الأستاذ

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- أكاديمية البحث العلمى
- صندوق الامم المتحدة للسكان
- المشروع العربى لصحة الاسرة بالجامعة العربية
- جميعة النور الخيرية (مغربى)

أدعو الله العزيز القدير أن يوفقنا جميعا لما فيه الخير لمصرنا العزيزة

والسلام عليكم ورحمة الله وبركاته

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أمانة المؤتمر والهيئة العلمية

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with $\Delta_1^0 = 1$, $\Delta_1^1 = 1 - r_1$, $\Delta_1^2 = (1 - r_1)(1 - r) - p_1 q$.

Solving the difference equation (a), we have $\Delta_i^j = c_1 \lambda_1^{j-i+1} + c_2 \lambda_2^{j-i+1}$,

where $\lambda_1 = p$, $\lambda_2 = q$ are the two roots of the corresponding characteristic equation.

Using the initial conditions: at $i = j$, $\Delta_i^j = 1 - r$, then $(1 - r) = c_1 \lambda_1 + c_2 \lambda_2$

and at $i = j + 1$, $\Delta_i^j = 1$, then $1 = c_1 + c_2$, and

$$c_1 = -\lambda_1 / (\lambda_2 - \lambda_1), \quad c_2 = \lambda_2 / (\lambda_2 - \lambda_1).$$

Immediately formula (4.7) follows.

Solving difference equation (b), we get $\Delta_1^i = c_3 \lambda_1^i + c_4 \lambda_2^i$, with $\lambda_1 = p$, $\lambda_2 = q$

Using the initial conditions

$$\Delta_1^0 = 1, \ \Delta_1^1 = 1 - r_1.$$

We obtain $c_3 = ((r - r_1) + \lambda_1)/(\lambda_1 - \lambda_2)$, and $c_4 = -[(r - r_1) + \lambda_2]/(\lambda_1 - \lambda_2)$, and formula (4.5) follows.

Solving the difference equation (c), using the solution (4.5) at $i = L - 1$, $i = L - 2$, we have

$$\Delta_1^L = \frac{1}{\lambda_1 - \lambda_2}\Big[(1 - r_L)(\lambda_1^L - \lambda_2^L) + (1 - r_L)(r - r_1)(\lambda_1^{L-1} - \lambda_2^{L-1}) \\ - q_L p(\lambda_1^{L-1} - \lambda_2^{L-1}) - q_L p(r - r_1)(\lambda_1^{L-2} - \lambda_2^{L-2})\Big]$$

$$\Delta_1^L = \frac{1}{\lambda_1 - \lambda_2}\Big[(1 - r_L)(\lambda_1^L - \lambda_2^L) + ((1 - r_L)(r - r_1) - q_L p)(\lambda_1^{L-1} - \lambda_2^{L-1}) \\ - q_L p(r - r_1)(\lambda_1^{L-2} - \lambda_2^{L-2})\Big],$$

and (4.4) follows.

Similarly, solving difference equation (d), we get

$$\Delta_i^L = c_5 \lambda_1^{L-i+1} + c_6 \lambda_2^{L-i+1}, \quad \lambda_1 = p, \lambda_2 = q.$$

Using the initial conditions $\Delta_{L+1}^L = 1$, $\Delta_L^L = 1 - r_L$, we get

$\therefore c_5 = (\lambda_1 + (r - r_L))/(\lambda_1 - \lambda_2)$, $c_6 = -(\lambda_2 + (r - r_L))/(\lambda_1 - \lambda_2)$.

Formula (4.6) immediately follows. This completes the proof.

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$$\prod_{k=s+1}^{j} p_k = \prod_{k=s+1}^{j} p_{kk+1} = \prod_{k=s+1}^{j} k(L-k)/L^2$$
$$=(-1)^{s+1-j}\left(\frac{1}{L^2}\right)^{j-s}\frac{\Gamma(1+j)\Gamma(1+j-L)}{\Gamma(s+1)\Gamma(s+1-L)},$$
$$\prod_{k=i}^{s} q_k = \prod_{k=i}^{s} p_{kk-1} = \prod_{k=i}^{s}\frac{k}{L}\left(1-\frac{k}{L}\right)$$
$$=(-1)^{i-s}\left(\frac{1}{L^2}\right)^{1-i+s}\frac{\Gamma(1+s)\Gamma(1+s-L)}{\Gamma(i)\Gamma(i-L)}.$$

Substituting into formula (4.23) we obtain the following explicit formula for the determinant Δ_i^j defined in (2.1) with (5.1)

$$\Delta_i^j = \sum_{s=i-1}^{j}\prod_{k=s+1}^{j} p_k \prod_{k=i}^{s} q_k = (2-i+j)(-1)^{1+i-j}\left(\frac{1}{L^2}\right)^{1-i+j}\frac{\Gamma(1+j)\Gamma(1+j-L)}{\Gamma(i)\Gamma(i-L)}.$$

Substituting into proposition 4.7 we obtain the following expressions for the absorption probabilities, starting from state i , for $1 \le i, j \le L$ are

$$B_0 = \frac{(1-i+L)}{(1+L)}, \quad \text{and} \quad B_{L+1} = \frac{i}{(1+L)}.$$

6. Conclusions. Closed form expressions for the entries of the fundamental matrix of the general MC-RW with absorbing boundaries have been presented using matrix theory in which the transition probabilities are $p_{0j} = \delta_{0j}$, $p_{L+1j} = \delta_{L+1j}$, $p_{1L} = \alpha$, $p_{L1} = \beta$, $p_{ii-1} = q_i$, $p_{ii} = r_i$, $p_{ii+1} = p_i$, with $p_1 + r_1 + q_1 = 1-\alpha$, $p_L + r_L + q_L = 1-\beta$, $0 \le \alpha, \beta < 1$, and $p_i + r_i + q_i = 1$ for $i \in \{2,...,L-1\}$. This leads also to closed form expressions for the mean number of step before absorption, the mean time to absorption as well as the absorption probabilities, for any initial state i , in the two cases (1)- when the states placed on a ring network and (2)-on one dimensional lattice segment, case $\alpha = 0 = \beta$. The results are generalizing previous works. Explicit expressions are also given for certain models. The present results are especially useful if, for example, the absorption time or the absorption probabilities for a specific initial state are of interest because only few entries of the fundamental matrix must be determined in this case.

Appendix. Proof of Proposition 4.1.

Formulae (4.1) and (4.2) follow immediately from Theorem 2.1 after inserting

$$p_i = \begin{cases} p_1 & i=1 \\ p & i=2,...,L-1 \end{cases}, q_i = \begin{cases} q & i=2,...,L-1 \\ q_L & i=L \end{cases}, r_i = \begin{cases} r_1 & i=1 \\ r & i=2,...,L-1 \\ r_L & i=L \end{cases},$$

and solving the following corresponding difference equations:

$$\Delta_i^j = (1-r)\Delta_{i+1}^j - qp\Delta_{i+2}^j, \qquad 2 \le i \le j \le L-1 \qquad \text{(a)}$$
$$\Delta_1^i = (1-r)\Delta_1^{i-1} - qp\Delta_1^{i-2}, \qquad i = 3,...,L-1 \qquad \text{(b)}$$
$$\Delta_1^L = (1-r_L)\Delta_1^{L-1} - q_L p\Delta_1^{L-2}, \qquad \text{(c)}$$
$$\Delta_i^L = (1-r)\Delta_{i+1}^L - pq\Delta_{i+2}^L, \qquad i = L-1,...,2,1 \qquad \text{(d)}$$

a clone of L ,$(i = 1, p + q = 1)$, proliferating cells is then found as the solution of a gambler's ruin problem:

$$P(L) = \frac{[q/p]-1}{[q/p]^L - 1}.$$

This probability shows that

(i) for $p > \frac{1}{2}$, the probability of exceeding L cells is nearly independent of L, being given by $(2p-1)/p$,

(ii) for $p < \frac{1}{2}$, probabilities are exceedingly small unless $L \le 10^3$, and

(iii) in the region of selective neutrality, $p \approx \frac{1}{2}$, a wide range of probabilities is found depending on the precise values of p and L.

(cf. Feller (1968), Bell (1976), Beyer and Waterman (1979) and El-Shehawey (1994))

A second biological model that also leads to a birth and death process is applied to an example taken from genetics.

5.2 Application to a genetic population model

We consider a genetic population model consists of a constant number of individuals and the state of the population is defined by the numbers of the various genotypes existing at a given time. The birth-death model postulates that at each unit of time, one individual is chosen at random to die, and is replaced by a new individual whose genotype is determined at random from those existing before the death. Thus the number of individuals of a given genotype-the state of the population-can take any of the values $0,1,...,L$, and can change by at most unity during one birth-death event. This model was introduced by Moran (1958). Actually, Moran considered as well the more general case when gene mutation was allowed. Here, if mutation is assumed absent, all individuals will eventually become of the same genotype because of random influences such as births, deaths, mating, selection, chromosome breakages and recombinations, i.e. there is no source for new genes, and once all individuals are of the same genotype the population state remains unchanged thereafter. The population behavior may in some circumstances be approximated by a MCRW with absorbing states.

The transition probabilities are (see Moran (1958) with a new notation), for $i,j = 0,1,...,L$,

$$\begin{aligned} p_{ii-1} &= p_{ii+1} = i\left(L-i\right)/L, \\ p_{ii} &= 1-2i\left(L-i\right)/L, \\ p_{ij} &= 0, \qquad |i-j| \ge 2 \end{aligned} \tag{5.1}$$

The states 0 and L are absorbing, those in-between are transient.

Eigenvectors of the transition matrix $P = \left(p_{ij}\right)$, in terms of orthogonal polynomials, and other properties have been studied by Watterson (1961).
Using the fact

$$p_{ii-1} = q_i = p_i = p_{ii+1} = i\left(L-i\right)/L^2,$$

we get

$$n_{ij} = \begin{cases} \dfrac{i(L+1-j)}{q(L+1)}, & i \le j, p = q \\ \dfrac{j(L+1-i)}{q(L+1)}, & i \ge j, p = q \end{cases} \tag{4.29}$$

$$\tau = \begin{cases} \dfrac{1}{q(a-1)}\left[(L+1)\dfrac{a^{L+1} - a^{L-i+1}}{a^{L+1}-1} - i\right], & p \ne q \\ \dfrac{i(L+1-i)}{2q}, & p = q \end{cases} \tag{4.30}$$

and

$$B_0 = \begin{cases} \dfrac{a^{L+1-i} - 1}{a^{L+1} - 1}, & p \ne q, a = p/q \\ \dfrac{L-i+1}{L+1}, & p = q \end{cases}, \qquad B_{L+1} = \begin{cases} \dfrac{(a^i - 1)a^{L+1-i}}{a^{L+1} - 1}, & p \ne q, a = p/q \\ \dfrac{i}{L+1}, & p = q. \end{cases} \tag{4.31}$$

These results, in the case $p+q=1$ agree with that of Kemeny and Snell (1976), and Iosifescu (1980) (see also Wittenburg (1998), and El-Shehawey $(2008)_a$, $(2008)_b$).

5. Applications

The present results are especially useful for studying many diverse areas, not only in computer science and engineering but also in other disciplines such as mathematics, probability and statistics, operation research, industrial engineering, electrical engineering, agriculture, economics, demographic, education, biology, genetics and so on. For example we consider two mathematical models for growth of cancer tumor, and genetics.

5.1 Application to a Cancer Tumor

The first mathematical model to which our result in formula (4.31) has been applied concerns the growth of a cancer tumor. A tumor is a non-inflamed abnormal mass of tissue. A cancer tumor is thought of as arising from one wayward cell that has lost the ability to control it self. This is the model used for certain simple cases. The wayward cell starts reproducing without regard to the presence of other cells. The cell progeny may all die before catastrophe overtakes the host organism, or may produce a family tree of descendants (called a clone) large enough to be noticeable to the host organism, in which case the descendants become a tumor. It is the tumor that is noticeable, not the early cells that died without progeny. It is of interest to estimate the time it takes for one wayward cell to reach tumor size. For information refer to the articles by Bell (1976) and El-Shehawey (1994).

In order to estimate the probability and the time for a single cell to become by chance a macroscopic clone. Consider a birth and death process in which each cell has probability p $(0 < p < 1)$ of dividing to produce two identical new cells, probability $r = 1 - p - q$ to stay without change, and probability q of dying or of becoming irreversibly committed to a non-dividing state. For normal cells in a tissue that is not undergoing net growth, $p = \frac{1}{2}$. The probability, $P(L)$, that a single cell ever becomes

Consider the perturbed particular case $p_{0j} = \delta_{0j}$, $p_{L+1j} = \delta_{L+1j}$, $p_{ii-1} = q$, $p_{ii} = r$, $p_{ii+1} = p$, for $i \in \{2,\dots,L-1\}$, $p+r+q=1$, $p_1+r_1+q_1=1$, and $p_L+r_L+q_L=1$. Formulae (4.20)- (4.23) can be written as:

$$n_{ij} = \frac{1}{\Delta_1^L}\begin{cases} \Delta_2^L, & i=j=1 \\ \Delta_{j+1}^L p^{j-2} p_1, & i=1, 2 \le j \le L-1 \\ p_1 p^{L-2}, & i=1, j=L \\ (\Delta_{j+1}^L \Delta_1^{i-1}) p^{j-i}, & 2 \le i \le j \le L-1 \end{cases} \tag{4.24}$$

$$n_{ij} = \frac{1}{\Delta_1^L}\begin{cases} (\Delta_{i+1}^L \Delta_1^{j-1}) q^{i-j}, & 2 \le j \le i \le L-1 \\ q_L q^{L-2}, & i=L, j=1 \\ \Delta_1^{j-1} q_L q^{L-j-1}, & i=L, 2 \le j \le L-1 \\ \Delta_1^{L-1}, & i=j=L \end{cases} \tag{4.25}$$

$$\tau = \frac{1}{\Delta_1^L}\begin{cases} \Delta_2^L + p_1 p^{L-2} + \sum_{j=2}^{L-1} p_1 p^{j-2} \Delta_{j+1}^L, & i=1 \\ \sum_{j=1}^{i} \Delta_{i+1}^L \Delta_1^{j-1} q^{i-j} + \sum_{j=i+1}^{L} \Delta_{j+1}^L \Delta_1^{i-1} p^{j-i}, & 2 \le i \le L-1, \\ \Delta_1^{L-1} + q_L q^{L-2} + \sum_{j=2}^{L-1} q_L q^{L-j-1} \Delta_1^{j-1}, & i=L \end{cases} \tag{4.26}$$

$$B_0 = \frac{q_1}{\Delta_1^L}\begin{cases} \Delta_2^L, & i=1 \\ \Delta_{i+1}^L q^{i-1}, & 2 \le i \le L-1, \\ q_L q^{L-2}, & i=L \end{cases} \text{ and } B_{L+1} = \frac{p_L}{\Delta_1^L}\begin{cases} p_1 p^{L-2}, & i=1 \\ \Delta_1^{i-1} p^{L-i}, & 2 \le i \le L-1, \\ \Delta_1^{L-1}, & i=L \end{cases} \tag{4.27}$$

where Δ, Δ_1^L, Δ_1^i, Δ_i^L, and Δ_i^j are given as in Proposition 4.1.

Formulae (4.24) and (4.25) agree with that of Wittenburg (1998) (case $\lambda_1 \neq \lambda_2$), (see also El-Shehawey $(2008)_b$).

Explicit expressions for formulae (4.20)- (4.23) can be obtained for the homogeneous case $p_{0j} = \delta_{0j}$, $p_{L+1j} = \delta_{L+1j}$, $p_{ii-1} = q$, $p_{ii} = r$, $p_{ii+1} = p$, $i \in \{1,2,\dots,L\}$, $p+r+q=1$, as:

$$n_{ij} = \begin{cases} \dfrac{\left(a^{L-i+1} - a^{j-i}\right)\left(a^i - 1\right)}{q\,(a-1)\left(a^{L+1}-1\right)}, & i \le j, p \neq q, a = p/q \\ \dfrac{\left(a^{L-i+1} - 1\right)\left(a^j - 1\right)}{q\,(a-1)\left(a^{L+1}-1\right)}, & i \ge j, p \neq q, a = p/q \end{cases} \tag{4.28}$$

$$n_{iL} = \frac{\sum\limits_{s=0}^{i-1}\prod\limits_{k=s+1}^{i-1} p_k \prod\limits_{k=1}^{s} q_k \prod\limits_{k=i}^{L-1} p_k}{\prod\limits_{k=1}^{j} q_k}, \tag{4.18}$$

$$\tau = \sum_{j=1}^{i} \frac{\sum\limits_{s=0}^{j-1}\prod\limits_{k=s+1}^{j-1} p_k \prod\limits_{k=1}^{s} q_k}{\prod\limits_{k=1}^{j} q_k} + \sum_{j=i+1}^{L-1} \frac{\left(\sum\limits_{s=0}^{i-1}\prod\limits_{k=s+1}^{i-1} p_k \prod\limits_{k=1}^{s} q_k\right)}{\prod\limits_{k=1}^{j} q_k} \prod_{k=i}^{j-1} p_k$$

$$+ \frac{\sum\limits_{s=0}^{i-1}\prod\limits_{k=s+1}^{i-1} p_k \prod\limits_{k=1}^{s} q_k}{\prod\limits_{k=1}^{L-1} q_k} \prod_{k=i}^{L-1} p_k\,, \tag{4.19}$$

and the absorption probability at the origin is $B_0 = 1$.

4.5. General One-Dimensional Lattice Segment MC-RW Between Two Absorbing Barriers

Consider a one-dimensional MC-RW on the nonnegative integers with absorbing boundaries at 0 and $L+1$ which has transition probabilities $p_{0j} = \delta_{0j}$, $p_{L+1j} = \delta_{L+1j}$, $p_{ii-1} = q_i$, $p_{ii} = r_i$, and $p_{ii+1} = p_i$, where $p_i + r_i + q_i = 1$ for $i \in \{1,2,\ldots,L\}$. In this case $Q \equiv T$ is a tridiagonal matrix of order L. The following results are immediately followed from Theorems 2.1, 3.1, and 3.2, with $\alpha = 0 = \beta$.

Proposition 4.7. Closed form expressions for the expected number of occurrences of state j prior to absorption, the expected time to absorption, and the absorption probabilities, starting from state i, respectively, for $1 \le i, j \le L$, are

$$n_{ij} = \begin{cases} \dfrac{D_{j+1}^{L} D_1^{i-1}}{D_1^{L}} \prod\limits_{k=i}^{j-1} p_k, & i \le j \\[2ex] \dfrac{D_{i+1}^{L} D_1^{j-1}}{D_1^{L}} \prod\limits_{k=j+1}^{i} q_k, & i \ge j \end{cases} \tag{4.20}$$

$$\tau = \frac{\Delta_{i+1}^{L}}{\Delta_1^{L}} \sum_{j=1}^{i} \left(\Delta_1^{j-1} \prod_{k=j+1}^{i} q_k \right) + \frac{\Delta_1^{i-1}}{\Delta_1^{L}} \sum_{j=i+1}^{L} \left(\Delta_{j+1}^{L} \prod_{k=i}^{j-1} p_k \right), \tag{4.21}$$

$$B_0 = \frac{\Delta_{i+1}^{L}}{\Delta_1^{L}} \prod_{k=1}^{i} q_k\,, \qquad \text{and} \qquad B_{L+1} = \frac{\Delta_1^{i-1}}{\Delta_1^{L}} \prod_{k=i}^{L} p_k\,, \tag{4.22}$$

where

$$\Delta_i^{j} = \sum_{s=i-1}^{j} \left(\prod_{k=s+1}^{j} p_k \right)\left(\prod_{k=i}^{s} q_k \right), \quad 1 \le i, j \le L. \tag{4.23}$$

Formulae (4.20) with (4.21) agrees with that of Rudolph (1999) (see also Usmani (1994), and El-Shehawey (2008)$_a$).

$$n_{ij} = \frac{\left(\sum_{s=0}^{j-1}\prod_{k=s+1}^{i-1} p_k \prod_{k=1}^{s} q_k\right)\left(\sum_{s=j}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=j+1}^{s} q_k\right)}{\sum_{s=0}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=1}^{s} q_k}\prod_{k=i}^{j-1} p_k, \text{ for } i \le j \qquad (4.14)_a$$

$$n_{ij} = \frac{\left(\sum_{s=0}^{j-1}\prod_{k=s+1}^{j-1} p_k \prod_{k=1}^{s} q_k\right)\left(\sum_{s=i}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=i+1}^{s} q_k\right)}{\sum_{s=0}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=1}^{s} q_k}\prod_{k=j+1}^{i} q_k, \text{ for } i \ge j \qquad (4.14)_b$$

$$\tau = \sum_{j=1}^{i} \frac{\left(\sum_{s=0}^{j-1}\prod_{k=s+1}^{j-1} p_k \prod_{k=1}^{s} q_k\right)\left(\sum_{s=i}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=i+1}^{s} q_k\right)}{\sum_{s=0}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=1}^{s} q_k}\prod_{k=j+1}^{i} q_k$$

$$+ \sum_{j=i+1}^{L} \frac{\left(\sum_{s=0}^{i-1}\prod_{k=s+1}^{i-1} p_k \prod_{k=1}^{s} q_k\right)\left(\sum_{s=j}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=j+1}^{s} q_k\right)}{\sum_{s=0}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=1}^{s} q_k}\prod_{k=i}^{j-1} p_k, \qquad (4.15)$$

and

$$B_0 = \frac{\sum_{s=i}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=1}^{s} q_k}{\sum_{s=0}^{L-1}\prod_{k=s+1}^{L} p_k \prod_{k=1}^{s} q_k}. \qquad (4.16)$$

Formulae $(4.14)_a$ and $(4.14)_b$ agree with that of Rudolph (1999) (see also El-Shehawey $(2008)_a$).

4.4. General One-Dimensional Lattice Segment MC-RW Between a Perfectly Reflecting and an Absorbing Barrier

Consider a one-dimensional MC-RW on the nonnegative integers with absorbing barrier at 0 and perfectly reflecting barrier at L with transition probabilities $p_{0j} = \delta_{0j}$, $p_{ii-1} = q_i$, $p_{ii} = r_i$, and $p_{ii+1} = p_i$, $q_L = 1$, where $p_i + r_i + q_i = 1$ for $i \in \{1,2,\ldots,L-1\}$. The following results are immediately follow from Theorems 2.1, 3.1, and 3.2, with $\alpha = 0$, $\beta = 0$ and $q_L = 1$.

Proposition 4.6. Closed form expressions for the expected number of occurrences of state j prior to absorption, the expected time to absorption, and the absorption probabilities, starting from state i , respectively, for $1 \le i, j \le L$, are

$$n_{ij} = \frac{1}{\prod_{k=1}^{j} q_k}\begin{cases} \sum_{s=0}^{i-1}\prod_{k=s+1}^{i-1} p_k \prod_{k=1}^{s} q_k \prod_{k=i}^{j-1} p_k, & i \le j \\ \sum_{s=0}^{j-1}\prod_{k=s+1}^{j-1} p_k \prod_{k=1}^{s} q_k & i \ge j \end{cases} \qquad (4.17)$$

4.2. General One-Dimensional Lattice Segment MC-RW Between Partially Segregating-Partially Reflecting and an Absorbing Barrier

Consider a one-dimensional MC-RW on the nonnegative integers with absorbing barrier at 0 and one partially segregating-partially reflecting barrier at L which has transition probabilities $p_{0j} = \delta_{0j}$, $p_{ii-1} = q_i$, $p_{ii} = r_i$, and $p_{ii+1} = p_i$, $r_L + q_L = 1$, where $p_i + r_i + q_i = 1$ for $i \in \{1,2,...,L-1\}$. In this case $Q \equiv T$ is a tridiagonal matrix of order L. The following results are immediately follow from Theorems 2.1, 3.1, and 3.2, with $\alpha = 0 = \beta$, and $r_L + q_L = 1$.

Proposition 4.4. Closed form expressions for the expected number of occurrences of state j prior to absorption, the expected time to absorption, and the absorption probabilities, starting from state i, respectively, for $1 \le i, j \le L$, are

$$n_{ij} = \frac{1}{\prod_{k=1}^{j} q_k} \begin{cases} \sum_{s=0}^{i-1} \prod_{k=s+1}^{i-1} p_k \prod_{k=1}^{s} q_k \prod_{k=i}^{j-1} p_k, & i \le j \\ \sum_{s=0}^{j-1} \prod_{k=s+1}^{j-1} p_k \prod_{k=1}^{s} q_k & i \ge j \end{cases} \tag{4.12}$$

$$\tau = \sum_{j=1}^{i} \frac{\sum_{s=0}^{j-1} \prod_{k=s+1}^{j-1} p_k \prod_{k=1}^{s} q_k}{\prod_{k=1}^{j} q_k} + \sum_{j=i+1}^{L} \frac{\sum_{s=0}^{i-1} \prod_{k=s+1}^{i-1} p_k \prod_{k=1}^{s} q_k}{\prod_{k=1}^{j} q_k} \prod_{k=i}^{j-1} p_k , \tag{4.13}$$

and the absorption probability at the origin is $B_0 = 1$.

Formula (4.12) agrees with that of Rudolph (2001) (see also El-Shehawey (2008)$_a$). We see that with the appropriate change of notation formula (4.13) in the homogeneous case $q_k = q, r_k = r, p_k = p$, $k \in \{1,2,...,L\}$, and $p + r + q = 1$ agrees with that given, for example, in El-Shehawey (2000, and 2008)$_b$, Neuts (1963) for the symmetric case $p = q$, $1 \le i \le L$, and for the non-symmetric case $p \ne q$ when $i = L$, with $r_L = \beta$, $q_L = \alpha$, $\alpha + \beta = 1$. The latter solution contains an error which is corrected in Hardin et al. (1970), and in the special case $q_L = q$, and $r_L = 0$ with Weesakul (1961), (see also Cox et al. (1965), Feller (1968), Srinivasan et al. (1978), Iosifescu (1980)).

4.3. General One-Dimensional Lattice Segment MC-RW Between Partially Segregating-Partially Absorbing and an Absorbing Barrier

Consider a one-dimensional MC-RW on the nonnegative integers with absorbing barrier at 0 and one partially segregating-partially absorbing barrier at L with transition probabilities $p_{0j} = \delta_{0j}$, $p_{ii-1} = q_i$, $p_{ii} = r_i$, and $p_{ii+1} = p_i$, $p_L + r_L = 1$, where $p_i + r_i + q_i = 1$ for $i \in \{1,2,...,L-1\}$. The following results are immediately follow from Theorems 2.1, 3.1, and 3.2, with $\alpha = 0 = \beta$, and $p_L + r_L = 1$.

Proposition 4.5. Closed form expressions for the expected number of occurrences of state j prior to absorption, the expected time to absorption, and the absorption probabilities, starting from state i, respectively, for $1 \le i, j \le L$, are

$$\Delta_1^L = \frac{1}{p-q}\Big[(1-r_L)(p^L-q^L)+\big((1-r_L)(r-r_1)-q_L p\big)(p^{L-1}-q^{L-1})$$
$$-q_L p(r-r_1)(p^{L-2}-q^{L-2})\Big], \qquad (4.4)$$

$$\Delta_1^i = \frac{1}{p-q}\Big[p^{i+1}-q^{i+1}+(r-r_1)(p^i-q^i)\Big], \quad i = 2,\dots,L-1 \qquad (4.5)$$

$$\Delta_i^L = \frac{1}{p-q}\Big[\big(p^{L-i+2}-q^{L-i+2}\big)+(r-r_L)(p^{L-i+1}-q^{L-i+1})\Big], i = L-1,\dots,2,1$$
$$(4.6)$$

$$\Delta_i^j = \frac{1}{p-q}\Big[p^{j-i+2}-q^{j-i+2}\Big], \quad 2\le i \le j \le L-1 \qquad (4.7)$$

Proof. See Appendix

Proposition 4.2. Closed form expressions for the expected time to absorption given the initial state i ,$1\le i \le L$, are

$$\tau = \frac{1}{\Delta}\begin{cases} \Delta_2^L + p_1 p^{L-2} + \alpha\Delta_2^{L-1} + \sum\limits_{j=2}^{L-1}\left(p_1 p^{j-2}\Delta_{j+1}^L + \alpha q_L q^{L-j-1}\Delta_2^{j-1}\right), & i=1 \\ \Delta_1^{L-1} + q_L q^{L-2} + \beta\Delta_2^{L-1} + \sum\limits_{j=2}^{L-1}\left(q_L q^{L-j-1}\Delta_1^{j-1} + \beta p_1 p^{j-2}\Delta_{j+1}^{L-1}\right), & i=L \end{cases} \qquad (4.8)$$

and for $2\le i \le L-1$

$$\tau = \frac{1}{\Delta}\sum_{j=1}^{i}\left(\left(\Delta_{i+1}^L\Delta_1^{j-1} - \beta\alpha\Delta_{i+1}^{L-1}\Delta_2^{j-1}\right)q^{i-j} + \beta\Delta_{j+1}^{i-1}p_1 p^{L+j-i-2}\right) +$$
$$\frac{1}{\Delta}\sum_{j=i+1}^{L}\left(\left(\Delta_{j+1}^L\Delta_1^{i-1} - \beta\alpha\Delta_{j+1}^{L-1}\Delta_2^{i-1}\right)p^{j-i} + \alpha\Delta_{i+1}^{j-1}q_L q^{L+i-j-2}\right), \qquad (4.9)$$

where D , Δ_1^L , Δ_1^i , Δ_i^L , and Δ_i^j are given as in Proposition 4.1.

Proposition 4.3. Closed form expressions for the probabilities, B_0 and B_{L+1}, of being absorbed at the boundary states 0 and L+1 given the initial state i ,$1\le i \le L$ respectively, are

$$B_0 = \frac{q_1}{\Delta}\begin{cases} \Delta_2^L, & i=1 \\ \Delta_{i+1}^L q^{i-1} + \beta\Delta_2^{i-1}p_1 p^{L-i-1}, & 2\le i\le L-1 \\ q_L q^{L-2} + \beta\Delta_2^{L-1}, & i=L \end{cases} \qquad (4.10)$$

and

$$B_{L+1} = \frac{p_L}{\Delta}\begin{cases} p_1 p^{L-2} + \alpha\Delta_2^{L-1}, & i=1 \\ \Delta_1^{i-1}p^{L-i} + \alpha\Delta_{i+1}^{L-1}q_L q^{i-2}, & 2\le i\le L-1 \\ \Delta_1^{L-1}, & i=L \end{cases} \qquad (4.11)$$

where D , Δ_1^L , Δ_1^i , Δ_i^L , and Δ_i^j are given as in Proposition 4.1.

Substituting from (2.8) into (3.6) and (3.7) we get

$$B_0 = q_1 \frac{\left(\Delta_{i+1}^{L}\Delta_1^{0} - \beta\alpha\Delta_{i+1}^{L-1}\Delta_2^{0}\right)\prod_{k=2}^{i} q_k + \beta\Delta_2^{i-1}\prod_{k=1}^{0} p_k \prod_{k=i}^{L-1} p_k}{\left[\Delta_1^{L} - \beta\alpha\Delta_2^{L-1} - \beta\prod_{k=1}^{L-1} p_k - \alpha\prod_{k=2}^{L} q_k\right]} \tag{3.8}$$

and

$$B_{L+1} = p_L \frac{\left(\Delta_{L+1}^{L}\Delta_1^{i-1} - \beta\alpha\Delta_{L+1}^{L-1}\Delta_2^{i-1}\right)\prod_{k=i}^{L-1} p_k + \alpha\Delta_{i+1}^{L-1}\prod_{k=2}^{i} q_k \prod_{k=L+1}^{L} q_k}{\left[\Delta_1^{L} - \beta\alpha\Delta_2^{L-1} - \beta\prod_{k=1}^{L-1} p_k - \alpha\prod_{k=2}^{L} q_k\right]}. \tag{3.9}$$

Since $\Delta_1^0 = 1, \Delta_2^0 = 0, \Delta_{L+1}^{L} = 1, \Delta_{L+1}^{L-1} = 0$ and $\prod_{k+1}^{k}(.) = 1$. Formulas (3.4) and (3.5) can be easily obtained from (3.8) and (3.9).

4. Particular Cases

Many interesting results can be introduced from the present results through an appropriate choice of the transition probabilities and the boundary probabilities. Some examples are listed below.

4.1. Perturbed MC-RW Around a Ring Network

The model presented in this subsection is a perturbed MC-RW around a ring network which has the transition probabilities $p_{0j} = \delta_{0j}, p_{L+1j} = \delta_{L+1j}, p_{1L} = \alpha, p_{L1} = \beta$, $p_{ii-1} = q$, $p_{ii} = r$, and $p_{ii+1} = p$, for $i \in \{2,\dots,L-1\}, p + r + q = 1$ with $p_1 + r_1 + q_1 = 1 - \alpha$, $p_L + r_L + q_L = 1 - \beta$, and $0 \le \alpha, \beta < 1$. In this case Q is a perturbed Toeplitz periodic tridiagonal matrix of order L, in which the boundary elements are different from the remaining elements. The following results are immediately obtained from Theorems 2.1, 3.1, and 3.2.

Proposition 4.1. Closed form expressions for the mean number of visiting state j from state i before absorption, for $1 \le i, j \le L$, are

$$n_{ij} = \frac{1}{\Delta}\begin{cases} \Delta_2^{L}, & i = j = 1 \\ \Delta_{j+1}^{L} p^{j-2} p_1 + \alpha\Delta_2^{j-1} q_L q^{L-j-1}, & i = 1, 2 \le j \le L-1 \\ p_1 p^{L-2} + \alpha\Delta_2^{L-1}, & i = 1, j = L \\ \left(\Delta_{j+1}^{L}\Delta_1^{i-1} - \beta\alpha\Delta_{j+1}^{L-1}\Delta_2^{i-1}\right)p^{j-i} + \alpha\Delta_{i+1}^{j-1} q_L q^{L+i-(j+2)}, & 2 \le i \le L-1, i \le j \le L \end{cases} \tag{4.1}$$

and

$$n_{ij} = \frac{1}{\Delta}\begin{cases} \left(\Delta_{i+1}^{L}\Delta_1^{j-1} - \beta\alpha\Delta_{i+1}^{L-1}\Delta_2^{j-1}\right)q^{i-j} + \beta\Delta_{j+1}^{i-1} p_1 p^{L+j-(i+2)}, & 2 \le i \le L-1, 1 \le j \le i \\ q_L q^{L-2} + \beta\Delta_2^{L-1}, & i = L, j = 1 \\ \Delta_1^{j-1} q_L q^{L-j-1} + \beta\Delta_{j+1}^{L-1} p_1 p^{j-2}, & i = L, 2 \le j \le L-1 \\ \Delta_1^{L-1}, & i = j = L \end{cases} \tag{4.2}$$

where

$$\Delta = \Delta_1^{L} - \beta\alpha\Delta_2^{L-1} - \beta p_1 p^{L-2} - \alpha q_L q^{L-2}, \tag{4.3}$$

3. Probability of Absorption and the Mean Time to Absorption

Let $N_{ij}(n)$ be the number of visits to state j by time n given that $X_0 = i$; then $\{N_{ij}(n), n \geq 1\}$ is a delayed renewal process. A renewal occurs whenever the process enters state j. For transient states i and j. Let $n_{ij} = E\left[N_{ij}(\infty)\right]$ be the expected total number of time periods spent in state j, starting from i. Therefore n_{ij} is the mean number of visits to transient state j from i before the particle enters any absorbing state. The probability of ever visiting state j, starting in state i, is given by $f_{ij} = \left(n_{ij}/n_{jj}\right)$. Let $t = \min\{k \geq 0 : X_k \in \{0, L+1\}\}$. Then $E\left[t \mid X_0 = i\right] = \tau_i$ denotes the expected absorption time for a MC-RW starting at the state i where τ_i is the i th entry of vector $\tau = Ne$. Thus, τ_i is just the sum of all entries of row i of the fundamental matrix. Note that in the general MC-RW model on a ring network the mean number of visits to transient state j from i before the particle enters any absorbing state is given in Theorem 2.1.

The next theorem is immediate from Theorem 2.1 and the fact

$$\tau = \sum_{j=1}^{i} n_{ij} + \sum_{j=i+1}^{L} n_{ij} . \tag{3.1}$$

Theorem 3.1. The expected time to absorption with initial state i, $1 \leq i \leq L$ is given by

$$\tau = \frac{1}{\Delta}\left[\sum_{j=1}^{i}\left(\left(\Delta_{i+1}^{L}\Delta_{1}^{j-1}\right)\prod_{k=j+1}^{i} q_k - \beta\alpha\Delta_{i+1}^{L-1}\Delta_{2}^{j-1}\prod_{k=j+1}^{i} q_k + \beta\Delta_{j+1}^{i-1}\prod_{k=1}^{j-1} p_k \prod_{k=i}^{L-1} p_k\right)\right.$$
$$\left. + \sum_{j=i+1}^{L}\left(\left(\Delta_{j+1}^{L}\Delta_{1}^{i-1} - \beta\alpha\Delta_{j+1}^{L-1}\Delta_{2}^{i-1}\right)\prod_{k=i}^{j-1} p_k + \alpha\Delta_{i+1}^{j-1}\prod_{k=2}^{i} q_k \prod_{k=j+1}^{L} q_k\right)\right], \tag{3.2}$$

where Δ_i^j is given as in Lemma 2.1, and

$$\Delta = \Delta_1^L - \beta\alpha\Delta_2^{L-1} - \beta\prod_{k=1}^{L-1} p_k - \alpha\prod_{k=2}^{L} q_k . \tag{3.3}$$

Theorem 3.2. The probabilities, B_0 and B_{L+1}, of being absorbed at the boundary states 0 and $L+1$ given the initial state i, $1 \leq i \leq L$ respectively, are

$$B_0 = \frac{1}{\Delta}\left(\Delta_{i+1}^{L}\prod_{k=1}^{i} q_k + \beta q_1 \Delta_2^{i-1}\prod_{k=i}^{L-1} p_k\right), \tag{3.4}$$

$$B_{L+1} = \frac{1}{\Delta}\left(\Delta_1^{i-1}\prod_{k=i}^{L} p_k + \alpha p_L \Delta_{i+1}^{L-1}\prod_{k=2}^{i} q_k\right) \tag{3.5}$$

where Δ_i^j and Δ are given, respectively, as in Lemma 2.1, and (3.3).

Proof. Let B_0, and B_{L+1} are, respectively, the probabilities of being absorbed at state 0 and $L+1$ given the starting state i, then for $1 \leq i \leq L$

$$B_0 = \Pr\{X_t = 0 \mid X_0 = i\} = n_{i1}.R_{10}, \tag{3.6}$$

$$B_{L+1} = \Pr\{X_t = L+1 \mid X_0 = i\} = n_{iL}.R_{LL+1}. \tag{3.7}$$

$$\Delta_i^j = \prod_{k=i}^{j} p_k \,. \tag{2.13}$$

Dividing both sides of equation (2.12) by the solution (2.13) , it is easy to see,

$$\frac{\Delta_i^j}{\prod_{k=i}^{j} p_k} - \frac{\Delta_i^{j-1}}{\prod_{k=i}^{j-1} p_k} = \frac{\prod_{s=i}^{j} q_s}{\prod_{k=i}^{j} p_k} \,. \tag{i}$$

Replacing $j-1$ instead of j in equation (i), we get

$$\frac{\Delta_i^{j-1}}{\prod_{k=i}^{j-1} p_k} - \frac{\Delta_i^{j-2}}{\prod_{k=i}^{j-2} p_k} = \frac{\prod_{s=i}^{j-1} q_s}{\prod_{k=i}^{j-1} p_k} \,. \tag{ii}$$

Replacing $j-2$ instead of $j-1$ in equation (ii), and repeating this process until reaching to the following equation

$$\frac{\Delta_i^{i}}{\prod_{k=i}^{i} p_k} - \frac{\Delta_i^{i-1}}{\prod_{k=i}^{i-1} p_k} = \frac{\prod_{s=i}^{i} q_s}{\prod_{k=i}^{i} p_k} \,. \tag{iii}$$

A further replacing $i-1$ instead of i in equation (iii), we get

$$\frac{\Delta_i^{i-1}}{\prod_{k=i}^{i-1} p_k} - \frac{\Delta_i^{i-2}}{\prod_{k=i}^{i-2} p_k} = \frac{\prod_{s=i}^{i-1} q_s}{\prod_{k=i}^{i-1} p_k} \,. \tag{iv}$$

Summing the equations from (i) until (iv), finally we get, a particular solution of (2.12) as

$$\Delta_i^j = \prod_{k=i}^{j}(p_k)\sum_{s=i}^{j}\left(\prod_{k=i}^{s}\left(\frac{q_k}{p_k}\right)\right). \tag{2.14}$$

From (2.13) and (2.14), the general solution of the difference equation (2.2) is

$$\Delta_i^j = g\prod_{k=i}^{j}(p_k) + \prod_{k=i}^{j}(p_k)\sum_{s=i}^{j}\left(\prod_{k=i}^{s}\left(\frac{q_k}{p_k}\right)\right),$$

where g is an arbitrary constant which can be determined from the initial condition $\Delta_i^{i-1} = 1$, we get $g = 1$. Then the general solution will be

$$\Delta_i^j = \prod_{k=i}^{j}(p_k) + \prod_{k=i}^{j}(p_k)\sum_{s=i}^{j}\left(\prod_{k=i}^{s}\left(\frac{q_k}{p_k}\right)\right), \tag{2.15}$$

which can be simplified to obtain (2.9).

Formula (2.9) can also be confirmed by induction.

$$\gamma_k = \begin{cases} 1-r_i, & k=i \\ (1-r_k) - \dfrac{p_{k-1}q_k}{\gamma_{k-1}}, & k=i+1, i+2, \dots, j, \end{cases} \tag{2.6}$$

$$\delta_k = \begin{cases} 1-r_j, & k=j \\ (1-r_k) - \dfrac{p_k q_{k+1}}{\delta_{k+1}}, & k=j-1, j-2, \dots, i. \end{cases} \tag{2.7}$$

Proceeding as in El-Shehawey et al. (2008)$_c$ (Theorem 3.1), with $n=L$; $a_i = 1-r_i$, $i=1,2,\dots,L$; $c_i = -p_i, i=1,2,\dots,L-1$; $d_i = q_i, i=2,3,\dots,L$; $d_1 = \alpha$ and $c_n = \beta$, the elements of the inverse $(I-Q)^{-1}$ can be computed according to the following theorem:

Theorem 2.1. Let Q be given as in (1.3). If $(I-Q)^{-1} = (n_{ij})_{ij}$ exists, then

$$n_{ij} = \begin{cases} \dfrac{\left(\Delta_{j+1}^{L}\Delta_1^{i-1} - \beta\alpha\Delta_{j+1}^{L-1}\Delta_2^{i-1}\right)\prod\limits_{k=i}^{j-1} p_k + \alpha\Delta_{i+1}^{j-1}\prod\limits_{k=2}^{i} q_k \prod\limits_{k=j+1}^{L} p_k}{\left[\Delta_1^{L} - \beta\alpha\Delta_2^{L-1} - \beta\prod\limits_{k=1}^{L-1} p_k - \alpha\prod\limits_{k=2}^{L} q_k\right]}, & i \le j \\[2ex] \dfrac{\left(\Delta_{i+1}^{L}\Delta_1^{j-1} - \beta\alpha\Delta_{i+1}^{L-1}\Delta_2^{j-1}\right)\prod\limits_{k=j+1}^{i} q_k + \beta\Delta_{j+1}^{i-1}\prod\limits_{k=1}^{j-1} p_k \prod\limits_{k=i}^{L-1} q_k}{\left[\Delta_1^{L} - \beta\alpha\Delta_2^{L-1} - \beta\prod\limits_{k=1}^{L-1} p_k - \alpha\prod\limits_{k=2}^{L} q_k\right]}, & i \ge j \end{cases} \tag{2.8}$$

where the sequence of determinants $\{\Delta_i^j\}$ is computed by the three-term recurrence relations (2.2)-(2.4).

With the assumptions that $p_i + r_i + q_i = 1, 2 \le i \le L-1, 0 < p_1 + r_1 < 1$, and $0 < q_L + r_L < 1$ explicit expression for the determinant Δ_i^j is obtained in the following lemma.

Lemma 2.1.

$$\Delta_i^j = \prod_{k=i}^{j}(p_k) + \sum_{s=i}^{j}\left(\prod_{k=s+1}^{j}(p_k)\right)\left(\prod_{k=i}^{s}(q_k)\right). \tag{2.9}$$

Proof:

Under the assumptions $q_1 = 1-(p_1+r_1)$, and $p_L = 1-(q_L+r_L)$, we can generalize the condition $p_i + r_i + q_i = 1$ from $2 \le i \le L-1$ to $1 \le i \le L$, and the relation (2.2) will be reduced to

$$\left(\Delta_i^j - p_j\Delta_i^{j-1}\right) = q_j\left(\Delta_i^{j-1} - p_{j-1}\Delta_i^{j-2}\right). \tag{2.10}$$

From (2.4),

$$\Delta_i^i = 1-r_i, \ \Delta_i^{i-1} = 1. \tag{2.11}$$

The difference equation (2.10) using (2.11) can be solved by a simple iteration as:

$$\Delta_i^j - p_j\Delta_i^{j-1} = \prod_{s-i}^{j} q_s. \tag{2.12}$$

The general solution to the associated homogeneous equation for (2.12) is fundamental matrix $N = \left(n_{ij}\right)_{ij}$, which equal $\left(I - Q\right)^{-1}$, where I is an identity matrix, and Q is a periodic tridiagonal matrix of order L :

$$Q \equiv Q\left(\alpha; r_k, p_k, q_k; \beta\right) = \begin{bmatrix} r_1 & p_1 & 0 & \cdots & 0 & \alpha \\ q_2 & r_2 & p_2 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & 0 \\ 0 & & & q_{L-1} & r_{L-1} & p_{L-1} \\ \beta & 0 & \cdots & 0 & q_L & r_L \end{bmatrix}. \tag{1.3}$$

The matrix $Q \equiv Q\left(\alpha; r_k, p_k, q_k; \beta\right)$ is called tridiagonal if $\alpha = \beta = 0$, and denoted by T, that is $T \equiv T\left(r_k, p_k, q_k\right) \equiv Q\left(0; r_k, p_k, q_k; 0\right)$.

Each entry n_{ij} of N gives the mean number of times starting in state i that the particle moves to state j before being absorbed. The mean times to absorption are given by the row sums of N, displayed as a vector τ. Thus $\tau = Ne$, where e denotes a column vector of all units. The probabilities of being absorbed into a particular state are given by the matrix product $B = NR$, where R is the lower left part of the transition probability matrix P.

2. Closed Form Expressions for the Fundamental Matrix

Determinant, Δ_i^j, in the following form shall be considered here:

$$D_i^j = \begin{vmatrix} 1-r_i & -p_i & 0 & L & L & 0 \\ -q_{i+1} & 1-r_{i+1} & -p_{i+1} & O & & M \\ 0 & O & O & O & O & M \\ M & O & O & O & O & 0 \\ M & & O & -q_{j-1} & 1-r_{j-1} & -p_{j-1} \\ 0 & L & L & 0 & -q_j & 1-r_j \end{vmatrix}. \tag{2.1}$$

By expanding the determinant Δ_i^j the three-term recurrence relations may be easily derived

$$\Delta_i^j = \left(1 - r_j\right)\Delta_i^{j-1} - q_j p_{j-1} \Delta_i^{j-2}, \tag{2.2}$$

and

$$\Delta_i^j = \left(1 - r_i\right)\Delta_{i+1}^j - q_{i+1} p_i \Delta_{i+2}^j, \tag{2.3}$$

where

$$\Delta_i^j = \begin{cases} 1 - r_i, & i = j \\ 1, & i = j+1 \\ 0, & i \ge j+2 \end{cases}. \tag{2.4}$$

Following for example, El_Shehawey (2008)a (see also Meurant (1992), and Usmani (1994)), it is easily to verify that the solutions of the recurrence relations (2.2)-(2.4) are

$$\Delta_i^j = \prod_{k=i}^{j} \gamma_k, \quad \text{and} \quad \Delta_i^j = \prod_{k=i}^{j} \delta_k, \tag{2.5}$$

where

(a) – The particle starts from the state i , $i \in \{1,2,...,L\}$,

(b) – From the current state k , $k \in \{2,...,L-1\}$ the one-step transition can occur only to one of the states $k-1, k$ or $k+1$ with respective probabilities q_k, r_k and p_k , $p_k + r_k + q_k = 1$,

(c) – When the particle reaches the state $k=1$, the one-step transition can occur only to one of the states $0,1,2$ or L with respective probabilities q_1, r_1, p_1 and α , $\alpha = 1-(p_1 + r_1 + q_1)$,

(d) - When the particle reaches the state $k = L$, the one-step transition can occur only to one of the states $L-1, L, L+1$ or 1 with respective probabilities q_L, r_L, p_L and $\beta, \beta = 1-(p_L + r_L + q_L)$.

This corresponds to the situation when, at the states 1 and L , the particle is either lost from the system with respective probabilities q_1 and p_L or turned back with respective probabilities r_1, p_1 and α to the states $1,2$ and L or r_L, q_L and β to the states $L, L-1$ and 1, and reduces to the problems of MC-RW on a ring network.

Such model is a generalization of many particular MC-RW problems, depending on appropriate choices of the probabilities $r_1, p_1, r_L, q_L, \alpha$ and β. The general MC-RW is a stochastic process $\{X_k : k \geq 0\}$, with state space $\{0,1,...,L+1\}$ and transition probability matrix P :

$$P = \begin{bmatrix} 1 & 0 & & \cdots & \cdots & & 0 & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & 0 & \alpha & \vdots \\ 0 & q_2 & r_2 & p_2 & \ddots & & 0 & \vdots \\ & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ & 0 & & \ddots & q_{L-1} & r_{L-1} & p_{L-1} & 0 \\ 0 & \beta & 0 & \cdots & 0 & q_L & r_L & p_L \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}, \tag{1.1}$$

such that $\Pr\{X_{k+1} = j \mid X_k = i\} = p_{ij}$, for $i, j = 0,1,...,L+1$, X_k represented the state of the particle at time k . P contains all the information on the geometry of the ring network and the kinetics of the MC-RW. Consequently, any information on the behavior of the MC-RW should be derivable from the properties of the transition matrix P .

The purpose of this paper is to find closed form expressions for the expected number of visits to each state before absorption, the mean time to absorption as well as the probabilities of being absorbed into different absorbing states for the MC-RW model. This work is a generalization to previous works; it is apparently not covered by the literature.

The matrix P defined in (1.1) can be written in the following canonical form:

$$P = \left[\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right]. \tag{1.2}$$

It is well known that (cf. Kemeny and Snell (1976) and Iosifescu (1980)) in order to find the quantities we are interested in, the key quantity to calculate is the

ON ABSORPTION PROBABILITIES RANDOM WALK

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Abstract

Presented is a more general form of Markov chain random walk with absorbing barriers has the transition probabilities $p_{0j} = \delta_{0j}$, $p_{L+1j} = \delta_{L+1j}$, $p_{1L} = \alpha$, $p_{L1} = \beta$, $p_{ii-1} = q_i$, $p_{ii} = r_i$, $p_{ii+1} = p_i$, $p_1 + r_1 + q_1 = 1 - \alpha$, $p_L + r_L + q_L = 1 - \beta$, $0 \le \alpha, \beta < 1$, and $p_i + r_i + q_i = 1$ for $i \in \{2,...,L-1\}$. Theoretical formulae are given for the mean number of step before absorption, the mean time to absorption as well as the absorption probabilities, when the states placed on a ring network and on one dimensional lattice segment. The results are generalizing previous works. Explicit expressions are also given for certain models, and two simple mathematical models in biology are discussed.

Key words: Transition matrix; Markov chain-random walk; Fundamental matrix; Mean time To absorption; Absorption probabilities
AMS subject classifications. 60G40, 60J80

1 Introduction

Markov chain random walk (MC-RW) is of great interest both for its central position in the theory of stochastic processes and because it has applications in very different research fields, including theory of potential, statistical field theory, structure of matter, biophysics, population biology, robotic, economics and others. Despite its long history, novel aspects continue to surface. For comprehensive treatments of MC-RW and their applications, cf. Bhat (1961), Weesakul (1961), Watterson (1961), Parzen (1962), Neuts (1963, and 1964), Cox and Miller (1965), Feller (1968), Hardin and Sweet (1970), Kao (1974), Kemeny and Snell (1976), Srinivasan and Mehata (1978), Beyer and Waterman (1979), Iosifescu (1980), Percus (1985), Grimmett et al. (1987), El_Shehawey (1994, 2000, 2008a , and 2008_b).

Higham (2003) studied the small-world properties of the ring network when it is reformulated as a Markov chain. Specifically, the initial ring network is identified with a one-dimensional symmetric periodic random walk. He analyzed the Markov chain model by combination of matrix perturbation theory and finite difference approximations to an underlying boundary value problem.

Catral, et al. (2005) investigated more explicitly, via purely matrix-theoretic methods, the Markov chain of the small-world properties of the ring network. He followed the terminology and notation of Higham (2003).

In this paper we consider a particle which executes a MC-RW on a ring network. The choice of a ring network instead of the more common MC-RW on an infinite long straight line is made for mathematical convenience, similar to the use of periodic boundary conditions in solid-state physics (cf. Franceschetti et al. (1998), and Kemeny et al. (1976)). A ring network consisting of L states labeled $1,2,...,L$, governed by two absorbing barriers at 0 and $L+1$. The states 0 and $L+1$ are placed outside the ring network. The model proceeds according the following assumptions:

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$Geo_1, Geo_2/G/1$	A $Geo/G/1$ queue with two types of arrivals.
$Geo^{[X]}/G/1$	A $Geo/G/1$ queue with batch arrival.
$Geo^{[X]}/G_H/1$	A $Geo^{[X]}/G/1$ queue with multiple types of service.
Repair time	The time it takes the server to return to work after breakdown and repair.
Multiple types of service	The server gives the customer the opportunity to choose his service time distribution from a pre-specified set of distributions.

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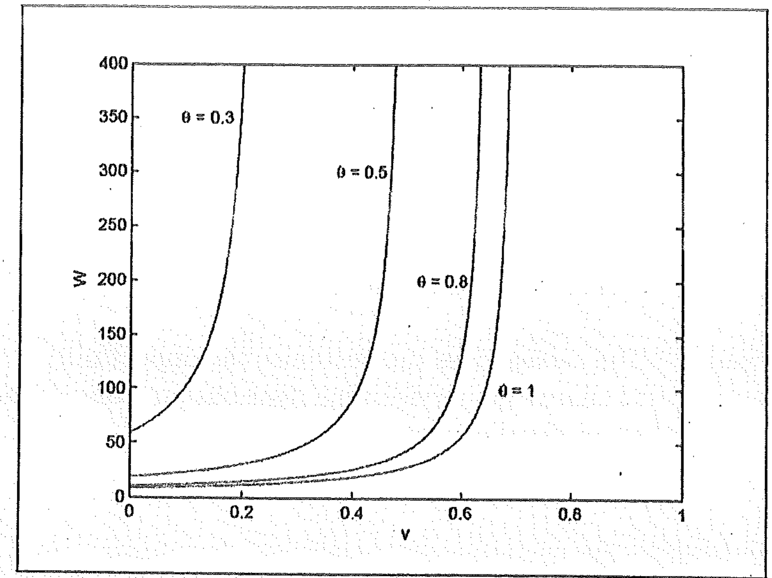

Fig. 4. The mean wait time in the system versus v for different θ.

also the probability generating functions of the orbit size and the system size distributions and this yielded expressions for the main performance characteristics of the system. The effect of different system parameters on the main performance measures was illustrated numerically.

The present work was a step towards extending the theory of discrete-time retrial queues with an unreliable server. We were concerned mainly with the generalization of the $Geo^{[X]}/G_H/1$ retrial queue with feedback to the case of a server with starting failures. We intend to make a similar treatment for a server subject to breakdowns. It remains to consider other features such as: impatient customers, waiting room existence, finite population, multiple servers, ...etc. working with an unreliable server.

Appendix A

Notation	Description
$Geo/Geo/1$	A single server queue with geometric inter-arrival times and geometric service times.
$Geo/Geo/c$	A c-server queue with geometric inter-arrival times and geometric service times.
$Geo/G/1$	A single server queue with geometric inter-arrival times and general service time.

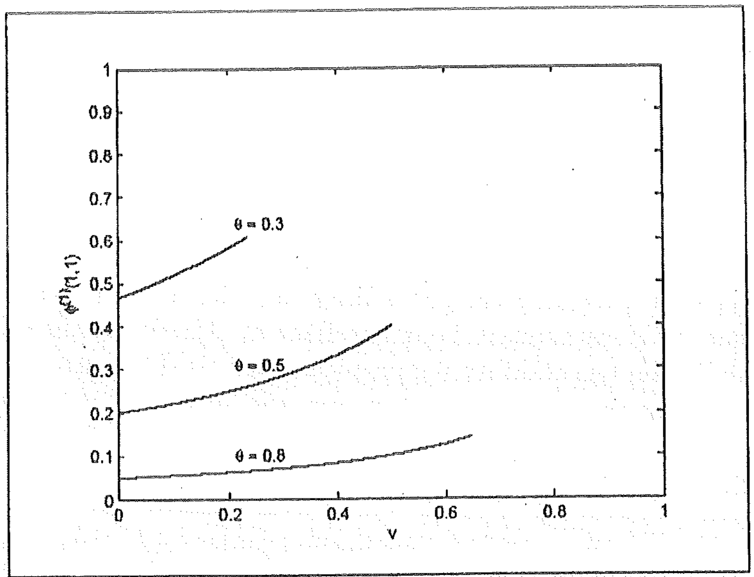

Fig. 2. The probability that the server is down versus v for different θ.

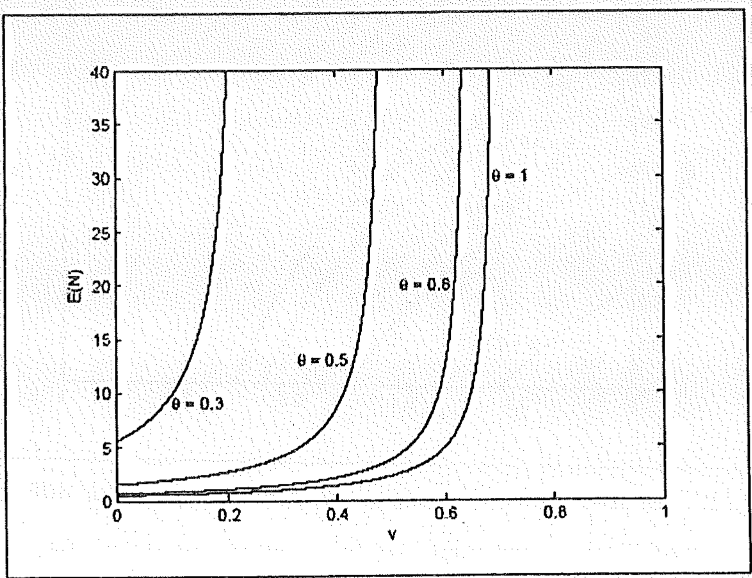

Fig. 3. The mean orbit size versus v for different θ.

5 Closing remarks

We have discussed a discrete-time $Geo^{[X]}/G_H/1$ retrial queue with feedback and starting failures. Retrial times are independent and have a common geometric distribution. Early arrival scheme was assumed to control the system evolution. Using supplementary variables technique, this queueing system was modelled using a Markov process. The generating function of the stationary distribution of this process was derived. Based on this result, we obtained of a feedback, decreases the probability that the server is idle while increases the probability that the server is down because feedback increases the working load. However, results of Figures 1 and 2 show that $\varphi^{(0)}(1)$ is more affected by changing v than $\varphi^{(1)}(1,1)$. In fact, Corollary 6 implies that this conclusion is valid in general. The expression of $\varphi^{(0)}(1)$ has an added term $(\frac{\rho_H}{v})$ that depends on v. On the other hand, Figures 1 and 2 reveal that θ has an equal (but inverse) effect on $\varphi^{(0)}(1)$ and $\varphi^{(1)}(1,1)$. Decreasing the server reliability (by decreasing the value of θ), clearly increases the probability that the server is down $\varphi^{(1)}(1,1)$ which implies an equal decrease in the probability that the server is idle $\varphi^{(0)}(1)$ because the probability that the server is busy $\varphi^{(H)}(1,1)$ is independent of θ (as appears from Corollary 6). Moreover, the effect of the server reliability on the stability region regarding v is illustrated in Figures 1 and 2. Decreasing the value of θ, decreases the server availability and this implies a noticeable reduction in the maximum allowable feedback probability v^* especially for small values of θ.

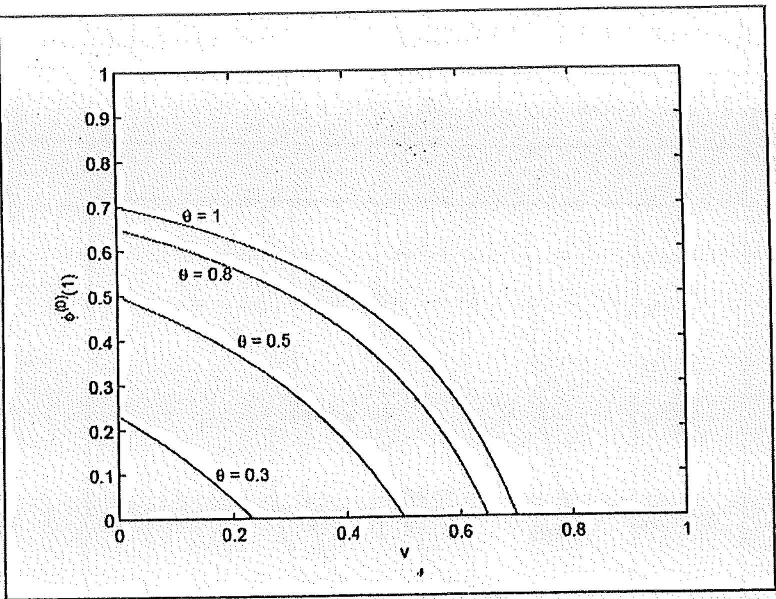

Fig. 1. The probability that the server is idle versus v for different θ.

Figure 3 depicts the behavior of the mean orbit size with parameters θ and v. It is assumed that $q_2 = 0.5$ and $\xi_1 = 2$. Clearly, $E(N)$ is an increasing function of v due to the increased working load caused by customer feedback. On the other side, $E(N)$ is a decreasing function of θ due to the decrease in the orbit joining rate when the server is more reliable $(\theta \to 1)$. The mean orbit size diverges when v tends to the upper limit of the stability region.

Figure 4 depicts the behavior of the mean wait time in the system W with parameters θ and v. We assume that $q_2 = 0.5$ and $\xi_1 = 2$. Decreasing the probability of a feedback, decreases the mean wait time while decreasing the probability of a successful start increases this time.

(2) The mean orbit size is given by

$$\begin{aligned} E(N) &= \Psi'(1) \\ &= \frac{2\,(\rho_1 + \frac{v}{\bar{v}}\rho_H^2 + \frac{v}{\bar{v}}\rho_1\rho_H) + p\xi_2(\frac{\bar{\theta}}{\theta}\beta_{1,1} + \beta_{.,1}) + p^2\xi_1^2(\frac{\bar{\theta}}{\theta}\beta_{1,2} + \beta_{.,2})}{2(\bar{v} - \rho)} \\ &+ \sum_{k=1}^{\infty} \frac{G'(r^k)}{G(r^k)}. \end{aligned}$$

(3) The mean system size is given by

$$\begin{aligned} E(L) &= \Phi'(1) \\ &= \frac{\rho(2 - 2\rho_H + \frac{\xi_1}{\xi_2}) + p^2\xi_1^2(\frac{\bar{\theta}}{\theta}\beta_{1,2} + \beta_{.,2})}{2(\bar{v} - \rho)} + \sum_{k=1}^{\infty} \frac{G'(r^k)}{G(r^k)}. \end{aligned}$$

(4) The mean time a customer spends in the system (including the service time) is given by

$$W = \frac{E(L)}{p}.$$

4 Numerical examples

In this section, we present a numerical study to illustrate the effect of different parameters on characteristics of the present queueing model. It is assumed that the arriving batch size follows a geometric distribution with arrival rate $p = 0.1$. The repair time is assumed to follow a geometric distribution with mean 1. There are two types of service whose distributions are geometrically distributed with means 1 and 2. It is also assumed that $r = 0.25$. For the system to be stable, the parameters θ and v are chosen to be in the intervals $[\theta^*, 1]$ and $[0, v^*]$, respectively, where

$$\theta^* = \frac{p\xi_1\beta_{1,1}}{p\xi_1(\beta_{1,1} - \beta_{.,1}) + \bar{v}},$$

and

$$v^* = 1 - \rho.$$

We consider four performance measures: the probability that the server is idle $\varphi^{(0)}(1)$, the probability that the server is down $\varphi^{(1)}(1,1)$, the mean orbit size $E(N)$ and the mean time a customer spends in the system W.

In Figures 1 and 2, $\varphi^{(0)}(1)$ and $\varphi^{(1)}(1,1)$ are plotted against v for different θ. We assume that $q_2 = 0.5$ and $\xi_1 = 2$. As expected, increasing the possibility

it easy to show that $F(z) \geq 0$ for $0 \leq z \leq 1$ if $\rho < \bar{v}$.
It is well known Conway (1978) [16] that the infinite product $\prod_{k=1}^{\infty}[1+F(r^k z)]$ with positive terms $F(r^k z)$ converges if and only if the infinite series $\sum_{k=1}^{\infty} F(r^k z)$ is convergent. The convergence of the latter is obvious since

$$\lim_{k\to\infty} \frac{F(r^{k+1}z)}{F(r^k z)} = r < 1.$$

The proof of Theorem 4 is now completed.

Based on Theorem 4, the generating functions of some important distributions can be derived. They are summarized in the following corollary.

Corollary 5

(1) Let N be the number of customers in the orbit with generating function $\Psi(z)$. Then

$$\begin{aligned}\Psi(z) &= \varphi^{(0)}(z) + \varphi^{(1)}(1,z) + \varphi^{(H)}(1,z)\\ &= \frac{\theta(1-z)(1-vS(\eta))}{\bar{\theta}zS_1(\eta) + \theta(\bar{v}+vz)S(\eta) - z}\varphi^{(0)}(z).\end{aligned}$$

(2) Let L be the number of customers in the system with generating function $\Phi(z)$. Then

$$\begin{aligned}\Phi(z) &= \varphi^{(0)}(z) + \varphi^{(1)}(1,z) + z\varphi^{(H)}(1,z)\\ &= \frac{\theta(1-z)\bar{v}S(\eta)}{\bar{\theta}zS_1(\eta) + \theta(\bar{v}+vz)S(\eta) - z}\varphi^{(0)}(z).\end{aligned}$$

The following corollary presents some performance measures of interest.

Corollary 6

(1) The server state distribution is given by

$$\begin{aligned}P[Idle] &= \varphi^{(0)}(1) = \frac{\bar{v}-\rho}{\bar{v}},\\ P[Busy] &= \varphi^{(H)}(1,1) = \frac{\rho_H}{\bar{v}},\\ P[Down] &= \varphi^{(1)}(1,1) = \frac{\rho_1}{\bar{v}}.\end{aligned}$$

Substituting for $\varphi_1^{(H)}(z)$ and $\varphi_1^{(1)}(z)$ in (11) leads to

$$\varphi^{(1)}(x,z) = p\bar{\theta}z\frac{S_1(x)-S_1(\eta)}{x-\eta}\frac{1-C(z)}{\bar{\theta}zS_1(\eta)+\theta(\bar{v}+vz)S(\eta)-z}x\varphi^{(0)}(z). \quad (22)$$

To evaluate $\varphi^{(0)}(z)$, we substitute in (6) for $\varphi_1^{(H)}(rz)$ and $\varphi_1^{(1)}(rz)$ as given by (17) and (18). This yields

$$\varphi^{(0)}(z) = \frac{\bar{\theta}rzS_1(\tilde{\eta})+\theta(\bar{v}+vrz)S(\tilde{\eta})-rz\tilde{\eta}}{\tilde{\eta}[\bar{\theta}rzS_1(\tilde{\eta})+\theta(\bar{v}+vrz)S(\tilde{\eta})-rz]}\bar{p}\,\varphi^{(0)}(rz), \quad (23)$$

where $\tilde{\eta} = \bar{p}+pC(rz) = \eta|_{z\to rz}$. Hence,

$$\varphi^{(0)}(z) = G(rz)\varphi^{(0)}(rz). \quad (24)$$

Applying (24) recursively, we get

$$\varphi^{(0)}(z) = \varphi^{(0)}(0)\prod_{k=1}^{\infty}G(r^kz).$$

Setting $z=1$, then

$$\varphi^{(0)}(0) = \frac{\varphi^{(0)}(1)}{\prod_{k=1}^{\infty}G(r^k)}. \quad (25)$$

From the normalizing condition

$$\varphi^{(0)}(1)+\varphi^{(1)}(1,1)+\varphi^{(H)}(1,1) = 1, \quad (26)$$

we get

$$\varphi^{(0)}(1) = \frac{\bar{v}-\rho}{\bar{v}}.$$

Hence,

$$\varphi^{(0)}(z) = \frac{\bar{v}-\rho}{\bar{v}}\frac{\prod_{k=1}^{\infty}G(r^kz)}{\prod_{k=1}^{\infty}G(r^k)}. \quad (27)$$

The remaining point is to prove the convergence of the infinite product $\prod_{k=1}^{\infty}G(r^kz)$. This product can be rewritten as

$$\prod_{k=1}^{\infty}G(r^kz) = \prod_{k=1}^{\infty}[1+F(r^kz)], \quad (28)$$

where

$$F(z) = \frac{\eta z - C(z)[\bar{\theta}zS_1(\eta)+\theta(\bar{v}+vz)S(\eta)]}{\eta[\bar{\theta}zS_1(\eta)+\theta(\bar{v}+vz)S(\eta)-z]}p.$$

Using Lemmas 1 and 3, and the clear inequality

$$\begin{aligned}\eta z - zC(z)\bar{\theta}S_1(\eta) - C(z)(\bar{v}+vz)\theta S(\eta) &\geq \eta z - z\bar{\theta}S_1(\eta) - z\theta S(\eta)\\ &= z[\theta(\eta - S(\eta)) + \bar{\theta}(\eta - S_1(\eta))]\\ &\geq 0,\end{aligned}$$

$$
\begin{aligned}
\frac{x-\eta}{x}\varphi^{(h)}(x,z) = & \eta\theta q_h S_h(x)\frac{\bar{v}+vz}{z}\varphi_1^{(H)}(z) + \eta\theta q_h S_h(x)\frac{1}{z}\varphi_1^{(1)}(z) \\
& - p\,\theta q_h S_h(x)\frac{1-C(z)}{z}\varphi^{(0)}(z) - \eta\varphi_1^{(h)}(z).
\end{aligned} \tag{12}
$$

Summing (12) over h, then

$$
\begin{aligned}
\frac{x-\eta}{x}\varphi^{(H)}(x,z) = & \eta(\theta S(x)\frac{\bar{v}+vz}{z} - 1)\varphi_1^{(H)}(z) + \eta\theta S(x)\frac{1}{z}\varphi_1^{(1)}(z) \\
& - p\,\theta S(x)\frac{1-C(z)}{z}\varphi^{(0)}(z),
\end{aligned} \tag{13}
$$

Setting $x=\eta$ in (11)-(13) yields

$$
\begin{aligned}
p\,\bar{\theta}S_1(\eta)(1-C(z))\varphi^{(0)}(z) = & \bar{\theta}\eta(\bar{v}+vz)S_1(\eta)\varphi_1^{(H)}(z) \\
& + \eta(\bar{\theta}S_1(\eta)-1)\varphi_1^{(1)}(z),
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
p\,\theta q_h S_h(\eta)\frac{(1-C(z))}{z}\varphi^{(0)}(z) = & \eta(\theta\frac{\bar{v}+vz}{z}q_h S_h(\eta)\varphi_1^{(H)}(z) \\
& + \eta\frac{\theta}{z}q_h S_h(\eta)\varphi_1^{(1)}(z) - \eta\varphi_1^{(h)}(z),
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
p\,\theta S(\eta)\frac{(1-C(z))}{z}\varphi^{(0)}(z) = & \eta(\theta\frac{\bar{v}+vz}{z}S(\eta)-1)\varphi_1^{(H)}(z) \\
& + \eta\frac{\theta}{z}S(\eta)\varphi_1^{(1)}(z).
\end{aligned} \tag{16}
$$

Solving (14) and (16) gives

$$
\varphi_1^{(H)}(z) = \frac{p\,\theta(1-C(z))S(\eta)}{\eta[\bar{\theta}zS_1(\eta)+\theta(\bar{v}+vz)S(\eta)-z]}\varphi^{(0)}(z), \tag{17}
$$

$$
\varphi_1^{(1)}(z) = \frac{p\,\bar{\theta}z(1-C(z))S_1(\eta)}{\eta[\bar{\theta}zS_1(\eta)+\theta(\bar{v}+vz)S(\eta)-z]}\varphi^{(0)}(z). \tag{18}
$$

By substituting for $\varphi_1^{(H)}(z)$ and $\varphi_1^{(1)}(z)$ into (15), we obtain

$$
\varphi_1^{(h)}(z) = \frac{p\,\theta q_h(1-C(z))S_h(\eta)}{\eta[\bar{\theta}zS_1(\eta)+\theta(\bar{v}+vz)S(\eta)-z]}\varphi^{(0)}(z). \tag{19}
$$

Taking into account Lemmas 1 and 2, then $\varphi_1^{(H)}(z)$, $\varphi_1^{(1)}(z)$ and $\varphi_1^{(h)}(z)$ are well-defined in [0,1) and are extended by continuity for $z=1$ if $\rho<\bar{v}$.
Inserting $\varphi_1^{(H)}(z)$, $\varphi_1^{(1)}(z)$ and $\varphi_1^{(h)}(z)$ into (12), we get

$$
\varphi^{(h)}(x,z) = p\,\theta q_h\frac{S_h(x)-S_h(\eta)}{x-\eta}\,\frac{1-C(z)}{\bar{\theta}zS_1(\eta)+\theta(\bar{v}+vz)S(\eta)-z}x\varphi^{(0)}(z). \tag{20}
$$

Hence,

$$
\varphi^{(H)}(x,z) = p\,\theta\frac{S(x)-S(\eta)}{x-\eta}\,\frac{1-C(z)}{\bar{\theta}zS_1(\eta)+\theta(\bar{v}+vz)S(\eta)-z}x\varphi^{(0)}(z). \tag{21}
$$

Theorem 4 *If $\rho < \bar{v}$, then the stationary distribution of the Markov process $\{Y_m, m = 0, 1, 2, ...\}$ has the following generating functions:*

$$\varphi^{(0)}(z) = \frac{\bar{v} - \rho}{\bar{v}} \frac{\prod_{k=1}^{\infty} G(r^k z)}{\prod_{k=1}^{\infty} G(r^k)},$$

$$\varphi^{(1)}(x, z) = p\bar{\theta}z \frac{S_1(x) - S_1(\eta)}{x - \eta} \frac{1 - C(z)}{\bar{\theta}zS_1(\eta) + \theta(\bar{v} + vz)S(\eta) - z} x\varphi^{(0)}(z),$$

$$\varphi^{(h)}(x, z) = p\theta q_h \frac{S_h(x) - S_h(\eta)}{x - \eta} \frac{1 - C(z)}{\bar{\theta}zS_1(\eta) + \theta(\bar{v} + vz)S(\eta) - z} x\varphi^{(0)}(z),$$

where

$$G(z) = \frac{\bar{\theta}zS_1(\eta) + \theta(\bar{v} + vz)S(\eta) - z\eta}{\eta[\bar{\theta}zS_1(\eta) + \theta(\bar{v} + vz)S(\eta) - z]}\bar{p}.$$

PROOF. Multiplying (1)-(3) by z^k and summing over k, we obtain

$$\varphi^{(0)}(z) = \bar{p}[\varphi^{(0)}(rz) + (\bar{v} + vrz)\varphi_1^{(H)}(rz) + \varphi_1^{(1)}(rz)], \tag{6}$$

$$\begin{aligned} \varphi_i^{(1)}(z) = \eta\Big[&\bar{\theta}s_{1,i}\big(\varphi^{(0)}(z) + (\bar{v} + vz)\varphi_1^{(H)}(z) + \varphi_1^{(1)}(z)\big) + \varphi_{i+1}^{(1)}(z)\Big] \\ &- \bar{p}\bar{\theta}s_{1,i}\big(\varphi^{(0)}(rz) + (\bar{v} + vrz)\varphi_1^{(H)}(rz) + \varphi_1^{(1)}(rz)\big), \end{aligned} \tag{7}$$

$$\begin{aligned} \varphi_i^{(h)}(z) = \eta\Big[&\frac{\theta q_h s_{h,i}}{z}\big(\varphi^{(0)}(z) + (\bar{v} + vz)\varphi_1^{(H)}(z) + \varphi_1^{(1)}(z)\big) + \varphi_{i+1}^{(h)}(z)\Big] \\ &- \frac{\bar{p}\theta q_h s_{h,i}}{z}\big(\varphi^{(0)}(rz) + (\bar{v} + vrz)\varphi_1^{(H)}(rz) + \varphi_1^{(1)}(rz)\big). \end{aligned} \tag{8}$$

Substituting (6) into (7) and (8), then

$$\begin{aligned} \varphi_i^{(1)}(z) = \eta\big(&\bar{\theta}s_{1,i}(\bar{v} + vz)\varphi_1^{(H)}(z) + \bar{\theta}s_{1,i}\varphi_1^{(1)}(z) + \varphi_{i+1}^{(1)}(z)\big) \\ &- p\bar{\theta}s_{1,i}\big(1 - C(z)\big)\varphi^{(0)}(z), \qquad i \geq 1, \end{aligned} \tag{9}$$

$$\begin{aligned} \varphi_i^{(h)}(z) = \eta\Big(&\theta q_h s_{h,i}\frac{\bar{v} + vz}{z}\varphi_1^{(H)}(z) + \theta q_h s_{h,i}\frac{1}{z}\varphi_1^{(1)}(z) + \varphi_{i+1}^{(h)}(z)\Big) \\ &- \theta q_h s_{h,i}\frac{1 - C(z)}{z}p\varphi^{(0)}(z), \qquad i \geq 1. \end{aligned} \tag{10}$$

Multiplying (9) and (10) by x^i and adding up, we get

$$\begin{aligned} \frac{x - \eta}{x}\varphi^{(1)}(x, z) = &\eta\bar{\theta}S_1(x)(\bar{v} + vz)\varphi_1^{(H)}(z) + \eta(\bar{\theta}S_1(x) - 1)\varphi_1^{(1)}(z) \\ &- p\bar{\theta}S_1(x)(1 - C(z))\varphi^{(0)}(z), \end{aligned} \tag{11}$$

$$
\begin{aligned}
\pi_{h,i,k} = & \, p\,\theta q_h s_{h,i} \sum_{l=0}^{k} c_{l+1}\pi_{0,k-l} + \bar{p}(1-r^{k+1})\theta q_h s_{h,i}\pi_{0,k+1} \\
& + \theta q_h s_{h,i}(\bar{v}pc_1 + v\bar{p}(1-r^{k+1}))\pi_{.,1,k} + \bar{p}\bar{v}(1-r^{k+1})\theta q_h s_{h,i}\pi_{.,1,k+1} \\
& + (1-\delta_{0,k})p\sum_{l=1}^{k} c_l\pi_{h,i+1,k-l} + \bar{p}\pi_{h,i+1,k} \\
& + (1-\delta_{0,k})p\,\theta q_h s_{h,i}\sum_{l=1}^{k}(vc_l + \bar{v}c_{l+1})\pi_{.,1,k-l} + \bar{p}(1-r^{k+1})\theta q_h s_{h,i}\pi_{1,1,k+1} \\
& + (1-\delta_{0,k})p\,\theta q_h s_{h,i}\sum_{l=0}^{k-1} c_{l+1}\pi_{1,1,k-l}, \qquad h = 2,3,\ldots H, i \geq 1, k \geq 0,
\end{aligned}
\tag{3}
$$

together with the normalizing condition

$$
\sum_{k=0}^{\infty}\pi_{0,k} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{1,i,k} + \sum_{h=2}^{H}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{h,i,k} = 1, \tag{4}
$$

where $\bar{p} = 1-p$ and $\delta_{a,b}$ denotes the Kronecker's delta.

The following lemmas will be used during the analytical solution of the above equations. The proof is straightforward and hence omitted. Interested reader is referred to Aboul-Hassan et al. (2008) [4].

Lemma 1 *If $\rho < \bar{v}$, then*

$$
\bar{\theta}zS_1(\eta) + \theta(\bar{v} + vz)S(\eta) - z > 0, \tag{5}
$$

for $0 \leq z < 1$ where $\eta = \bar{p} + pC(z)$.

Lemma 2 *The following limit is positive if and only if $\rho < \bar{v}$:*

$$
\lim_{z\to 1}\frac{1-C(z)}{\bar{\theta}zS_1(\eta) + \theta(\bar{v}+vz)S(\eta) - z} = \frac{\xi_1}{\theta(\bar{v}-\rho)}.
$$

Lemma 3 *The following inequalities hold:*

$$
\begin{array}{lll}
S_1(x) \leq x & for & 0 \leq x \leq 1, \\
S(x) \leq x & for & 0 \leq x \leq 1, \\
C(z) \leq z & for & 0 \leq z \leq 1.
\end{array}
$$

The following theorem discusses the solution of the Kolmogorov equations (1)-(4).

$$
\begin{aligned}
\varphi^{(0)}(z) &= \sum_{k=0}^{\infty} \pi_{0,k} z^k, \\
\varphi_i^{(1)}(z) &= \sum_{k=1}^{\infty} \pi_{1,i,k} z^k, \qquad i \geq 1, \\
\varphi^{(1)}(x,z) &= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \pi_{1,i,k} z^k x^i = \sum_{i=1}^{\infty} \varphi_i^{(1)}(z) x^i, \\
\varphi_i^{(h)}(z) &= \sum_{k=0}^{\infty} \pi_{h,i,k} z^k \qquad h = 2,3,...,H, i \geq 1, \\
\varphi^{(h)}(x,z) &= \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{h,i,k} z^k x^i = \sum_{i=1}^{\infty} \varphi_i^{(h)}(z) x^i, h = 2,3,...,H, \\
\varphi_i^{(H)}(z) &= \sum_{k=0}^{\infty} \pi_{.,i,k} z^k = \sum_{h=2}^{H} \varphi_i^{(h)}(z), \qquad i \geq 1, \\
\varphi^{(H)}(x,z) &= \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{.,i,k} z^k x^i = \sum_{h=2}^{H} \varphi^{(h)}(x,z),
\end{aligned}
$$

where $\pi_{.,i,k} = \sum_{h=2}^{H} \pi_{h,i,k}$ is the stationary probability that, the remaining service time is i slots and the orbit contains k customers.

The Kolmogorov equations of the Markov process $\{Y_m, m = 0,1,2,...\}$ are given by

$$\pi_{0,k} = \bar{p}r^k\pi_{0,k} + \bar{p}\bar{v}r^k\pi_{.,1,k} + (1-\delta_{0,k})\bar{p}vr^k\pi_{.,1,k-1} + (1-\delta_{0,k})\bar{p}r^k\pi_{1,1,k}, \; k \geq 0, \quad (1)$$

$$
\begin{aligned}
\pi_{1,i,k} = {} & p\bar{\theta}s_{1,i}\sum_{l=1}^{k} c_l\pi_{0,k-l} + \bar{p}(1-r^k)\bar{\theta}s_{1,i}\pi_{0,k} + \bar{\theta}s_{1,i}(pc_1\bar{v} + \bar{p}v(1-r^k))\pi_{.,1,k-1} \\
& + (1-\delta_{l,k})p\bar{\theta}s_{1,i}\sum_{l=1}^{k}(\bar{v}c_{l+1} + vc_l)\pi_{.,1,k-l-1} + \bar{p}\bar{v}(1-r^k)\bar{\theta}s_{1,i}\pi_{.,1,k} \\
& + (1-\delta_{1,k})p\bar{\theta}s_{1,i}\sum_{l=1}^{k-1} c_l\pi_{1,1,k-l} + \bar{p}(1-r^k)\bar{\theta}s_{1,i}\pi_{1,1,k} \\
& + (1-\delta_{1,k})p\sum_{l=1}^{k-1} c_l\pi_{1,i+1,k-l} + \bar{p}\pi_{1,i+1,k}, \qquad i \geq 1, k \geq 1,
\end{aligned}
\quad (2)
$$

A customer who finishes service returns to the orbit to seek another service with probability υ or leaves the system forever with probability $\bar{\upsilon} = 1 - \upsilon$.

Each customer in the orbit forms an independent retrial source with rate $1-r$. Therefore, the retrial time follows a geometric distribution with parameter $1 - r$. If more than one (primary and/or returning) customer arrive at the same slot and the server is idle, one of them is randomly chosen to access the server and the others join the orbit.

The repair time of the server has general distribution $\{s_{1,i}\}_{i=1}^{\infty}$, generating function $S_1(x) = \sum_{i=1}^{\infty} s_{1,i}x^i$ and nth factorial moment $\beta_{1,n}$.

Inter-arrival times, retrial times, service times and repair times are assumed to be mutually independent and independent of the system state.

The load of the system is given by $\rho = \rho_1 + \rho_H$ where $\rho_1 = p\frac{\bar{\theta}}{\theta}\xi_1\beta_{1,1}$, $\rho_H = p\xi_1\beta_{.,1}$ and $\beta_{.,1} = \sum_{h=2}^{H} q_h\beta_{h,1}$.

3 Generating functions analysis

At time m^+, the state of the system can be described by the Markov process

$$\{Y_m = (C_m, \zeta_{1,m}, \zeta_{h,m}, N_m), m = 0, 1, 2, \ldots, h = 2, 3, ..., H\},$$

where C_m denotes the server state (0, 1 or h if the server is idle, down or making a service of type h respectively) and N_m is the number of customers in the orbit. If $C_m = 1$, then $\zeta_{1,m}$ represents the remaining repair time and if $C_m = h$, then $\zeta_{h,m}$ represents the remaining service time of the customer currently being served.
The state space of the process $\{Y_m, m = 0, 1, 2, ...\}$ is

$$\{(0,k) : k \geq 0; (1,i,k) : i \geq 1, k \geq 1; (h,i,k) : i \geq 1, k \geq 0, h = 2, 3, ..., H\}.$$

The aim of this paper is to find the stationary distribution

$$\begin{aligned}
\pi_{0,k} &= \lim_{m\to\infty} P[C_m = 0, N_m = k], & k \geq 0,\\
\pi_{1,i,k} &= \lim_{m\to\infty} P[C_m = 1, \zeta_{1,m} = i, N_m = k], & i \geq 1, k \geq 1,\\
\pi_{h,i,k} &= \lim_{m\to\infty} P[C_m = h, \zeta_{h,m} = i, N_m = k], & i \geq 1, k \geq 0.
\end{aligned}$$

We define the following generating functions:

measures of the system. A numerical study is built up to investigate the effect of the system parameters on selected performance measures.

The rest of the paper is organized as follows. In the next section, a detailed description of the queueing system under study and some basic notations are given. We analyze the Markov process describing the system state using the generating functions technique in Section 3. The effect of different system parameters on the system performance is analyzed numerically in Section 4. Conclusion and suggested future work are presented in Section 5.

2 Model description

We consider a single server retrial queueing system working in discrete-time setting. Time axis is divided into intervals of equal length called slots and all system events are assumed to occur at the slot boundaries. Multiple events may occur at the same time epoch. Hence, it is necessary to determine the order in which events take place. Throughout this work, it is assumed that departures occur immediately before the slot boundaries (in the interval (m^-, m)), whereas arrivals and retrials occur immediately after the slot boundaries (in the interval (m, m^+)). This assumption is known as early arrival scheme (EAS) or departure first (DF) policy (Gravey and Hébuterne (1992) [19]; Hunter (1983) [21]). Batches of customers arrive at the system according to a geometrical process with parameter p. In other words, during any time slot a batch of customers arrives at the system with probability p. The distribution of the bath size is given by $\{c_l\}_{l=1}^{\infty}$. Let $C(z) = \sum_{l=1}^{\infty} c_l z^l$ be the generating function of the batch size distribution and ξ_n be the nth factorial moment. If the server is busy or down (sent for repair), all theses customers join the orbit. If the server is idle, only one of the arriving customers is allowed to access the server and the others join the orbit.

When a customer (primary or returning) is allowed to access the server, he must start (turn on) the server, which takes negligible time. If the server is activated successfully (with a probability θ), the customer begins his service immediately and leaves the system forever after service completion. If he cannot turn on the server (with a probability $\bar{\theta} = 1 - \theta$), the server is sent for repair and the customer joins the orbit.

Service time has a distribution whose type is defined at the beginning of each service. More specifically, immediately after starting the server successfully, the customer chooses a service of type h with probability q_h, where $\sum_{h=2}^{H} q_h = 1$. Service time of type h has a general distribution $\{s_{h,i}\}_{i=1}^{\infty}$, with generating function $S_h(x) = \sum_{i=1}^{\infty} s_{h,i} x^i$ and nth factorial moment $\beta_{h,n}$ where $h = 2, 3, ..., H$. Let $S(x) = \sum_{h=2}^{H} q_h S_h(x)$.

the time. From a practical point of view, the server always suffers from breakdowns, starting failures, periodical maintenances or other types of events that prevent it from being available whenever needed. It is clear that the system performance is highly affected by the server reliability. There is a few number of papers that focused on this phenomenon. Atencia and Moreno (2006a) [11] discussed a discrete-time $Geo/G/1$ retrial queue with the server subject to breakdowns and repairs. Atencia and Moreno (2006b) [12] studied a discrete-time $Geo/G/1$ retrial queue with the server subject to starting failures. Moreno (2006) [26] studied a discrete-time retrial queue with an unreliable server and general server lifetime. Wang and Zhao (2007a) [29] introduced a discrete-time $Geo/G/1$ retrial queue with general retrial times and starting failures. Wang and Zhao (2007b) [30] discussed a discrete-time $Geo/G/1$ retrial queue with starting failures in which all the arriving customers require a first essential service while only some of them ask for a second optional service.

Our intention in this work is to extend the theory of discrete-time retrial queues with an unreliable server. As far as we know, all the models in this research area assume a single arrival stream. For applications such as digital communication networks, a batch arrival stream is more appropriate. Messages are divided into packets which are served independently. Hence, the first line of extension here is to analyse a system with a batch arrival stream. Our second line of extension is related to the phenomenon of feedback which is not included before (as far as we know) in discrete-time retrial queues with an unreliable server. In fact, it is only the paper of Atencia and Moreno (2004b) [10] which considered the phenomenon of feedback in discrete-time retrial queues (but with a reliable server). We consider here (as in Atencia and Moreno (2004b) [10]) Bernoulli feedback in which any served customer returns back to the system with a fixed probability. Such feature is related to application such as communication networks in which corrupted packets are sent again. Finally, we assume a single (unreliable) server with multiple types of service. More specifically, an arriving customer which is allowed to enter the service facility chooses first his (random) service characteristics. Communication systems that are shared by different data types with different quality of service requirements make it natural to assume that arriving customers are not identical in their service requirements. Such feature was considered by Atencia and Moreno (2004b) [10] for the reliable server case. In fact, the present work can be viewed as an extension of the work that appeared in Atencia and Moreno (2004b) [10] to the unreliable server case.

Our model is a discrete-time $Geo/G/1$ retrial queue with feedback and starting failures. Moreover, we have a batch arrival stream and multiple types of service. No queue is allowed to form and retrial times are assumed to be geometrically distributed. We derive the generating function of the stationary distribution of the system state. Moreover, we obtain the generating functions of the orbit size and the system size distributions and the main performance this customer to be lost. He returns back to the customer population form which the stream of arrivals is generated. In the retrial queueing theory (e.g. Artalejo and Gómez- Corral (2008) [8]; Falin and Templeton (1997) [18]), it is assumed that this customer doesn't return back to the customer population. Instead, he waits for some time (perhaps doing another job) and then returns to the system to retry getting the service. Following the standard terminology, such customers are assumed to join a hypothetical place called "orbit" from which a stream of retrials arrives at the system besides the original stream of customers. Retrial queues appear naturally in applications such as telephone switching system, communication networks and stacked air crafts waiting to land.

The analysis of retrial queueing systems focused mainly on the continuous-time setting (e.g. Artalejo and Gómez-Corral (2008) [8]; Falin (1990) [17]; Falin and Templeton (1997) [18]; Kulkarni and Liang (1997) [23]). Nowadays, there is a great interest in analyzing queueing systems working in discrete-time setting. They are more appropriate for modeling communication systems working in slotted-time environment such as Asynchronous Transfer Mode (ATM) based systems Bruneel and Kim (1993) [14] and IEEE 802.11 MAC based wireless networks Bianchi (2000) [13]. Moreover, discrete-time models can be used to derive results for the continuous-time models but not vice versa Takagi (1993) [27].

The first paper in the area of discrete-time retrial queues is due to Yang and Li (1995) [31]. They considered a $Geo/G/1$ retrial queue and gave a methodology for studying discrete-time retrial queues which was applied by other authors. Choi and Kim (1997) [15] considered a discrete-time $Geo_1, Geo_2/G/1$ retrial queue with two types of calls. Li and Yang (1998) [24] studied a discrete-time $Geo/G/1$ retrial queue with Bernoulli schedule in which the blocked customers either join the infinite waiting space or leave the server and enter the orbit. Using a matrix analytic method, Li and Yang (1999) [25] studied a discrete-time single server retrial queue with phase-type interarrival times and geometric service times. Takahashi et al. (1999) [28] introduced a $Geo^{[X]}/G/1$ retrial queue with non-preemptive priority. Atencia and Moreno (2004a) [9] considered a discrete-time $Geo/G/1$ retrial queue in which the retrial time has a general distribution and the orbit is viewed as a FCFS queue. Artalejo et al. (2005) [5] studied a discrete-time $Geo/G/1$ retrial queue with batch arrival in which individual arriving customers have a control of admission. Aboul-Hassan et al. (2005a [1], 2005b [2], 2006 [3], 2008 [4]) considered a discrete-time $Geo/G/1$ retrial queue with balking customers. Artalejo and Lopez-Herrero (2007) [7] gave a simulation study of a discrete-time multiserver retrial queue with finite population. Artalejo et al. (2008) [6] introduced algorithmic analysis of a $Geo/Geo/c$ retrial queue.

In all the above mentioned models, it is assumed that the server is available all

A discrete-time $Geo^{[X]}/G_H/1$ retrial queue with unreliable server

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Abstract

This paper discusses a discrete-time $Geo^{[X]}/G_H/1$ [2] retrial queue with server subject to starting failures. We study the underlying Markov process and derive the generating function of its stationary distribution. A set of generating functions regarding the orbit and the system size distributions is also derived. Hence, expressions for the main performance characteristics of the system are obtained. Finally, a set of numerical examples is presented to illustrate the effect of different system parameters on the main performance characteristics.

Key words: Discrete-time retrial queues, Batch arrivals, Starting failures

1 Introduction

When a customer arrives at a full queueing system, the classical queueing theory (e.g. Gross and Harris (1998) [20]; Kleinrock (1975) [22]) considers

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[2] Appendix A contains a list of notations and terminologies that are used throughout this work.

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Differencing: 1 regular difference
Number of observations: Original series 192, after differencing 191
Residuals: SS = 1.83372 (backforecasts excluded)
MS = 0.00975 DF = 188

Modified Box-Pierce (Ljung-Box) Chi-Square statistic

Lag	12	24	36	48
Chi-Square	32.1	52.7	71.6	83.0
DF	9	21	33	45
P-Value	0.000	0.000	0.000	0.000

Where Z, X and Y represents the variables as the above theorem.

Note that, the sum of squares for X plus the sum of squares for Y less than the sum of squares for Z.

(Ministry of tourisms in Egypt is the source of data)

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taken. We use the MathCAD 14 and Minitab 13 for the analysis. The output results as the following:

ARIMA Model for Z

Final Estimates of Parameters

Type	Coef	SE Coef	T	P
AR 1	-1.3047	0.0121	-107.94	0.000
AR 2	-0.9875	0.0138	-71.41	0.000
MA 1	-1.2745	0.0177	-71.93	0.000
MA 2	-0.9874	0.0013	-781.57	0.000
Constant	-0.01771	0.03710	-0.48	0.634

Differencing: 1 regular difference
Number of observations: Original series 192, after differencing 191
Residuals: SS = 5.54557 (backforecasts excluded)
MS = 0.02981 DF = 186

Modified Box-Pierce (Ljung-Box) Chi-Square statistic

Lag	12	24	36	48
Chi-Square	27.3	41.6	50.6	72.8
DF	7	19	31	43
P-Value	0.000	0.002	0.015	0.003

ARIMA Model for X

Final Estimates of Parameters

Type	Coef	SE Coef	T	P
AR 1	0.1513	0.0719	2.10	0.037
Constant	-0.004427	0.009357	-0.47	0.637

Differencing: 1 regular difference
Number of observations: Original series 192, after differencing 191
Residuals: SS = 3.16043 (backforecasts excluded)
MS = 0.01672 DF = 189

Modified Box-Pierce (Ljung-Box) Chi-Square statistic

Lag	12	24	36	48
Chi-Square	24.1	29.2	31.1	36.5
DF	10	22	34	46
P-Value	0.007	0.140	0.611	0.841

ARIMA Model for Y

Final Estimates of Parameters

Type	Coef	SE Coef	T	P
MA 1	0.7407	0.0119	62.49	0.000
MA 2	0.2455	0.0666	3.69	0.000
Constant	0.0000938	0.0003929	0.24	0.812

$\sum(X_t - \hat{X}_{t\,x})^2 + \sum(Y_t - \hat{Y}_{t\,y})^2 \leq \sum(Z_t - \hat{Z}_{t\,z})^2$, where $\hat{X}_{t\,x}$ is the forecasting of X at time t by using ARMA model on X and $\hat{Y}_{t\,y}$ is the forecasting of Y at time t by using ARMA model on Y.

From step 3

$$\hat{X}_{t\,x} = \sum_{s=1}^{t-1} g_{t-s}(\hat{\Phi}_{1x}, \hat{\Phi}_{2x}, \ldots, \hat{\Phi}_{px}, \hat{\theta}_{1x}, \hat{\theta}_{2x}, \ldots, \hat{\theta}_{qx}) X_{t-s}$$

where

$g_{t-s}(\Phi_{1x}, \Phi_{2x}, \ldots, \Phi_{px}, \theta_{1x}, \theta_{2x}, \ldots, \theta_{qx})$ is function of parameters ARMA(p,q) on X

then

$\sum(X_t - \sum_{s=1}^{t-1} g_{t-s}(\Phi_{1x}, \Phi_{2x}, \ldots, \Phi_{px}, \theta_{1x}, \theta_{2x}, \ldots, \theta_{qx}) X_{t-s})^2$ is minimum

Therefore

$$\sum(X_t - \sum_{s=1}^{t-1} g_{t-s}(\Phi_{1x}, \Phi_{2x}, \ldots, \Phi_{px}, \theta_{1x}, \theta_{2x}, \ldots, \theta_{qx}) X_{t-s})^2$$

$$\leq \sum(X_t - \sum_{s=1}^{t-1} g_{t-s}(\Phi_{1z}, \Phi_{2z}, \ldots, \Phi_{pz}, \theta_{1z}, \theta_{2z}, \ldots, \theta_{qz}) X_{t-s})^2$$

i.e.

$$\sum(X_t - \hat{X}_{tx})^2 \leq \sum(X_t - \hat{X}_{tz})^2 \qquad (4)$$

Similarly

$$\sum(Y_t - \hat{Y}_{ty})^2 \leq \sum(Y_t - \hat{Y}_{tz})^2 \qquad (5)$$

then, by addition inequalities (4) and (5) we have

$$\sum(X_t - \hat{X}_{tx})^2 + \sum(Y_t - \hat{Y}_{ty})^2 \leq \sum(X_t - \hat{X}_{tz})^2 + \sum(Y_t - \hat{Y}_{tz})^2$$

Finally, from step (3) we find that

$$\sum(X_t - \hat{X}_{tx})^2 + \sum(Y_t - \hat{Y}_{ty})^2 \leq \sum(Z_t - \hat{Z}_{tz})^2 .$$

(5) Application example

In this section, we will analyze the numbers of tourists whom coming to Egypt monthly through the period (1990 – 2006) as the time series by using wavelet technique after the natural logarithm of values and twelfth seasonal difference are

$$\operatorname{cov}(\hat{X},Y)=\frac{1}{n}\sum\hat{X}_tY_t-\left(\frac{\sum\hat{X}}{n}\right)\left(\frac{\sum Y}{n}\right)=0$$

$$=\sum\hat{X}Y=0 \quad \textbf{(step 4)}$$

Step (5) proof that $\sum\hat{Y}_{tz}X_t=0$, where $\hat{Y}_{tz}$ is the forecasting of Y at time t by using ARMA model of Z

Proof

From step 4

$\sum Y_t=0$

then, $\sum\hat{Y}_t$ must be approximately equal zero

Since X and Y are uncorrelated. Then, X and $\hat{Y}$ are uncorrelated i.e.

$$\operatorname{cov}(X,\hat{Y})=\frac{1}{n}\sum X_t\hat{Y}_t-\left(\frac{\sum X}{n}\right)\left(\frac{\sum\hat{Y}}{n}\right)=0$$

$$=\sum X\hat{Y}=0 \quad \textbf{(step 5)}$$

From steps 1, 4 and 5

$$\operatorname{cov}(\hat{X},\hat{Y})=\frac{1}{n}\sum\hat{X}_{tz}\hat{Y}_{tz}-\left(\frac{\sum\hat{X}_{tz}}{n}\right)\left(\frac{\sum\hat{Y}_{tz}}{n}\right)=0$$

$$=\sum\hat{X}_{tz}\hat{Y}_{tz}=0 \quad \textbf{(step 6)}$$

$$\sum(Z_t-\hat{Z}_{tz})^2=\sum(X_t+Y_t-\hat{X}_{tz}-\hat{Y}_{tz})^2$$

$$=\sum(X_t-\hat{X}_{tz})^2+\sum\left(Y_t-\hat{Y}_{tz}\right)^2+2\sum\left((X_t-\hat{X}_{tz})\left(Y_t-\hat{Y}_{tz}\right)\right)$$

from steps 1, 3, 4 and 5

$$\sum\left((X_t-\hat{X}_{tz})\left(Y_t-\hat{Y}_{tz}\right)\right)=0$$

then

$$\sum(Z_t-\hat{Z}_{tz})^2=\sum(X_t-\hat{X}_{tz})^2+\sum(Y_t-\hat{Y}_{tz})^2 \quad \textbf{(step 7)}$$

finally, we proof that

$$\hat{Z}_{t\,z} = -\sum_{s=1}^{t-1} f_{t-s}(\hat{\theta}_1,\hat{\theta}_2) Z_{t-s}$$

$$= -\sum_{s=1}^{t-1} f_{t-s}(\hat{\theta}_1,\hat{\theta}_2)(X_{t-s} + Y_{t-s})$$

$$= -\sum_{s=1}^{t-1} f_{t-s}(\hat{\theta}_1,\hat{\theta}_2) X_{t-s} - \sum_{s=1}^{t-1} f_{t-s}(\hat{\theta}_1,\hat{\theta}_2) Y_{t-s}$$

$$= \hat{X}_{t\,z} + \hat{Y}_{t\,z}$$

by the same way we can prove at the case of MA(q) as above.

From addition firstly and secondly proof we can prove at case of mixed autoregressive and moving average ARMA(p,q) as the following equation:

$$Z_t = \phi_1 Z_{t-1} + \ldots + \phi_p Z_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \ldots - \theta_q \varepsilon_{t-q}$$

and let

$$\hat{Z}_{t\,z} = \sum_{s=1}^{t-1} g_{t-s}(\hat{\Phi}_{1z},\hat{\Phi}_{2z},\ldots,\hat{\Phi}_{pz},\hat{\theta}_{1z},\hat{\theta}_{2z},\ldots,\hat{\theta}_{qz}) Z_{t-s}$$

where

$g_{t-s}(\Phi_{1z},\Phi_{2z},\ldots,\Phi_{pz},\theta_{1z},\theta_{2z},\ldots,\theta_{qz})$ is function of parameters ARMA(p,q) on Z

then

$$\hat{Z}_{t\,z} = \sum_{s=1}^{t-1} g_{t-s}(\hat{\Phi}_{1z},\hat{\Phi}_{2z},\ldots,\hat{\Phi}_{pz},\hat{\theta}_{1z},\hat{\theta}_{2z},\ldots,\hat{\theta}_{qz}) X_{t-s}$$

$$+ \sum_{s=1}^{t-1} g_{t-s}(\hat{\Phi}_{1z},\hat{\Phi}_{2z},\ldots,\hat{\Phi}_{pz},\hat{\theta}_{1z},\hat{\theta}_{2z},\ldots,\hat{\theta}_{qz}) Y_{t-s}$$

$$= \hat{X}_{t\,z} + \hat{Y}_{t\,z} \qquad \textbf{(step 3)}$$

From properties of threshold the time series Y represent as the noise has mean zero i.e.

$$\sum Y_t = 0$$

Since X and Y are uncorrelated

Therefore, $\hat{X}$ and Y are uncorrelated also, and the covariance of $\hat{X}$ and Y equal zero i.e.

$$\hat{Z}_{t\,z} = -\sum_{s=1}^{t-1} (\hat{\theta}_z)^s Z_{t-s}$$

$$\hat{Z}_{t\,z} = -\sum_{s=1}^{t-1} (\hat{\theta}_z)^s (X_{t-s} + Y_{t-s})$$

$$\hat{Z}_{t\,z} = -\sum_{s=1}^{t-1} (\hat{\theta}_z)^s X_{t-s} - \sum_{s=1}^{t-1} (\hat{\theta}_z)^s Y_{t-s}$$

$$= \hat{X}_{t\,z} + \hat{Y}_{t\,z}$$

where $\hat{\theta}_z$ is the estimation of parameters MA(1) due to the analysis of time series Z.

Let q =2 then

$$Z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

set $\varepsilon_0 = \varepsilon_1 = 0$

then

$\varepsilon_2 = Z_2$, and

$$\varepsilon_t = Z_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

$$= Z_t + \theta_1 \left(Z_{t-1} + \theta_1 \varepsilon_{t-2} + \theta_2 \varepsilon_{t-3} \right) + \theta_2 \left(Z_{t-2} + \theta_1 \varepsilon_{t-3} + \theta_2 \varepsilon_{t-4} \right)$$

$$= Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_1^2 \varepsilon_{t-2} + 2\theta_1 \theta_2 \varepsilon_{t-3} + \theta_2^2 \varepsilon_{t-4}$$

$$= Z_t + \theta_1 Z_{t-1} + (\theta_2 + \theta_1^2) Z_{t-2} + (2\theta_1 \theta_2 + \theta_1^3) \varepsilon_{t-3} + (\theta_2^2 + \theta_1^2 \theta_2) \varepsilon_{t-4}$$

and so on

we can write the error ε_t as the form

$$\varepsilon_t = \sum_{s=0}^{t-1} f_{t-s}(\theta_1, \theta_2) Z_{t-s}$$

where

$$f_t(\theta_1, \theta_2) = 1$$

$$f_{t-1}(\theta_1, \theta_2) = \theta_1$$

$$f_{t-2}(\theta_1, \theta_2) = \theta_2 + \theta_1^2$$

$$\vdots$$

and so on

set $\varepsilon_t = 0$, then

$$= X_t + Y_t - 2(X_{t-1} + Y_{t-1}) + X_{t-2} + Y_{t-2}$$

$$= \Delta^2 X_t + \Delta^2 Y_t$$

$$\vdots$$

and so on

$\Delta^s Z_t = \Delta^s X_t + \Delta^s Y_t$ **(step 2)**

Now we consider Z is stationary time series. Firstly we proof at the case of Autoregressive of order p process AR(p) as the following equation

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \ldots + \phi_p Z_{t-p} + \varepsilon_t$$

(see Box and Jenkins (1976))

then

$$\hat{Z}_{t\,z} = \sum_{s=0}^{p} \hat{\Phi}_{zs} Z_{t-s}$$

where $\hat{\Phi}_{zs}$'s the estimation of parameters AR(p) due to the analysis of time series Z.

then

$$\hat{Z}_{t\,z} = \sum_{s}^{p} \hat{\Phi}_{zs} (X_{t-s} + Y_{t-s})$$

$$= \sum_{s}^{p} \hat{\Phi}_{zs} X_{t-s} + \sum_{s}^{p} \hat{\Phi}_{zs} Y_{t-s}$$

$$= \hat{X}_{t\,z} + \hat{Y}_{t\,z}$$

Secondly we proof at the case of Moving Average of order q process MA(q) as the following equation

$$Z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q} \, .$$

And let q =1 then

$$Z_t = \varepsilon_t - \theta \varepsilon_{t-1}$$

set $\varepsilon_0 = 0$

$$\varepsilon_t = Z_t + \sum_{s=1}^{t-1} (\theta)^s Z_{t-s}$$

set $\varepsilon_t = 0$

then

where X_t and Y_t denote the signal and noise, respectively, then the sum of the two sum square residuals from ARIMA model of X_t plus the sum square residual from ARIMA model of Y_t less than or equal the sum of the sum square residual from ARIMA model of Z_t under condition the analysis of each time series Z_t, X_t and Y_t done individually.

Proof

The proof of the previous theorem depend on proven the following steps

From equation (1) the DWT of Z is

$$\mathbf{d} = WZ$$

by using hard threshold in equation (2) and let

$$\mathbf{d1}_i = \begin{cases} \mathbf{d}_i & \text{if } \mathbf{d}_i > |\lambda| \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{d2}_i = \begin{cases} \mathbf{d}_i & \text{if } \mathbf{d}_i < |\lambda| \\ 0 & \text{otherwise} \end{cases}$$

it is clear that **d1** and **d2** are uncorrelated

use the inverse discrete wavelet transformation (IDWT)

$X = W^T \mathbf{d1}$ and $Y = W^T \mathbf{d2}$

where the matrix W is orthonormal, then

$$X^T Y = \mathbf{d1}^T W W^T \mathbf{d2} = \mathbf{d1}^T \mathbf{d2}$$

Therefore

X_t and Y_t are uncorrelated **(step 1)**

The first difference backward of Z_t is

$$\Delta Z_t = Z_t - Z_{t-1}$$
$$= X_t + Y_t - X_{t-1} - Y_{t-1}$$

then

$$\Delta Z_t = \Delta X_t + \Delta Y_t$$

also,

$$\Delta^2 Z_t = Z_t - 2Z_{t-1} + Z_{t-2}$$

and $\tilde{d}_{jk}$ is distributed as

$$\tilde{d}_{jk} \sim N\left(d_{jk}, \sigma^2\right).$$

Based on these assumptions, Donoho and Johnstone (1994, 1995) suggested two types of thresholding methods: hard and soft thresholding. Hard thresholding sets all the wavelet coefficients to be 0 if their absolute values are below a certain threshold $\lambda \geq 0$:

$$\hat{d}_{jk} = \eta_\lambda\left(\tilde{d}_{jk}\right) = \tilde{d}_{jk} I\left(\left|\tilde{d}_{jk}\right| > \lambda\right) \quad \text{(hard thresholding)} \qquad (3)$$

Soft thresholding shrinks the wavelet coefficients that are larger than the threshold by λ:

$$\hat{d}_{jk} = \eta_\lambda\left(\tilde{d}_{jk}\right) = \operatorname{sgn}\left(\tilde{d}_{jk}\right)\max\left(0, \left|\tilde{d}_{jk}\right| - \lambda\right) \quad \text{(soft thresholding)}$$

Figure (1): Hard thresholding (left) and soft thresholding (right)

Choices of Threshold

Too large a threshold might cut off important parts of the true function underlying the data, whereas too small a threshold may excessively retains noise in the reconstruction. Universal Threshold Donoho and Johnstone (1994) proposed the universal threshold:

$$\lambda = \sigma\sqrt{2\log(n)}$$

When σ is unknown, σ may be replaced by a robust estimate $\hat{\sigma}$, such as the median absolute deviation (MAD) of the wavelet coefficients at the finest level $J = \log(N) - 1$ divided by 0.6745 and can be expressed as

$$\hat{\sigma} = MAD\left\{d_{jk}, k = 1, \ldots, 2^J\right\} / 0.6745$$

(3) Wavelet and ARIMA Model

In this Section, we study the effect of wavelet transformation on the sum of square error for the ARIMA model.

Theorem:

Suppose that the time series Z_t has constant variance and non seasonality and it is contaminated by correlated noise; using wavelet transformation and threshold technique, this time series is decomposed into two time series as following

$$Z_t = X_t + Y_t$$

$$d = Wg \qquad (1)$$

where d is an $n \times 1$ vector comprising both discrete scaling coefficients, $c_{j_0 k}$, and discrete wavelet coefficients, d_{jk}, and W is an orthogonal $n \times n$ matrix associated with the orthonormal wavelet basis chosen.. Note that, because of orthogonality of W, the inverse DWT (IDWT) is simply given by

$$g = W^T d \qquad (2)$$

where W^T denotes the transpose of W.

If $n = 2^J$ for some positive integer J, the DWT and IDWT may be performed through a computationally fast algorithm developed by Mallat (1989) that requires only order n operations. In this case, for a given j_0 and under periodic boundary conditions, the DWT of g results in an n-dimensional vector **d** comprising both discrete scaling coefficients $c_{j_0 k}$, $k = 0,1,\ldots,2^{j_0} - 1$ and discrete wavelet coefficients d_{jk}, $\mathrm{j} = \mathrm{j}_0,\ldots,J-1$; $k = 0,1,\ldots,2^j - 1$.

We do not provide technical details here of the order DWT algorithm mentioned above. Essentially the algorithm is a fast hierarchical scheme for deriving the required inner products which at each step involves the action of low and high pass filters, followed by a decimation (selection of every even member of a sequence). The IDWT may be similarly obtained in terms of related filtering operations. For excellent accounts of the DWT and IDWT in terms of filter operators we refer to Nason & Silverman (1995), Strang & Nguyen (1996), or Burrus, Gonipath & Guo (1998).

2.3 Classical Threshold Schemes

Since the wavelet representation of many kinds of function is very economical, it is reasonable to assume that there are a few large value wavelet coefficients concentrated near the areas of major spatial activity, e.g. discontinuities, but the majority of wavelet coefficients are small. Also, owning to the fact that the wavelet transform is orthogonal, if the ε_i are assumed to be independent Gaussian noise, then the wavelet coefficients will also be contaminated with independent Gaussian noise. So in this case, the empirical wavelet coefficients can be written as

$$\tilde{d}_{jk} = d_{jk} + \varepsilon_{jk}$$

2. A short background on wavelets

In this section we give a brief overview of some relevant material on the wavelet series expansion and a fast wavelet transform that we need further.

2.1. The wavelet series expansion

The term wavelets is used to refer to a set of orthonormal basis functions generated by dilation and translation of a compactly supported scaling function (or father wavelet), ϕ, and a mother wavelet, ψ, associated with an r-regular multiresolution analysis of $L^2(R)$. A variety of different wavelet families now exist that combine compact support with various degrees of smoothness and numbers of vanishing moments (see, Daubechies (1992) and Mallat (1999)), and these are now the most intensively used wavelet families in practical applications in statistics. Hence, many types of functions encountered in practice can be sparsely (i.e. parsimoniously) and uniquely represented in terms of a wavelet series. Wavelet bases are therefore not only useful by virtue of their special structure, but they may also be (and have been!) applied in a wide variety of contexts.

An usual assumption underlying the use of periodic wavelets is that the function to be expanded is assumed to be periodic. However, such an assumption is not always realistic and periodic wavelets exhibit a poor behavior near the boundaries (they create high amplitude wavelet coefficients in the neighborhood of the boundaries when the analyzed function is not periodic). However, periodic wavelets are commonly used because the numerical implementation is particular simple. While, as Johnstone (1994) has pointed out, this computational simplification affects only a fixed number of wavelet coefficients at each resolution level, we will also present later on an effective method, developed recently by Oh and Lee (2005), combining wavelet decompositions with local polynomial regression, for correcting the boundary bias introduced by the inappropriateness of the periodic assumption.

2.2 The Discrete Wavelet Transform

In statistical settings we are more usually concerned with discretely sampled, rather than continuous, functions. It is then the wavelet analogy to the discrete Fourier transform which is of primary interest and this is referred to as the discrete wavelet transform (DWT). Given a vector of function values $g = \left(g\left(t_1\right),\ldots,g\left(t_n\right)\right)'$ at equally spaced points t_i, the discrete wavelet transform of g is given by

The Effect of Wavelet Transformation on the Sum of Square Error for the ARIMA Model

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Abstract

The objective of this article is to study the effect of wavelet filter on the time series data. By using wavelet transformation, the ARIMA model decomposed into sum of two ARIMA models and the relation between sum of square errors for them will be discussed. An applied example will be illustrated.

Keywords: Wavelet transformation; Threshold; Autoregressive, moving-average, ARIMA model; Sum square error.

1. Introduction

The wavelet transform is a powerful mathematical tool that is receiving more and more attention by the statistical community. While most work is being done in the engineering and physical sciences, wavelet transforms have already proven useful in well established statistical fields such as nonparametric regression, classification, and time series analysis. The groundbreaking work of Donoho and co-workers (Donoho 1993; Donoho and Johnstone1994; Donoho 1995; Donoho, Johnstone, Kerkyacharian, and Picard 1995) introduced statisticians to wavelet transforms in the context of signal estimation and wavelet shrinkage.

The past two decades have witnessed the development of wavelet analysis, Donoho and Johnstone (1995), Johnstone and Silverman (1997), Nason and von Sachs (1999), Priestley (1996), Percival, and Walden, (1999), have applied wavelet theory to the estimation of the functions whose observations are contaminated by noise as well as time series analysis either in the time domain or in the observation are equally spaced and independent. Most of these applications are based on the assumption that the observations are equally spaced and independent. However, for time series data the observations are likely to be dependent.

In **Section** 2 the short background on wavelet will be introduced. In **Section** 3, we study the effect of wavelet transformation on the sum of square error for the ARIMA model. In **Section** 4, the application example will be illustrated

Table (2)
MSE of the different estimators

When $\alpha = 2, \beta = 3$ and R=0.4											
n	m	r_1	r_2	R_1	R_2	R_3	R_4	MSE_1	MSE_2	MSE_3	MSE_4
5	4	3	2	4.81E-01	6.63E-03	3.11E-01	7.90E-03	2.90E-02	1.38E-01	2.83E-01	1.40E-02
5	5	4	3	4.01E-01	3.77E-07	4.88E-01	2.28E-03	2.30E-02	1.42E-01	8.72E-01	2.23E-01
10	5	6	5	4.18E-01	3.30E-04	3.99E-01	2.86E-07	4.80E-02	1.24E-01	7.88E-01	1.51E-01
10	10	7	6	4.66E-01	4.39E-03	4.91E-01	8.22E-03	2.90E-02	1.37E-01	7.03E-01	9.20E-02
10	15	8	7	4.56E-01	3.13E-03	3.73E-01	7.50E-04	2.90E-02	1.38E-01	9.38E-01	2.89E-01
10	20	9	10	4.72E-01	5.13E-03	4.74E-01	5.45E-03	3.60E-02	1.33E-01	5.42E-01	2.00E-02
15	5	6	5	4.96E-01	9.17E-03	2.76E-01	5.86E-04	5.70E-02	1.17E-01	3.70E-01	8.90E-04
15	10	7	6	4.73E-01	5.33E-03	4.87E-01	7.43E-03	5.00E-02	1.22E-01	3.18E-01	6.71E-03
15	15	8	7	4.81E-01	6.54E-03	3.65E-01	1.25E-03	2.70E-02	1.25E-01	4.37E-01	1.40E-03
15	20	10	10	4.77E+00	5.93E-03	379E-01	4.41E-04	3.10E-01	1.36E-01	6.33E-01	5.40E-02
20	10	16	9	4.79E+00	6.24E-03	3.88E-01	1.44E-04	2.60E-02	1.40E-01	6.58E-01	6.70E-02
When $\alpha = 2, \beta = 2$ and R=0.5											
n	m	r_1	r_2	R_1	R_2	R_3	R_4	MSE_1	MSE_2	MSE_3	MSE_4
5	4	3	2	4.89E-01	1.19E-04	4.80E-01	4.20E-04	5.30E-02	2.00E-01	3.55E-01	2.10E-02
5	5	4	3	4.60E-01	1.57E-03	4.18E-01	6.68E-03	2.60E-02	2.25E-01	1.87E-01	9.80E-02
10	5	6	5	5.31E-01	9.44E-04	4.91E-01	8.10E-05	3.80E-02	2.14E-01	8.43E-01	1.18E-01
10	10	7	6	5.85E-01	7.18E-03	4.97E-01	8.86E-06	5.80E-02	1.95E-01	3.53E-01	2.20E-02
10	15	8	7	4.23E-01	5.88E-03	5.65E-01	4.24E-03	4.50E-02	2.06E-01	2.70E-01	5.30E-02
10	20	9	10	4.21E-01	6.22E-03	4.35E-01	4.18E-03	4.00E-02	2.11E-01	4.41E-01	3.46E-03
15	5	6	5	5.45E-01	2.07E-03	4.68E-01	1.03E-03	5.80E-02	1.95E-01	3.02E-01	3.90E-02
15	10	7	6	4.80E-01	3.90E-04	5.17E-01	2.81E-04	6.10E-02	1.93E-01	1.79E-01	1.03E-01
15	15	8	7	4.74E-01	6.52E-04	5.38E-01	1.45E-03	5.10E-02	2.01E-01	3.08E-01	3.70E-02
15	20	10	10	4.94E-01	3.21E-05	4.55E-01	2.05E-03	3.90E-02	2.13E-01	9.29E-01	1.84E-01
20	10	16	9	5.20E-01	3.93E-04	4.92E-01	6.30E-05	3.30E-02	2.18E-01	9.22E-01	1.97E-01

Where both α and β are the scale parameters for X and Y respectively. R is the reliability function ,also. MSE_1 is The mean square error of $\hat{R}_1$, MSE_2 is The mean square error of $\hat{R}_2$ MSE_3 is The mean square error of $\hat{R}_3$ and MSE_4 is The mean square error of $\hat{R}_4$.

Table (1)
MSE of the different estimators

When $\alpha = 2 , \beta = 3$ and R=0.4											
n	m	r_1	r_2	R_1	R_2	R_3	R_4	MSE_1	MSE_2	MSE_3	MSE_4
5	5	3	3	4.89E-01	4.46E-01	1.22E-01	1.77E-01	7.89E-03	2.14E-03	7.70E-02	5.00E-02
5	5	4	4	4.86E-01	4.55E-01	1.00E-01	2.00E-01	7.47E-03	7.47E-03	9.00E-02	4.00E-02
10	10	6	6	4.74E-01	4.01E-01	8.50E-02	4.65E-01	5.47E-03	1.35E-06	9.90E-02	4.23E-03
10	10	7	7	4.81E-01	4.48E-01	6.80E-02	2.11E-01	6.54E-03	2.35E-03	1.10E-01	3.60E-02
10	10	8	8	4.55E-01	4.23E-01	5.40E-02	3.69E-01	3.03E-03	5.33E-04	1.19E-01	9.87E-04
10	10	9	9	4.90E-01	4.81E-01	5.00E-02	4.08E-01	8.11E-03	6.63E-03	1.23E-01	6.98E-05
15	15	12	12	4.64E-01	4.43E-01	3.50E-02	9.20E-01	4.16E-03	1.87E-03	1.33E-01	2.17E-01
15	15	13	13	4.53E-01	3.83E-01	5.00E-02	3.84E-01	2.82E-03	3.04E-04	1.23E-01	2.51E-04
20	20	15	15	4.91E-01	4.82E-01	5.10E-02	4.92E-01	8.22E-03	6.69E-03	1.22E-01	8.39E-03
20	20	16	16	4.93E-01	4.88E-01	5.00E-02	7.91E-01	8.73E-03	7.70E-03	1.23E-01	1.53E-01
20	20	17	17	4.89E-01	4.46E-01	1.22E-01	1.77E-01	7.89E-03	2.14E-03	7.70E-02	5.00E-02
When $\alpha = 2 , \beta = 2$ and R=0.5											
n	m	r_1	r_2	R_1	R_2	R_3	R_4	MSE_1	MSE_2	MES_3	MSE_4
5	5	3	3	5.17E-01	5.52E-01	4.70E-02	3.95E-01	3.05E-04	2.67E-03	2.05E-01	1.10E-02
5	5	4	4	5.07E-01	5.11E-01	4.90E-02	2.04E-01	5.28E-05	1.24E-04	2.04E-01	8.80E-02
10	10	6	6	5.69E-01	4.48E-01	5.50E-02	7.09E-01	4.81E-03	2.75E-03	1.98E-01	4.40E-02
10	10	7	7	5.57E-01	5.85E-01	5.60E-02	6.04E-01	3.22E-03	7.16E-03	1.97E-01	1.10E-02
10	10	8	8	4.74E-01	4.60E-01	3.20E-02	6.87E-01	6.71E-04	1.59E-03	2.19E-01	3.50E-02
10	10	9	9	4.73E-01	4.68E-01	3.40E-02	3.28E-01	7.05E-04	1.03E-03	2.22E-01	2.00E-02
15	15	12	12	4.89E-01	4.86E-01	4.20E-02	7.48E-01	1.12E-04	1.92E-04	2.10E-01	6.20E-02
15	15	13	13	5.02E-01	5.03E-01	3.90E-02	9.42E-01	5.46E-06	7.97E-06	2.12E-01	1.95E-01
20	20	15	15	4.67E-01	4.55E-01	4.60E-02	7.77E-01	1.10E-03	2.07E-03	2.06E-01	7.70E-02
20	20	16	16	4.85E-01	4.80E-01	4.40E-02	9.54E-01	2.40E-04	3.97E-04	2.08E-01	2.06E-01
20	20	17	17	5.49E+02	5.60E-01	4.20E-02	8.88E-01	2.43E-03	3.59E-03	2.10E-01	1.51E-01

At the end of this study, we can find that, the results of the case of $r_1 \neq r_2$ which represented in table (2) is smaller than that's represented in table (1) for the case $r_1 = r_2$.

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$$L_1 = \hat{R}_1 - z_{(1-\frac{\alpha}{2})}\sigma_{\hat{R}_1} \tag{5.3}$$

and

$$U_1 = \hat{R}_1 + z_{(1-\frac{\alpha}{2})}\sigma_{\hat{R}_1} \tag{5.4}$$

where $z_{(1-\frac{\alpha}{2})}$ is $(1-\frac{\alpha}{2})$th quantile of the standard normal distribution and $\hat{R}_1$ is given by Eq. (2.8), see Surles and Padjett (2001).

6. Numerical study for the different estimators

In this section, we perform some simulation experiments to observe the baehavior of the different methods for different sample sizes and for different parameter values. We used the software package MathCad 2001 for this purpose. We compare, in terms of the mean square error, the performances of the MLE, UMVUE and Bayes estimates with respect to squares error loss function. The following steps will be considered to obtain the estimators:

Step (1): Generate random samples $X_1,...,X_r$ from exponential distribution, we consider the following sample sizes (n, m) = (5,5), (10,10), (15,15), (20,20), (5,4), (10,5), (10,15), (10,20), (15,5), (15,10), (15,20), (20,10), and the following parameter values $\alpha = 2$, with different type II censoring at 60%, 70%, 80% and 90%. We will generate 1000 random samples from exponential distribution.

Step (2): Similarly, we generate samples for exponential distribution, with parameter $\beta = 3, 2$.

Step (3): Using the Equation (2.8) to find the MLE of R and the Equation (3.3) to find the UMVUE of R . Also using the equation (4.4) the values of Bayes estimator of R is obtained using Conjugate prior distribution. Finally, the equation (4.9) gives the estimators of R using non informative prior distribution. The results are based on 1000 replications.

Step (4): we take the average of the simulated values and calculate the mean square error of R . The results are reported in Tables (1) and (2).

Form table (1), it is clear that MSE of MLE ($\hat{R}_1$) is the smallest values among the other values of MSE's of [($\hat{R}_2$),($\hat{R}_3$)and($\hat{R}_4$)] expect at some points the MSE of UMVUE $\hat{R}_2$ has advantage over the other estimators, and at some points MSE of Bayes estimator with conjugate $\hat{R}_3$ is better than MSE of Bayes estimator with Non-informative $\hat{R}_4$ .also All mean square errors decrease as α and β increase.

Form table (2), it is clear that MSE of UMVUE ($\hat{R}_2$) is the smallest values among the other values of MSE's of [($\hat{R}_1$),($\hat{R}_3$)and($\hat{R}_4$)] expect at some points the MSE of MLE $\hat{R}_1$ has advantage over the other estimators, and at some points MSE of Bayes estimator with conjugate $\hat{R}_3$ is better than MSE of Bayes estimator with Non-informative $\hat{R}_4$ .also All mean square errors decrease as α and β increase.

$$F_1 = \frac{(1-\hat{R}_1)}{\hat{R}_1}\frac{R}{(1-R)}$$

Using F_1 as a pivotal quantity, we obtain a $(1-\alpha)100\%$ confidence interval for R as (L_2, U_2), where

$$L_2 = F_{1-\frac{\alpha}{2}}(2r_1,2r_2)(\frac{V}{Z}+F_{1-\frac{\alpha}{2}}(2r_1,2r_2))^{-1} \tag{5.1}$$

and

$$U_2 = F_{\frac{\alpha}{2}}(2r_1,2r_2)(\frac{V}{Z}+F_{\frac{\alpha}{2}}(2r_1,2r_2))^{-1} \tag{5.2}$$

where F_α is $(1-\alpha)$th quantile of an F distribution random variables with $(2r_2,2r_1)$ degrees of freedom .

5.2. Approximate confidence interval

The exact results obtained in the previous subsection are theoretically important but too time consuming for practical case. Therefore we need to develop good approximate methods that can be used in practical applications.

Let $X_1,\ldots,X_{r_1}$ and $Y_1,\ldots,Y_{r_2}$ be random samples with size r_1, r_2 from exponential distribution with parameter α , β , respectively, we can easy show that the maximum likelihood function of R, $\hat{R}_1$ is asymptotically normal distribution with mean R and variance-covariance matrices $\mathrm{I}^{-1}(\alpha,\beta)$ where $\mathrm{I}(\alpha,\beta)$ is the Fisher information matrix and given by:

$$\mathrm{I}(\alpha,\beta) = \begin{pmatrix} \frac{r_1}{\alpha^2} & 0 \\ 0 & \frac{r_2}{\beta^2} \end{pmatrix}$$

The asymptotic variance-covariance matrix is obtained by replacing its values by their maximum likelihood estimators. Hence, the asymptotic variance-covariance matrix will be

$$\mathrm{I}^{-1}(\alpha,\beta) = \begin{pmatrix} \frac{\beta^2}{r_2} & 0 \\ 0 & \frac{\alpha^2}{r_1} \end{pmatrix}$$

Thus the maximum likelihood estimator, $\hat{R}_1$ is asymptotically normal distribution with mean R and variance

$$\sigma^2_{\hat{R}_1} = \frac{r_1 r_2(\alpha+\beta)^2}{\beta^2\alpha^2} = \frac{r_1 r_2}{\beta^2 R^2}$$

Hence, an approximate $(1-\alpha)100\%$ confidence interval for R would be (L_1, U_1), where

Similarly, if $Y_1,...,Y_{r_2}$ is a random sample from exponential distribution with parameter β, the prior distribution of β , $\pi_{02}(\beta)$,will be proportional to $\frac{1}{\beta}$,i.e.,

$$\pi_{02}(\beta) \propto \frac{1}{\beta} \tag{4.7}$$

Assuming that α and β are independent, the posterior distribution of α and β is

$$\pi_2(\alpha,\beta \mid x_1,....,x_{r_1},y_1,....,y_{r_2}) \propto L(x_1,....,x_{r_1}|\alpha)L(y_1,....,y_{r_2}|\beta)\pi_{01}(\alpha)\pi_{02}(\beta) \tag{4.8}$$

then

$$\pi_2(\alpha,\beta \mid x_1,....,x_{r_1},y_1,....,y_{r_2}) = \frac{\eta^{r_1}\varepsilon^{r_2}}{\alpha^{r_1+1}\beta^{r_2+1}\Gamma(r_1)\Gamma(r_2)} e^{\frac{-\eta}{\theta_1}} e^{\frac{-\varepsilon}{\theta_2}}, \alpha,\beta > 0$$

where $\eta = (x_{r_1}(n-r_1) + \sum_{i=1}^{r_1} x_i)$ and $\varepsilon = (y_{r2}(m-r_2) + \sum_{j=1}^{r_2} y_j)$.

Therefore, the Bayes estimator with respect to mean square error loss function will be

$$\hat{R}_4 = E(R \mid x, y) = \frac{\eta^{r_1}\varepsilon^{r_2}\Gamma(r_1+r_2+3)}{\Gamma(r_1)\Gamma(r_2)} \int_0^1 \frac{(1-R)^{r_1+1} R^{r_2+2}}{(\eta(1-R)+R\varepsilon)^{r_1+r_2+3}} dR \tag{4.9}$$

Notice that using non-informative prior distribution in (4.6) and (4.7) for α and β, respectively, makes estimating the prior parameters (v_1,u_1) and (v_2,u_2) unnecessary. It also overcomes the sensitivity of $\hat{R}_3$ to the parameter of the prior distributions.

5. Interval Estimation of R

5.1. Exact confidence interval

Let $X_1,...,X_{r_1}$ and $Y_1,...,Y_{r_2}$ be random samples with size r_1 , r_2 from exponential distribution with parameter α , β respectively.

Recalling that $Z = \sum_{i=1}^{r_1} \ln e^{x_i}$ and $V = \sum_{j=1}^{r_2} \ln e^{y_j}$ are independent with gamma distribution with parameters (r_1,α) and (r_2,β) , respectively, and $2\alpha \ln e^{\sum_{i=1}^{r_1} x_i}$ and $2\beta \ln e^{\sum_{j=1}^{r_2} y_j}$ are independent with chi -square distributions with degree of freedom $2r_1$ and $2r_2$ hence

$$\hat{R}_1 = (1+\frac{\beta}{\alpha})^{-1}$$

we know that $F_1 = \frac{V\alpha}{Z\beta}$ has F-distribution with $(2r_1,2r_2)$ degrees of freedom , Since

$\hat{R}_1 = (1+\frac{\beta}{\alpha}F_1)^{-1}$, which can be written as

$$L_1(x_1,\dots,\ x_{r_1}) \propto (\frac{1}{\alpha})^{r_1} e^{\frac{-(\sum_{i=1}^{r_1} x_i + x_{r_1}(n-r_1))}{\alpha}} \qquad (4.1)$$

and

$$L_2(y_1,\dots,\ y_{r_2}) \propto (\frac{1}{\beta})^{r_2} e^{\frac{-(\sum_{j=1}^{r_2} y_j + y_{r_2}(m-r_2))}{\beta}} \qquad (4.2)$$

Assuming that α and β are independent having prior gamma distributions, the posterior distributions of α and β will be gamma with parameters (δ_1,λ_1) and (δ_2,λ_2), respectively, where

$$\delta_1 = r_1 + u_1, \qquad \lambda_1 = v_1 + \sum_{i=1}^{r_1} x_i + x_{r_1}(n-r_1),$$

$$\delta_2 = r_2 + u_2 \text{ and } \lambda_2 = v_2 + \sum_{j=1}^{r_2} y_j + y_{r_2}(m-r_2)$$

The posterior distribution of α and β given $x_1,\dots,\ x_{r_1}$ and $y_1,\dots,\ y_{r_2}$ will be

$$\pi_1(\alpha,\beta|x_1,\dots,x_{r_1};y_1,\dots,y_{r_2}) = \frac{\lambda_1^{\delta_1}\lambda_2^{\delta_2}}{\Gamma(\delta_1)\Gamma(\delta_2)\alpha^{\delta_1-1}\beta^{\delta_2-1}} e^{-\frac{\lambda_1}{\alpha}-\frac{\lambda_2}{\beta}} \qquad (4.3)$$

Hence Bayes estimator of R with respect to the mean square error loss function is

$$\hat{R}_3 = E(R \mid x_1,\dots,x_{r_1};y_1,\dots,y_{r_2})$$

$$\hat{R}_3 = \frac{\lambda_1^{\delta_1}\lambda_2^{\delta_2}\Gamma(\delta_1+\delta_2-1)}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_0^1 \frac{(1-R)^{\delta_1} R^{\delta_2+1}}{((1\text{-}R)\lambda_1 + R\lambda_2)^{\delta_1+\delta_2-1}} dR \qquad (4.4)$$

4.2 Non-Informative Prior Distributions

Let $X_1,\dots,X_{r_1}$ be a random sample from exponential distribution with parameter α. The prior distribution, $\pi_{01}(\alpha)$, of α is proportional to $\sqrt{\mathrm{I}(\alpha)}$, where $\mathrm{I}(\alpha)$ is Fisher's information of the sample about α, and is given by

$$\mathrm{I}(\alpha) = \frac{1}{\alpha^2} \qquad (4.5)$$

Thus

$$\pi_{01}(\alpha) \propto \frac{1}{\alpha} \qquad (4.6)$$

Let W be the indicator variable $I_{[0,v_1)}(z_1)$. It could be seen that W be unbiased estimator for R, by using Rao-Black Well and Lehmann-Scheff e$'$ we have $\hat{R}_2$ is UMVUE for R. (see Mood et al (1974)).

$$\hat{R}_2 = E(W / Z, V)$$

$$\hat{R}_2 = \int_{z_1} \int_{v_1} w f(z_1, v_1 / Z, V) dv_1 dz_1$$

Where $f(z_1,v_1 / Z,V)$ is the conditional pdf of z_1, v_1 given Z, V.

Notice that z_1 and v_1 are independent exponential random variables with parameters α and β, respectively, and that Z and V are independent gamma random variables with parameters (r_1,α) and (r_2,β), respectively.

We see that $Z-z_1$ and $V-v_1$ are impendent gamma random with parameters (r_1-1,α) and (r_2-1,β), respectively. Moreover $Z-z_1$ and z_1 are independent, as well as $V-v_1$ and v_1 are also independent. We see that

$$\hat{R}_2 = \int_{z_1}\int_{v_1} w \frac{(r_1-1)(z-z_1)^{(r_1-2)}}{z^{r_1-1}} \frac{(r_2-1)(v-v_1)^{(r_2-2)}}{v^{r_2-1}} dv_1 dz_1 \tag{3.1}$$

$$\hat{R}_2 = \frac{(r_1-1)(r_2-1)}{z^{(r_1-1)} v^{(r_2-1)}} \begin{cases} \int_0^v \int_{v_1}^z (v-v_1)^{r_2-2}(z-z_1)^{r_1-2} dz_1 dv_1 & ,v<z \\ \int_0^z \int_0^{z_1} (z-z_1)^{r_1-2}(v-v_1)^{r_2-2} dv_1 dz_1 & ,v \geq z \end{cases} \tag{3.2}$$

Using Binomial expansion, the UMVU of R is given by

$$\hat{R}_2 = \begin{cases} (r_1-1)!(r_2-1)! \sum_{j=1}^{r_1-1} \dfrac{(-1)^j (\frac{v}{z})^j}{(r_1-1-j)!(r_2+j-1)!}, & v<z \\ 1-(r_1-1)!(r_2-1)! \sum_{j=1}^{r_1-1} \dfrac{(-1)^j (\frac{z}{v})^j}{(r_2-1-j)!(r_1+j-1)!} & ,v \geq z \end{cases} \tag{3.3}$$

4. Bayes Estimator

We obtain Bayes Estimator of R with respect to the mean square error loss function based on conjugate and non-informative prior distributions.

4.1 Conjugate prior distribution

Let $X_1,...,X_{r_1}$ and $Y_1,...,Y_{r_2}$ be the first r_1 and r_2 failure observations from $X_1,....,X_n$ and $Y_1,...,Y_m$, respectively. Where both of them have exponential distribution with parameters α and β, respectively. The likelihood function of each sample will be respectively

Now, we shall study some prosperities of $\hat{R}_1$.

$$E(\hat{R}_1) = \frac{\alpha}{\alpha + \frac{\beta r_1}{(r_1 - 1)}}[1 - \frac{(r_1 + r_2 - 1)}{r_2(r_1 - 2)}(1 - \frac{\alpha}{\alpha + \frac{\beta r_1}{(r_1 - 1)}})^2] \quad (2.10)$$

For fixed r_2,

$$\lim_{r_1 \to \infty} E(\hat{R}_1) = R[1 - \frac{1}{r_2}(1-R)^2] \quad (2.11)$$

and

$$\lim_{r_1, r_2 \to \infty} E(\hat{R}_1) = R \quad (2.12)$$

Then, $\hat{R}_1$ asymptotically unbiased estimator of R.

Also,

$$Var(\hat{R}_1) = \frac{(r_1 + r_2 - 1)}{r_2(r_1 - 2)}[\frac{\frac{\beta r_1}{\alpha(r_2 - 1)}}{\frac{\beta\alpha(r_1 + 1)}{\beta(r_2 - 1)}}]^2[\frac{1}{1 + \frac{\beta r_1}{\alpha(r_1 - 1)}}]^2 \quad (2.13)$$

For fixed r_2,

$$\lim_{r_1, r_2 \to \infty} Var(\hat{R}_1) = R^2[\frac{\beta}{\alpha}]^4 \lim_{r_2 \to \infty} \frac{1}{r_2} \quad (2.14)$$

then

$$\lim_{r_1, r_2 \to \infty} Var(\hat{R}_1) = 0 \quad (2.15)$$

Hence, $\hat{R}_1$ is a consistent estimator for R.

3. Uniform Minimum Variance unbiased Estimator of R

Let $X_1, \ldots, X_{r_1}$ and $Y_1, \ldots, Y_{r_2}$ be two independent random samples, of size r_1 and r_2, drawn from exponential distributions with parameters α and β, respectively,

Define

$$z_i = \ln e^{x_i}, \quad v_i = \ln e^{y_i}; \; i = 1, \ldots, r_1 \text{ and } j = 1, \ldots, r_2,$$

$$Z = \sum_{i=1}^{r_1} z_i, \text{ and } V = \sum_{i=1}^{r_2} v_i$$

From Eq. (2.5) and (2.7) we can show that (Z, V) is a complete sufficient statistic for (α, β).

Now, we have

$$P(v_1 < z_1) = P(\ln e^{y_1} < \ln e^{x_1})$$
$$= P(y_1 < x_1) = R$$

$$\hat{\alpha} = \frac{\sum_{i=1}^{r_1} x_i + (n-r_1)x_{r_1}}{r_1} \tag{2.5}$$

Also, suppose that r_2 are the first failure observations with stress $Y_j; j=1,\ldots,r_2$, where $r_2 \leq m$ each of which having exponential distribution with parameter β respectively, the likelihood function based on type-II censored sample will be

$$L = \frac{m!}{(m-r_2)!} \prod_{j=1}^{r_2} \left(\frac{1}{\beta} e^{-\frac{y_j}{\beta}}\right)\left[e^{-\frac{y_{r_2}}{\beta}}\right]^{m-r_2}$$

$$L = \frac{m!}{(m-r_2)!} \left(\frac{1}{\beta}\right)^{r_2} e^{\frac{-\left(\sum_{j=1}^{r_2} y_j + y_{r_2}(m-r_2)\right)}{\beta}} \tag{2.6}$$

$$\frac{d \ln L}{d\beta} = \frac{-r_2}{\beta} + \frac{\sum_{j=1}^{r_2} y_j}{\beta^2} + \frac{(m-r_2)y_{r_2}}{\beta^2} = 0$$

The MLE of β will be

$$\hat{\beta} = \frac{\sum_{j=1}^{r_2} y_j + (m-r_2)y_{r_2}}{r_2} \tag{2.7}$$

Once we obtain α and β ,the MLE of R becomes

$$\hat{R}_1 = \frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}} \tag{2.8}$$

Using (2.5) and (2.7) in (2.8)

$$\hat{R}_1 = \frac{1}{1 + \frac{r_1\left(\sum_{j=1}^{r_2} y_j + (m-r_2)y_{r_2}\right)}{r_2\left(\sum_{i=1}^{r_1} x_i + (n-r_1)x_{r_1}\right)}} \tag{2.9}$$

Since $2\alpha\left(\sum_{i=1}^{r_1} x_i + (n-r_1)x_{r_1}\right)$ and $2\beta\left(\sum_{j=1}^{r_2} y_j + (m-r_2)y_{r_2}\right)$ have chi-square distribution with $(2(r_1+1), 2(r_2+1))$ degrees of freedom respectively, Then

$$\frac{\alpha}{\beta}\left(\frac{1}{\hat{R}_1} - 1\right) = \frac{r_1\alpha\left(\sum_{j=1}^{r_2} y_j + (m-r_2)y_{r_2}\right)}{r_2\beta\left(\sum_{i=1}^{r_1} x_i + (n-r_1)x_{r_1}\right)} \sim F(2(r_1+1), 2(r_2+1))$$

type II censored. In section 2, we derive maximum likelihood estimator (MLE) of R and some properties of the estimator will be discussed. Uniform minimum variance unbiased estimator, (UMVUE) of R is given in section 3. Bayesian estimators with mean square error loss functions corresponding to conjugate and non-informative priors are presented in section 4. In section 5, we obtain the Interval estimators of R. A Monte Carlo simulation study is presented in section 6 to compare the various estimators obtained.

2. Maximum Likelihood Estimator of R with Type II Censored Samples

Let X be the strength of a component and Y be the stress acting on it. Let X and Y be exponential independent random variables with parameters α and β, respectively. That is, the probability density functions (pdf) of X and Y are, respectively,

$$f(x;\alpha)=\frac{1}{\alpha}e^{-\frac{x}{\alpha}}, \quad x>0, \qquad \alpha>0 \tag{2.1}$$

and

$$f(y;\beta)=\frac{1}{\beta}e^{-\frac{y}{\beta}}, \quad y>0, \qquad \beta>0 \tag{2.2}$$

The reliability function is defined as

$$R = P(Y<X) = \frac{1}{\alpha}\int_0^\infty (1-e^{\frac{-x}{\beta}})e^{\frac{-x}{\alpha}}\,dx$$
$$= \frac{\alpha}{\alpha+\beta}. \tag{2.3}$$

If α and β are known then R is simply calculated using Eq. (2.3).
Now to compute the MLE of R, first we need to obtain the MLE of α and β.

Suppose that r_1 are the first failure observations with strengths X_i; $i=1,\ldots,r_1$, where $r_1 \leq n$ each of which having exponential distribution with parameter α respectively, the likelihood function based on type-II censored sample will be

$$L = \frac{n!}{(n-r_1)!}\prod_{i=1}^{r_1}\left(\frac{1}{\alpha}e^{-\frac{x_i}{\alpha}}\right)\left[e^{-\frac{x_{r_1}}{\alpha}}\right]^{n-r_1}$$

$$L = \frac{n!}{(n-r_1)!}\left(\frac{1}{\alpha}\right)^{r_1}e^{\frac{-(\sum_{i=1}^{r_1}x_i+x_{r_1}(n-r_1))}{\alpha}}$$

$$\frac{d\ln L}{d\alpha} = \frac{-r_1}{\alpha}+\frac{\sum_{i=1}^{r_1}x_i}{\alpha^2}+\frac{(n-r_1)x_{r_1}}{\alpha^2}=0 \tag{2.4}$$

then

Estimation of $P(Y < X)$ in Exponential Case Based on type-II Censored Samples

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Abstract: In this article, the estimation of reliability of a system is discussed when strength, X, and stress, Y, are independent exponentially distributed random variables with different scale parameters based on type II censored samples. Different methods for estimating the reliability are applied. The point estimators obtained are maximum likelihood estimator, uniformly minimum variance unbiased estimator, and Bayesian estimators based on conjugate and non-informative prior distributions. A comparison between the different estimates obtained is performed. Interval estimators of the reliability are also discussed

Keywords: Maximum likelihood estimator; Unbiasedness; Consistency; Uniform minimum variance unbiased estimator; Bayesian estimator; Pivotal quantity; Fisher information.

1. Introduction

In life testing, it is often the items drawn from a population are put on test and their times of failure are recorded. For a number of reasons, such as budget or limited time, it is often necessary to terminate the test before all the failure times have been observed; In this case the data become available in some ordered manner. The test is usually terminated after a fixed time or a fixed number of failures are observed giving a censored sample .

In stress-strength model, the stress Y and the strength X are treated as random variables and the reliability of a component during a given period $(0,T)$ is taken to be the probability that the strength exceeds the stress during the entire interval. the reliability R of a component is $R = P(Y < X)$.

Several authors have considered different studies for stress and strength with exponential distribution with complete sample. Tong (1974) discussed the estimation of $P(Y < X)$ in the exponential case.

Tong (1977) had a look at the estimation of $P(Y < X)$ for exponential families. Sathe and Shah (1981) studied estimation of $P(Y > X)$ for the exponential distribution. Chao (1982) provided simple approximations for bias and mean square error of the maximum likelihood estimators of reliability when stress and strength are independent exponentially distributed random variables. Awad and Charraf (1986) studied three different estimators for reliability has a bivariate exponential distribution. Bai and Hong (1992) estimated $P(Y \le X)$ in the exponential case with common location parameter. Siu-Keung and Geoffrey (1996) studied the shrinkage estimation of reliability for exponential distributed lifetime. A note on the UMVUE on $P(Y \le X)$ in the exponential case discussed in Cramer and Kamps (1997). Shrinkage estimation of $P(Y \le X)$ in the exponential case discussed by Ayman and Walid (2003). Tachen (2005) deals with the empirical Bayes testing the reliability of an exponential distribution.

In this article, we studied different estimation procedures for the reliability, R when X and Y two independent exponential distribution when the available sample is

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