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A Generalized Burr Distribution with Hazard Power Parameter

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Abstract

A new Burr distribution is presented based on the hazard power parameter family. It has two positive shape parameters namely θ and α, and a scale positive parameter namely β. It is implied from the probability density function, which can be unimodal and the hazard rate function can be either decreasing, increasing or upside-down shaped. To illustrate the flexibility of the new distribution, an application on three sets of real data (prices, time failure and taxes revenues) using the maximum likelihood method for estimating the parameters of distribution is presented. The estimation of parameters of the new distribution using Bayesian estimation and maximum likelihood method is performed in complete data. A trial to obtain the MLEs mathematically but it is difficult to obtain them in complete samples, the fisher matrix is obtained. The estimates of the parameters are obtained using simulated data. A trial to obtain the Bayes estimators depending on gamma priors and using MCMC to obtain the estimators mathematically is performed.

1. Introduction

Burr (1942) has suggested the Burr family that has twelve types of cumulative distribution functions that are given in Table 1. The Bur distributions related to Lomax distribution, Weibull distribution, Logistic distribution, Log Logistic distribution and Kappa distribution. Family of Burr distributions is a very popular distribution family for modelling lifetime data and for modelling phenomenon with unimodal failure rates. Many researchers used many methods for obtaining new distribution. Chiodo and Pasquale (2016) proposed the inverse Burr distribution as an alternative for the probabilistic modelling for extreme values of wind speed that is validated on several real wind data sets. Afify (2018) presented a Four-Parameter Burr XII distribution namely the Weibull Burr XII distribution using the Weibull-G (W-G) family. Jamal et al. (2020) presented minimum Gambel Burr distribution by compounding the Gumbel Type II and the Burr XII. Abuzaid et al. (2021) presented a new modified Burr type III distribution based on the Pearson differential equations. Baharith (2022) presented the Alpha Power Kumaraswamy Burr III distribution based on combining two well-known techniques namely the alpha power transformation and Kumaraswamy generated family.

One of the transformations for adding parameter is the hazard power parameter transformation. Marshall and Olkin (2007) introduced the hazard power parameter family (HPP) that depends on the hazard function. Also, Muhammed (2022)[a] introduced the families of bivariate distributions depending on hazard and reversed hazard function. Muhammed (2022)[b] introduced a generalized Gompertz distribution and its bivariate extensions. In this paper, we will introduce a new generalization for the Burr distribution by adding a hazard power parameter based on the idea that discussed by the author that mentioned above. The proposed distribution is called hazard power parameter Burr (HPPB) Distribution and its probability density function is unimodal and its hazard function can be increasing, decreasing and upside down failure rate.

Table 1: The twelve cumulative types of the Burr distribution

Types	$F(X)$	Interval
I	x	$(0,1)$
II	$(1+e^{-x})^{-\beta}$	$(-\infty,\infty)$
III	$\left(1+x^{-\theta}\right)^{-\beta}$	$(0,\infty)$
IV	$\left\{1+[x^{-1}(\theta-x)]^{-\theta}\right\}^{-\beta}$	$(0,\theta)$
V	$\left\{1+\theta e^{-tan(x)}\right\}^{-\beta}$	$\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$
VI	$\left\{1+\theta e^{-rsihn(x)}\right\}^{-\beta}$	$(-\infty,\infty)$
VII	$2^{-\beta}[1+tanh(x)]^{\beta}$	$(-\infty,\infty)$
VIII	$[2\pi^{-1}arctan(e^{x})]^{-\beta}$	$(-\infty,\infty)$
IX	$1-2\left\{2+\theta\left[(1+e^{x})^{\beta}-1\right]\right\}^{-1}$	$(-\infty,\infty)$
X	$\left(1-e^{-x^2}\right)^{\beta}$	$(0,\infty)$
XI	$[x-(2\pi)^{-1}sin(2\pi x)]^{\beta}$	$(0,1)$
XII	$1-\left(1+x^{\theta}\right)^{-\beta}$	$(0,\infty)$

The Burr XII distribution has two shape positive parameters namely θ and β. The PDF of the Burr distribution as follows

$$f(x;\theta,\beta)=\beta\theta\left(1+x^{\theta}\right)^{-(\beta+1)}x^{\theta-1} \qquad 0\le x\le\infty \quad \theta\ ,\beta>0, \tag{1}$$

The corresponding CDF of the Burr distribution is given as follows

$$F(x;\theta,\beta)=\ 1-\left(\frac{1}{1+x^{\theta}}\right)^{\beta}, \tag{2}$$

And the survival function, cumulative hazard rate and hazard rate function are given respectively as follows

$$S(x) = \left(1 + x^{\theta}\right)^{-\beta}, \tag{3}$$

$$H(x) = -\log\left(S(x)\right) = \beta \log\left(1 + x^{\theta}\right), \tag{4}$$

And

$$h(x) = \beta\theta\left(1 + x^{\theta}\right)^{-1} x^{\theta-1}. \tag{5}$$

Recently, Marshall and Olkin (2007) introduced the hazard power parameter (HPP) family of distributions that is given as follows

If the survival function can be written as follows

$$S(x) = 1 - F(x) = e^{-H(x)}.$$

Where $H(x)$ is the cumulative hazard function

Thus,

$$S_{HPP}(x) = e^{\{-[H_B(x)]^{\alpha}\}} \qquad \alpha > 0, \tag{6}$$

$$F_{HPP}(x) = 1 - S_{HPP}(x),$$

Where $[H_B(x)]^{\alpha}$ is accumulative hazard function.

Defines a survival function for all $\alpha > 0$, and $\mathrm{S_{HPP}(x)}$, $\alpha > 0$ is a semi-parametric family. The parameter α is called hazard power parameter family. So the corresponding PDF of the HPP family is given as

$$f_{\mathrm{HPP}}(x) = \alpha\, h_B(x)[H_B(x)]^{\alpha-1} e^{\{-[H_B(x)]^{\alpha}\}}, \tag{7}$$

Where α is a shape parameter $H_B(x)$ is the baseline cumulative hazard function and $h_B(x)$ is the baseline hazard function. Respectively the hazard function for HPP family is given by

$$\mathrm{h_{HPP}(x)} = \alpha\, h_B(x)[H_B(x)]^{\alpha-1}. \tag{8}$$

It is follows if h_B increasing and $\alpha \geq 1$, then the hazard function for HPP family is increasing, and if h_B decreasing and $0 < \alpha < 1$, then the hazard function for HPP family is decreasing. For more information about this family see Marshall and Olkin (2007) and Muhammed $(2022)^{\alpha}$.

The paper is organized as follows, in Section 2, the mathematical distributional properties of the new distribution were discussed. In Section 3, estimation the parameters using the maximum likelihood estimation and the Bayesian estimation methods were discussed. In Section 4, a simulation study is performed. In Section 5, an application on a real data set and a comparison between the Burr distribution with two parameters, the Generalized Burr distribution with three parameters, Weibull distribution, Singh-Maddala and the hazard power parameter Burr (HPPB) distribution with three parameters are performed.

2. The Distributional Properties of the HPPB Distribution

In this section, we will obtain the mathematical properties of the hazard power parameter Burr (HPPB) distribution, including the PDF, CDF, survival function, hazard function, cumulative hazard function, reversed hazard function, cumulative reversed hazard function, mode, quantile function, quartiles, the distribution of order statistics and the stochastic ordering of the HPPB distribution.

2.1 The PDF and CDF of HPPB Distribution

By substituting Eq (2), (3), (4) and (5) into Eq (6) and (7), it is obtained the PDF and CDF of the HPPB distribution. The random variable X is said to have the HPPB distribution if its CDF is of the form

$$F_{HPPB}(x;\alpha,\beta,\theta) = 1 - e^{-\left[\beta\, log\left(1+x^{\theta}\right)^{\alpha}\right]}, \tag{9}$$

The corresponding PDF of HPPB distribution is

$$f_{HPPB}(\mathrm{x};\alpha,\beta,\theta) = \theta\alpha\beta\left(1+x^{\theta}\right)^{-1}x^{\theta-1}\left[\beta\log\left(1+x^{\theta}\right)\right]^{\alpha-1} e^{-\left[\beta\log\left(1+x^{\theta}\right)\right]^{\alpha}} \quad x \geq 0 \quad \beta,\alpha,\theta > 0. \tag{10}$$

Where α and θ are shape positive parameters, and β is scale positive parameter, note that the Figure 1 illustrates some of the possible shapes of the probability density function of the HPPB distribution with parameters β, α and θ for $(\alpha = 0.5, \beta = 0.5, \theta = 4), (\alpha = 0.5, \beta = 2, \theta = 4),$

$(\alpha = 6, \beta = 0.5, \theta = 3) and \ (\alpha = 6, \beta = 0.5, \theta = 2).$

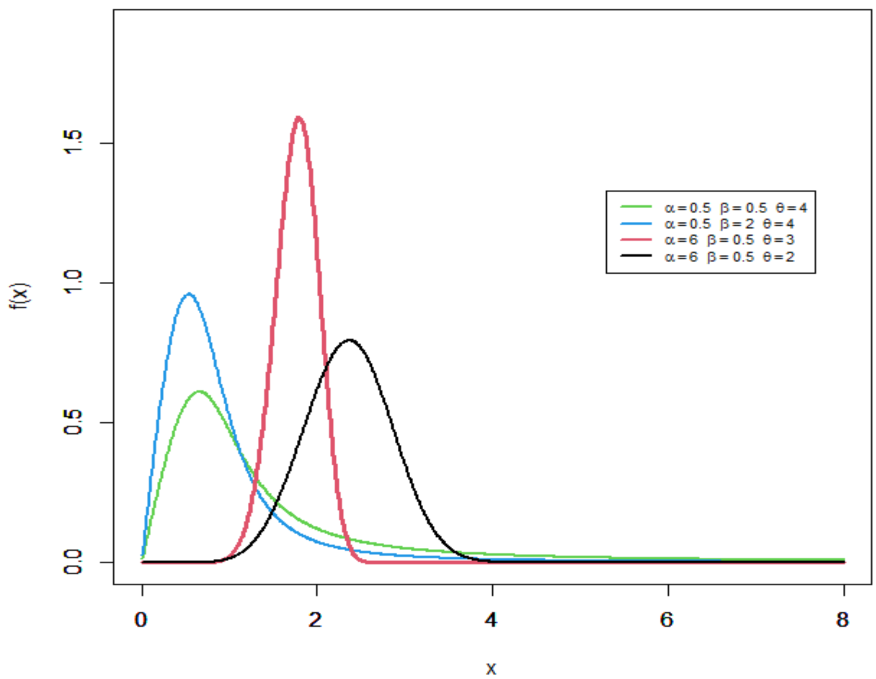

Figure 1: Plots of the PDF of HPPB Distribution for Selected Values of the Parameters

From Figure 1, it is found that the distribution is unimodal. The HPPB distribution with parameters $\alpha = 0.5, \beta = 0.5, \theta = 4$ and $\alpha = 0.5, \beta = 2, \theta = 4$ is positive skewed to the right. Thus, *f(x)* has a flat and relatively long right-hand tail, meaning that longer lifetimes and less probable than shorter lifetimes and that the mean life is greater than the median life. The distribution with parameters $\alpha = 6\,, \beta = 0.5\,, \theta = 3$ and $\alpha = 6\,, \beta = 0.5\,, \theta = 2$ is Semi-Symmetric distribution.

Figure 2 illustrates some of the possible shapes of the cumulative density function of the HPPB distribution with parameters β, α and θ for $(\alpha = 0.5\,, \beta = 0.5\,, \theta = 4), (\alpha = 0.5\,, \beta = 2\,, \theta = 4), (\alpha = 6\,, \beta = 0.5\,, \theta = 3) and\ (\alpha = 6\,, \beta = 0.5\,, \theta = 2)$.

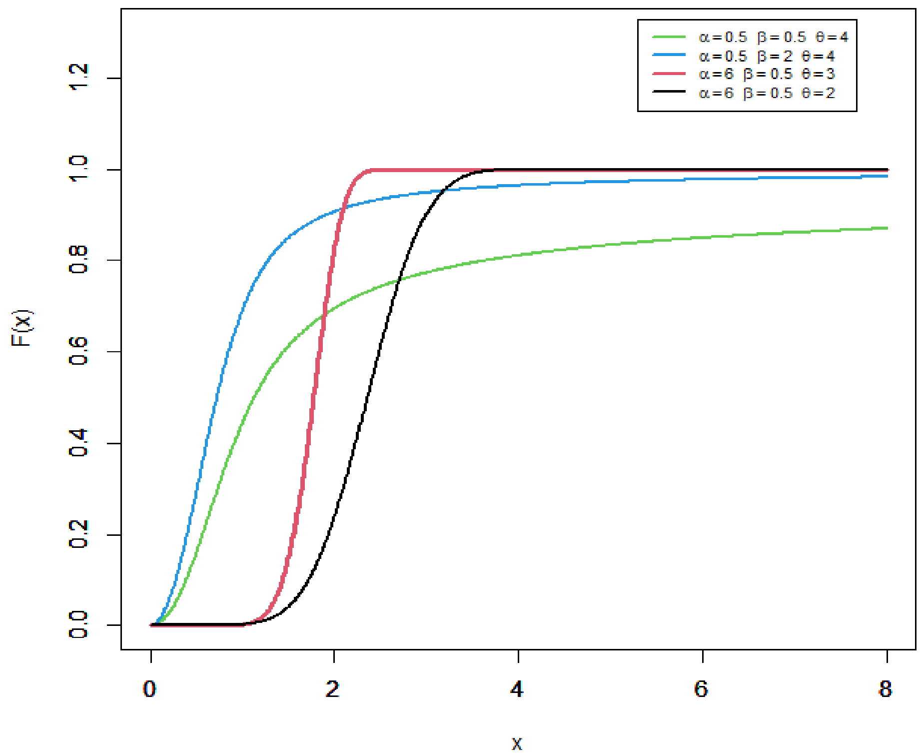

Figure 2: Plots of the CDF of HPPB Distribution for Selected Values of the Parameters

2.2 The Survival Function

It is also called reliability function that is the probability of exceeding *x* or the probability that an individual survives beyond time *x*. it is given as following

$$S(x;\alpha,\beta,\theta) = \int_x^{\infty} f(x;\alpha,\beta,\theta)dx = 1 - \int_0^{x} f(x;\alpha,\beta,\theta)dx = 1 - F(x;\alpha,\beta,\theta), \tag{11}$$

By substituting Eq (9) into Eq (11), it is obtained that

$$S(x;\alpha,\beta,\theta) = e^{-[\beta\, log(1+x^{\theta})]^{\alpha}}. \tag{12}$$

Figure 3 illustrates some of the possible shapes of the survival function of the HPPB distribution with parameters β, α and θ for $(\alpha = 0.5\,, \beta = 0.5\,, \theta = 4), (\alpha = 0.5\,, \beta = 2\,, \theta = 4),$

$(\alpha = 6\,, \beta = 0.5\,, \theta = 3)\ and\ (\alpha = 6\,, \beta = 0.5\,, \theta = 2).$

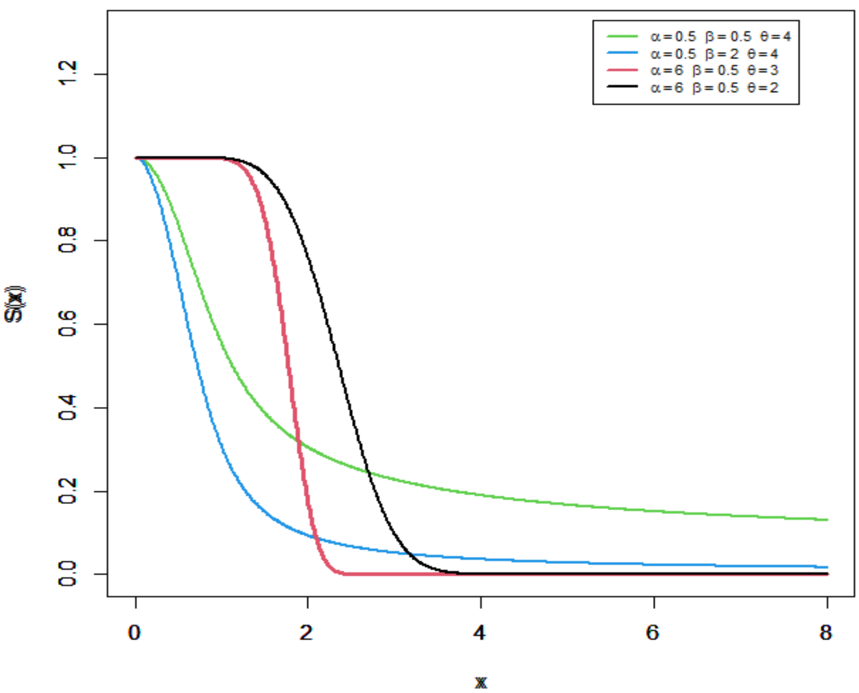

Figure 3: Plots of the Survival Function of HPPB Distribution for Selected Values of the Parameters

2.3 The Hazard and the Cumulative Hazard Functions for HPPB Model

A quantity that is often used along with the survival function is the hazard function. It is describes the intensity of the death at the time *x* given that the individual has already survived pas time *x***.** it is given as following

$$h(x;\alpha,\beta,\theta)=\frac{f(x;\alpha,\beta,\theta)}{s(x;\alpha,\beta,\theta)}, \tag{13}$$

By substituting Eq (10) and (12) into Eq (13), it is obtained that

$$h(x;\alpha,\beta,\theta) \ = \theta\alpha\beta\left(1+x^{\theta}\right)^{-1} x^{\theta-1}\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha-1}.$$

The cumulative hazard function is defined as the cumulative amount of hazard up to time *x*. The cumulative hazard function is given in terms of the survival function as follows:

$$H(x;\alpha,\beta,\theta)=\int_0^x h(x)dx=-log(1-\ F(x;\alpha,\beta,\theta))=-\,log\big(S(x;\alpha,\beta,\theta)\big), \tag{14}$$

By substituting Eq (12) into Eq (14), it is obtained the cumulative hazard function of the HPPB model as:

$$H(x;\alpha,\beta,\theta)=\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}.$$

Figure 4 illustrates some of the possible shapes of the hazard function of the HPPB distribution with parameters $\beta,\alpha\ and\ \theta$ for $(\alpha=0.2\ ,\beta=0.5\ ,\theta=0.8)$, $(\alpha=2\ ,\beta=0.5\ ,\theta=0.8)$,

$(\alpha=0.7\ ,\beta=0.5\ ,\theta=0.8)$, $(\alpha=3\ ,\beta=1\ ,\theta=0.4)$, $(\alpha=0.5\ ,\beta=0.5\ ,\theta=4)$ and

$(\alpha=0.5\ ,\beta=2\ ,\theta=4)$.

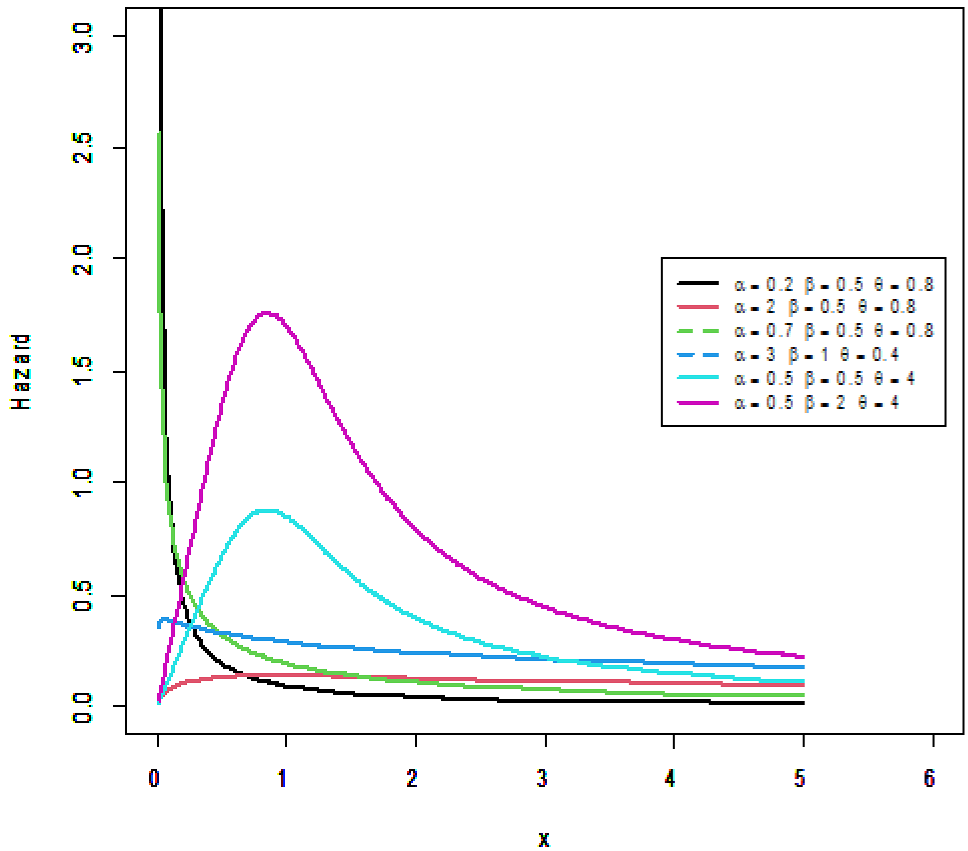

Figure 4: Plots of the Hazard Function of the HPPB Distribution for Selected Values of the Parameters

There are four states from Figure 4 that describe the behavior of the hazard function as following

1. If $\alpha < 1$ and $\theta < 1$, the hazard function has decreased failure rate.
2. If $\alpha > 1$ and $\theta > 1$, the hazard function has an increased failure rate.
3. If $\alpha < 1$ and $\theta > 1$, the hazard function has upside down failure rate.
4. If $\alpha > 1$ and $\theta < 1$, the hazard function has upside down failure rate.

2.4 The Reversed Hazard and the Cumulative Reversed Hazard Functions

The reversed hazard function is defined as the ratio of the density to its distribution function as follows

$$r(x;\alpha,\beta,\theta) = \frac{f(x;\alpha,\beta,\theta)}{F(x;\alpha,\beta,\theta)}. \tag{15}$$

By substituting Eq (9) and (10) into Eq (15), it is obtained the reversed hazard function of the HPPB model that is given by

$$r(x;\alpha,\beta,\theta) = \frac{\theta\alpha\beta\left(1+x^{\theta}\right)^{-1} x^{\theta-1}\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha-1} e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}}{1-e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}}.$$

The cumulative reversed hazard function is defined as the total amount of failures accumulated after the time point x. it is defined as

$$R(x;\alpha,\beta,\theta) = log\left(F(x)\right) \tag{16}$$

By substituting Eq (9) into Eq (16), it is obtained the cumulative reversed hazard function that is given by

$$R(x;\alpha,\beta,\theta) = log\left(1-e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right).$$

Figure 5 illustrates some of the possible shapes of the reversed hazard function of the HPPB distribution with parameters β, α and θ for $(\alpha = 0.5\,, \beta = 0.5\,, \theta = 4)$, $(\alpha = 0.5\,, \beta = 2\,, \theta = 4)$, $(\alpha = 6\,, \beta = 0.5\,, \theta = 3)$ and $(\alpha = 6\,, \beta = 0.5\,, \theta = 2)$.

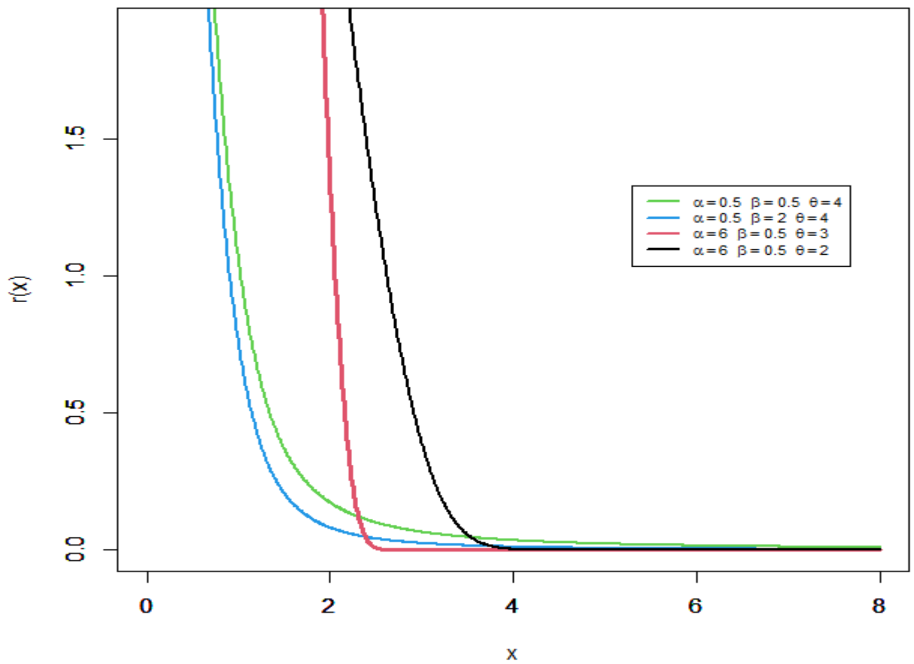

Figure 5: Plots of the reversed hazard function of the HPPB distribution for selected values of the parameters

2.5 The Mode of the HPPB Distribution

The mode of this distribution can be found by solving the following equation,

$$\frac{d}{dx} f(x; \alpha, \beta, \theta) = 0.$$

By taking the derivative of Eq (10) and equating it to zero and solving for x, we get the following equation

$$\frac{d}{dx} f(x;\alpha,\beta,\theta)$$

$$= -(\theta\alpha\beta^{\alpha})^2\left(1+x^{\theta}\right)^{-2}x^{2(\theta-1)}\ \left[log\left(1+x^{\theta}\right)\right]^{2\alpha-2}\ e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}$$
$$+\ \theta^2\alpha\beta^{\alpha}\left(1+x^{\theta}\right)^{-2}x^{2(\theta-1)}\ e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}$$
$$-\ \theta^2\alpha\beta^{\alpha}\left(1+x^{\theta}\right)^{-2}x^{2(\theta-1)}\left[log\left(1+x^{\theta}\right)\right]^{\alpha-1}\ e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}$$
$$+\ \theta\alpha\beta^{\alpha}(\theta-1)\left[log\left(1+x^{\theta}\right)\right]^{\alpha-1}\left(1+x^{\theta}\right)^{-1}x^{(\theta-1)} = 0.$$

This equation can be solved numerically for x. Figure 1 illustrates that the HPPB distribution is unimodal.

2.6 The Quantile Function of HPPB Distribution

The quantile function specifies the value of the random variable such that the probability of the variable being less than or equal to that value equals the given probability. It used for the simulation of the HPPB distribution. It is obtained as following:

By inverting the CDF of the distribution,

$F(x;\alpha,\beta,\theta) = U,$

The quantile function is given as following

$$x_q = e^{\left\{\frac{[-log(1-U)]^{\frac{1}{\alpha}}}{\beta}\right\}^{1/\theta}}. \tag{17}$$

Where q is the quantile, and u is $0 < u < 1$. It is easily to obtain the median and the quartiles as following

If u=0.25, it is obtained the first quantile that equal

$$Q_1 = e^{\left\{\frac{[-log(0.75)]^{\frac{1}{\alpha}}}{\beta}\right\}^{1/\theta}}.$$

If u=0.5, it is obtained the second quantile that equal the median

$$Q_2 = e^{\left\{\frac{[-log(0.5)]^{\frac{1}{\alpha}}}{\beta}\right\}^{1/\theta}} .$$

If u=0.75, it is obtained the third quantile that equal

$$Q_3 = e^{\left\{\frac{[-log(0.25)]^{\frac{1}{\alpha}}}{\beta}\right\}^{1/\theta}} .$$

2.7 The Distribution of Order Statistics

Suppose $x_1, x_2, \ldots \ldots \ldots . x_n$ is a random sample from the HPPB distribution and $x_{i:n}$ denotes the i^{th} order statistics, thus the PDF $f_{i:n}(i = 1,2, \ldots \ldots \ldots n)$ of the i^{th} order statistics is expressed as

$$f_{i:n}(x, \alpha, \beta, \theta) = \frac{n!}{(i-1)!\,(n-i)!}\, f(x, \theta, \beta, \alpha) * \ [F(x, \theta, \beta, \alpha)]^{i-1} * [1 - F(x, \theta, \beta, \alpha)]^{n-i}, \tag{18}$$

Where $x_{i:n} = x$. By substituting Eq (9) and (10) into Eq (18)

$$f_{i:n}(x, \alpha, \beta, \theta) = \frac{n!}{(i-1)!\,(n-i)!}\left\{\theta\alpha\beta^{\alpha}\left(1 + x^{\theta}\right)^{-1} x^{\theta-1}\left(log\left(1 + x^{\theta}\right)\right)^{\alpha-1} e^{-\left(\beta\, log\left(1+x^{\theta}\right)\right)^{\alpha}}\right\}$$

$$.\left\{1 - e^{-\left(\beta\, log\left(1+x^{\theta}\right)\right)^{\alpha}}\right\}^{i-1} \left\{e^{-\left(\beta\, log\left(1+x^{\theta}\right)\right)^{\alpha}}\right\}^{n-i}, \tag{19}$$

By using the binomial series expansion that can expressed as

$$(1 - x)^n = \sum_{k=0}^{n}(-1)^k \binom{n}{k} x^k$$

If n is neither a natural number nor zero

So

$$\left\{1 - e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}^{i-1} = \sum_{k=0}^{i-1}(-1)^k \binom{i-1}{k} \left\{e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}^{k}, \tag{20}$$

By substituting Eq (20) into Eq (19), it is obtained the following

$$f_{i:n}(x,\alpha,\beta,\theta) = n!\,\theta\alpha\beta^{\alpha}\left(1+x^{\theta}\right)^{-1} x^{\theta-1}\left[log\left(1+x^{\theta}\right)\right]^{\alpha-1}$$

$$.\sum_{k=0}^{i-1}\frac{(-1)^{k}\binom{i-1}{k}}{(i-1)!(n-i)!}\left\{e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}^{k+n-i+1}.$$

Where PDF the smallest order statistic when i=1 is given by

$$f_{1:n}(x;\alpha,\beta,\theta) = n\, f(x,\alpha,\beta,\theta) *\ \{1-F(x,\alpha,\beta,\theta)\}^{n-1}, \tag{21}$$

By substituting Eq (9) and (10) into Eq (21), it is obtained that

$$f_{1:n}(x;\alpha,\beta,\theta) = n\left\{\theta\alpha\beta^{\alpha}\left(1+x^{\theta}\right)^{-1} x^{\theta-1}\left[log\left(1+x^{\theta}\right)\right]^{\alpha-1} e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}$$

$$.\left\{e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}^{n-1}$$

$$= n\left\{\theta\alpha\beta^{\alpha}\left(1+x^{\theta}\right)^{-1}\ x^{\theta-1}\left[log\left(1+x^{\theta}\right)\right]^{\alpha-1}\right\}\left\{e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}^{n}.$$

Where PDF the largest order statistic when i=n is given by

$$f_{n:n}(x;\alpha,\beta,\theta) = n\, f(x,\alpha,\beta,\theta) *\ \{F(x,\alpha,\beta,\theta)\}^{n-1}, \tag{22}$$

By substituting Eq (9) and (10) into Eq (22), it is obtained that

$$f_{n:n}(x;\alpha,\beta,\theta) = n\left\{\theta\alpha\beta^{\alpha}\left(1+x^{\theta}\right)^{-1}\ x^{\theta-1}\left[log\left(1+x^{\theta}\right)\right]^{\alpha-1} e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}$$

$$.\left\{1-e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}^{n-1},$$

$$f_{n:n}(x;\alpha,\beta,\theta) = n\theta\alpha\beta^{\alpha}\left(1+x^{\theta}\right)^{-1}\ x^{\theta-1}\left[log\left(1+x^{\theta}\right)\right]^{\alpha-1}$$

$$.\sum_{k=0}^{n-1}(-1)^{k}\binom{n-1}{k}\left\{e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}}\right\}^{k+1}.$$

2.8 The Stochastic Ordering

The stochastic ordering of positive continuous random variables is an important tool for judging their comparative behaviour. It is interested in the concept of random variable X being bigger than another. X is said to be smaller than Y in the

- Stochastic order $X \leq_{ST} Y \; if \; F_X(x) \geq F_Y(x) for \; all \; x.$
- Hazard rate order $X \leq_{hr} Y \; if \; h_X(x) \geq h_Y(x) for \; all \; x.$
- Mean residual life order $X \leq_{mrl} Y \; if \; m_X(x) \geq m_Y(x) for \; all \; x.$
- Likelihood ratio order $X \leq_{lr} Y \; if \frac{f_X(x)}{f_Y(x)} \; decrease \; in \; x.$

The following stochastic ordering relationships due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of continuous distributions.

$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{ST} Y$

HPPB distribution is ordered with respect to the strongest order as shown in the following

Suppose random variables X and Y are distributed according to $HPPB(\alpha_1, \beta_1, \theta_1)$ and $HPPB(\alpha_2, \beta_2, \theta_2)$. Then, the following results hold true.

If $X \sim HPPB(\alpha_1, \beta_1, \theta_1) and$ Y$\sim HPPB(\alpha_2, \beta_2, \theta_2) \; if \; \theta_2 = \theta_1 = \theta \; and \; \beta_1 = \beta_2 = \beta \; and \; \alpha_1 \geq \alpha_2 then \; X \leq_{lr} Y, X \leq_{hr} Y \; and \; X \leq_{ST} Y.$

The likelihood ratio is $\frac{f_x(x)}{f_y(y)}$.

It can be proved that

$$\frac{f_x(x)}{f_y(y)} = \frac{\theta_1 \alpha_1 {\beta_1}^{\alpha_1} \left(1 + x^{\theta_1}\right)^{-1} x^{\theta_1 - 1} \left[log\left(1 + x^{\theta_1}\right)\right]^{\alpha_1 - 1} e^{-\left[\beta_1 \, log\left(1 + x^{\theta_1}\right)\right]^{\alpha_1}}}{\theta_2 \alpha_2 {\beta_2}^{\alpha_2} (1 + x^{\theta_2})^{-1} x^{\theta_2 - 1} [log(1 + x^{\theta_2})]^{\alpha_2 - 1} e^{-\left[\beta_2 \, log\left(1 + x^{\theta_2}\right)\right]^{\alpha_2}}} \; .$$

$$\begin{aligned} log \frac{f_x(x)}{f_y(y)} = & \; log \, \theta_1 + log \, \alpha_1 + \alpha_1 \, log \, \beta_1 - log\left(1 + x^{\theta_1}\right) + (\theta_1 - 1) \, log \, x \\ & + (\alpha_1 - 1) \, log\left(log\left(1 + x^{\theta_1}\right)\right) - \left[\beta_1 \, log\left(1 + x^{\theta_1}\right)\right]^{\alpha_1} - log \, \theta_2 - log \, \alpha_2 \\ & - \alpha_2 \, log \, \beta_2 + log\left(1 + x^{\theta_2}\right) - (\theta_2 - 1) \; log \, x \\ & - (\alpha_2 - 1) \, log\left(log\left(1 + x^{\theta_2}\right)\right) + \left[\beta_2 \, log\left(1 + x^{\theta_2}\right)\right]^{\alpha_2}. \end{aligned}$$

$$\frac{d}{dx}log\frac{f_x(x)}{f_y(y)} = -\frac{\theta_1 x^{\theta_1-1}}{1+x^{\theta_1}} + \frac{\theta_1 - 1}{x} + \frac{(\alpha_1-1)\theta_1 x^{\theta_1-1}}{(1+x^{\theta_1})\,log(1+x^{\theta_1})}$$

$$-\frac{\alpha_1{\beta_1}^{\alpha_1}\theta_1 x^{\theta_1-1}\left[log\left(1+x^{\theta_1}\right)\right]^{\alpha_1-1}}{1+x^{\theta_1}} + \frac{\theta_2 x^{\theta_2-1}}{1+x^{\theta_2}} - \frac{\theta_2-1}{x}$$

$$-\frac{(\alpha_2-1)\theta_2 x^{\theta_2-1}}{(1+x^{\theta_2})\,log(1+x^{\theta_2})} + \frac{\alpha_2{\beta_2}^{\alpha_2}\theta_2 x^{\theta_2-1}\left[log\left(1+x^{\theta_2}\right)\right]^{\alpha_2-1}}{1+x^{\theta_2}},$$

If $if\ \theta_2 = \theta_1 = \theta\ and\ \beta_1 = \beta_2 = \beta\ and\ \alpha_1 \geq \alpha_2.$

$$\frac{d}{dx}log\frac{f_x(x)}{f_y(y)} = \frac{(\alpha_1-1)\theta_1 x^{\theta_1-1}}{(1+x^{\theta_1})\,log(1+x^{\theta_1})} - \frac{\alpha_1{\beta_1}^{\alpha_1}\theta_1 x^{\theta_1-1}\left[log\left(1+x^{\theta_1}\right)\right]^{\alpha_1-1}}{1+x^{\theta_1}}$$

$$-\frac{(\alpha_2-1)\theta_2 x^{\theta_2-1}}{(1+x^{\theta_2})\,log(1+x^{\theta_2})} + \frac{\alpha_2{\beta_2}^{\alpha_2}\theta_2 x^{\theta_2-1}\left[log\left(1+x^{\theta_2}\right)\right]^{\alpha_2-1}}{1+x^{\theta_2}}$$

$$= \frac{(\alpha_1-1)\theta_1 x^{\theta_1-1} - (\alpha_2-1)\theta_2 x^{\theta_2-1}}{(1+x^{\theta_1})\,log(1+x^{\theta_1})}$$

$$+\frac{\alpha_2{\beta_2}^{\alpha_2}\theta_2 x^{\theta_2-1}\ \left[log\left(1+x^{\theta_2}\right)\right]^{\alpha_2-1} - \alpha_1{\beta_1}^{\alpha_1}\theta_1 x^{\theta_1-1}\left[log\left(1+x^{\theta_1}\right)\right]^{\alpha_1-1}}{1+x^{\theta_1}}$$

$$= \frac{(\alpha_1-\alpha_2)\theta_1 x^{\theta_1-1}}{(1+x^{\theta_1})\,log(1+x^{\theta_1})}$$

$$-\frac{\theta_1 x^{\theta_1-1}\left\{\alpha_1{\beta_1}^{\alpha_1}\left[log\left(1+x^{\theta_1}\right)\right]^{\alpha_1-1} - \alpha_2{\beta_2}^{\alpha_2}\left[log\left(1+x^{\theta_2}\right)\right]^{\alpha_2-1}\right\}}{1+x^{\theta_1}}.$$

Then $\frac{d}{dx}\log\frac{f_x(x)}{f_y(y)} < 0$, which implies that $X \leq_{lr} Y$ and hence $X \leq_{mrl} Y$, $X \leq_{hr} Y$, $X \leq_{ST} Y$.

3. The Maximum Likelihood and Bayesian Estimation under Complete Data

In these sections, the point estimation and confidence interval based on maximum likelihood are proposed for simulated complete data. It is also obtained the Bayes estimates of the known parameters under the assumption of independent gamma priors. The Bayes estimates of the known parameters cannot be obtained in closed form. Markov chain Monte Carlo (MCMC) method has been used to compute the approximate Bayes estimates under the squared error loss function and also to construct the highest posterior density (HPD).

3.1Maximum Likelihood Estimation for HPPB Distribution

The maximum likelihood estimation is popular and important method that used for estimating the unknown parameters for the distribution. Suppose $x_1, x_2, \ldots\ldots\ldots . x_n$ is a random sample from the HPPB distribution, the likelihood function is given as following

$$L(\alpha, \beta, \theta) = \prod_{i=1}^{n} f(x_i; \alpha, \beta, \theta) \tag{23}$$

By substituting Eq (10) into Eq (23) it is obtained the following

$$L = L(\alpha, \beta, \theta) = (\alpha)^n (\beta)^{\alpha n} \theta^n \left(\prod_{i=1}^{n} x_i^{\theta-1}\right) e^{-\beta^\alpha \sum_{i=1}^{n}\left[log\left(1+x_i^\theta\right)\right]^\alpha} \left[\prod_{i=1}^{n}\left(1+x^\theta\right)^{-1}\right]$$

$$.\left\{\prod_{i=1}^{n}\left[log\left(1+x_i^\theta\right)\right]^{\alpha-1}\right\}.$$

The log- likelihood function is given as following, suppose $\ln(L) \equiv \ln[L(\alpha, \beta, \theta)]$

$$\ln(L) = n\ln(\alpha) + n\ln(\theta) + n\alpha\ln(\beta) + (\theta-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\ln\left(1+x_i^\theta\right)$$

$$+(\alpha-1)\sum_{i=1}^{n}\ln\left[log\left(1+x_i^{\theta}\right)\right]-\beta^{\alpha}\sum_{i=1}^{n}\left[log\left(1+x_i^{\theta}\right)\right]^{\alpha}. \qquad (24)$$

By differentiating Eq (24) with respect to β, α and θ respectively, for obtaining the estimated parameters $\hat{\beta}, \hat{\alpha}$ *and* $\hat{\theta}$.

$$\frac{\partial \ln(L)}{\partial\beta}=\frac{n\alpha}{\beta}-\alpha\,\beta^{\alpha-1}\sum_{i=1}^{n}\left[log\left(1+x_i^{\theta}\right)\right]^{\alpha},$$

$$\frac{\partial \ln(L)}{\partial\alpha}=\frac{n}{\alpha}+n\ln\beta-\sum_{i=1}^{n}\left[\beta\, log\left(1+x_i^{\theta}\right)\right]^{\alpha}\ln\left[\beta\, log\left(1+x_i^{\theta}\right)\right]$$

$$+\sum_{i=1}^{n}\ln\left[log\left(1+x_i^{\theta}\right)\right],$$

And

$$\frac{\partial \ln(L)}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\left(\frac{x_i^{\theta}\ln x_i}{1+x_i^{\theta}}\right)+(\alpha-1)\sum_{i=1}^{n}\left(\frac{x_i^{\theta}\ln x_i}{\left(1+x_i^{\theta}\right)log\left(1+x_i^{\theta}\right)}\right)$$

$$+\sum_{i=1}^{n}\ln x_i-\alpha\beta^{\alpha}\sum_{i=1}^{n}\left[log\left(1+x_i{}^{\theta}\right)\right]^{\alpha-1}\left(\frac{x_i^{\theta}\ln x_i}{1+x_i^{\theta}}\right),$$

By equating $\frac{\partial \ln(L)}{\partial\beta}, \frac{\partial \ln(L)}{\partial\alpha}$ and $\frac{\partial \ln(L)}{\partial\theta}$ to zero and solving for $\hat{\alpha}$ *and* $\hat{\theta}$, the MLE of these parameters are the numerical solution of the following two equations:

$$\frac{n\hat{\alpha}}{\hat{\beta}}-\hat{\alpha}\,\hat{\beta}^{\,\hat{\alpha}-1}\sum_{i=1}^{n}\left[log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}}=0,$$

$$\frac{n}{\hat{\alpha}} + n\ln\hat{\beta} - \sum_{i=1}^{n}\left[\hat{\beta}\,log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}}\ln\left[\hat{\beta}\,log\left(1+x_i^{\hat{\theta}}\right)\right] + \sum_{i=1}^{n}\ln\left[log\left(1+x_i^{\hat{\theta}}\right)\right] = 0,$$

And

$$\frac{n}{\hat{\theta}} + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i^{\hat{\theta}}\ln x_i}{1+x_i^{\hat{\theta}}}\right) + (\hat{\alpha}-1)\sum_{i=1}^{n}\left[\frac{x_i^{\hat{\theta}}\ln x_i}{\left(1+x_i^{\hat{\theta}}\right)log\left(1+x_i^{\hat{\theta}}\right)}\right]$$

$$-\hat{\alpha}\,\hat{\beta}^{\hat{\alpha}}\sum_{i=1}^{n}\left[log\left(1+x_i{}^{\hat{\theta}}\right)\right]^{\hat{\alpha}-1}\left(\frac{x_i^{\hat{\theta}}\ln x_i}{1+x_i^{\hat{\theta}}}\right) = 0,$$

So

$$\hat{\beta} = \left[\frac{n}{\sum_{i=1}^{n}\left[log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}}}\right]^{\frac{1}{\hat{\alpha}}}$$

Due to the equations are not in closed form, the MLEs of these parameters can be obtained numerically as will be seen in following sections

We can obtain the asymptotic variance covariance matrix of $\alpha, \beta\ and\ \theta$ by inverting the information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood function, which is given by Cohen in 1965 as following

$$\left.\begin{bmatrix} \frac{\partial^2\ln(\mathrm{L})}{\partial\alpha^2} & \frac{\partial^2\ln(\mathrm{L})}{\partial\alpha\,\partial\beta} & \frac{\partial^2\ln(\mathrm{L})}{\partial\alpha\,\partial\theta} \\ \frac{\partial^2\ln(\mathrm{L})}{\partial\beta\,\partial\alpha} & \frac{\partial^2\ln(\mathrm{L})}{\partial\beta^2} & \frac{\partial^2\ln(\mathrm{L})}{\partial\beta\,\partial\theta} \\ \frac{\partial^2\ln(\mathrm{L})}{\partial\theta\,\partial\alpha} & \frac{\partial^2\ln(\mathrm{L})}{\partial\theta\,\partial\beta} & \frac{\partial^2\ln(\mathrm{L})}{\partial\theta^2} \end{bmatrix}^{-1}\right|_{\emptyset=\hat{\emptyset}} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}. \qquad (25)$$

Where $\emptyset = (\alpha, \beta, \theta)$ and $\hat{\emptyset} = \left(\hat{\alpha}, \hat{\beta}, \hat{\theta}\right)$. The elements of the observed information matrix are given as follows:

$$\left.\frac{\partial^2 \ln(L)}{\partial \beta^2}\right|_{\emptyset=\hat{\emptyset}} = \frac{-\hat{\alpha}\, n}{\hat{\beta}^2} - \hat{\alpha}\,(\hat{\alpha}-1)\hat{\beta}^{\hat{\alpha}-2} \sum_{i=1}^{n} \left[log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}},$$

$$\left.\frac{\partial^2 \ln(L)}{\partial \alpha^2}\right|_{\emptyset=\hat{\emptyset}} = \frac{-n}{\hat{\alpha}^2} - \sum_{i=1}^{n} \left[\hat{\beta}\, log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}} \left\{\ln\left[\hat{\beta}\, log\left(1+x_i^{\hat{\theta}}\right)\right]\right\}^2,$$

$$\left.\frac{\partial^2 \ln(L)}{\partial \theta^2}\right|_{\emptyset=\hat{\emptyset}} = \frac{-n}{\hat{\theta}^2} - \sum_{i=1}^{n} \left[\frac{x_i^{\hat{\theta}}\,(\ln x_i)^2\left(1+x_i^{\hat{\theta}}\right) - \left(x_i^{\hat{\theta}}\,\ln x_i\right)^2}{\left(1+x_i^{\hat{\theta}}\right)^2}\right]$$

$$+(\hat{\alpha}-1)\sum_{i=1}^{n}\left\{\frac{x_i^{\hat{\theta}}\,(\ln x_i)^2\,\left(1+x_i^{\hat{\theta}}\right) log\left(1+x_i^{\hat{\theta}}\right) - (\ln x_i)^2\left(x_i^{\hat{\theta}}\right)^2\left[1+log\left(1+x_i^{\hat{\theta}}\right)\right]}{\left[\left(1+x_i^{\hat{\theta}}\right) log\left(1+x_i^{\hat{\theta}}\right)\right]^2}\right\}$$

$$-\hat{\alpha}\,\hat{\beta}^{\hat{\alpha}}\sum_{i=1}^{n}\left\{\frac{\left[log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}-1}\left[x_i^{\hat{\theta}}\,(\ln x_i)\left(1+x_i^{\hat{\theta}}\right) - \left(x_i^{\hat{\theta}}\,\ln x_i\right)^2\right]}{\left[\left(1+x_i^{\hat{\theta}}\right)\right]^2}\right\}$$

$$-\hat{\alpha}\,\hat{\beta}^{\hat{\alpha}}\sum_{i=1}^{n}\left\{\frac{(\hat{\alpha}-1)\left[log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}-2}(\ln x_i)\left(x_i^{\hat{\theta}}\right)^2}{\left[\left(1+x_i^{\hat{\theta}}\right)\right]^2}\right\},$$

$$\left.\frac{\partial^2 \ln(L)}{\partial \alpha\, \partial \beta}\right|_{\emptyset=\hat{\emptyset}} = \frac{n}{\hat{\beta}} - \hat{\alpha}\,\hat{\beta}^{\hat{\alpha}-1}\sum_{i=1}^{n}\left[log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}} \ln\left[\hat{\beta}\, log\left(1+x_i^{\hat{\theta}}\right)\right]$$

$$-\hat{\beta}^{\hat{\alpha}-1}\sum_{i=1}^{n}\left[\log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}},$$

$$\left.\frac{\partial^2 \ln(L)}{\partial \alpha\, \partial \theta}\right|_{\emptyset=\hat{\emptyset}} = \sum_{i=1}^{n}\left[\frac{x_i^{\hat{\theta}}\,\ln x_i}{\left(1+x_i^{\hat{\theta}}\right) log\left(1+x_i^{\hat{\theta}}\right)}\right] - \sum_{i=1}^{n}\left\{\frac{\hat{\beta}^{\hat{\alpha}}\,x_i^{\hat{\theta}}\,\ln x_i\left[log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}-1}}{\left(1+x_i^{\hat{\theta}}\right) log\left(1+x_i^{\hat{\theta}}\right)}\right\}$$

$$-\sum_{i=1}^{n}\left\{\frac{\hat{\alpha}\,\hat{\beta}x_i^{\hat{\theta}}\,\ln x_i\left[\hat{\beta}\,log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}-1}\ln\left[\hat{\beta}\,log\left(1+x_i^{\hat{\theta}}\right)\right]}{\left(1+x_i^{\hat{\theta}}\right)}\right\},$$

And

$$\left.\frac{\partial^2\ln(L)}{\partial\beta\,\partial\theta}\right|_{\emptyset=\hat{\emptyset}}=-\hat{\alpha}^{\,2}\hat{\beta}^{\hat{\alpha}\,-1}\sum_{i=1}^{n}\left\{\frac{x_i^{\hat{\theta}}\,\ln x_i\left[\,log\left(1+x_i^{\hat{\theta}}\right)\right]^{\hat{\alpha}-1}}{\left(1+x_i^{\hat{\theta}}\right)}\right\}.$$

Where $\frac{\partial^2\ln(L)}{\partial\beta\,\partial\theta}=\frac{\partial^2\ln(L)}{\partial\theta\,\partial\beta},\frac{\partial^2\ln(L)}{\partial\alpha\,\partial\theta}=\frac{\partial^2\ln(L)}{\partial\theta\,\partial\alpha}$ and $\frac{\partial^2\ln(L)}{\partial\alpha\,\partial\beta}=\frac{\partial^2\ln(L)}{\partial\beta\,\partial\alpha}$.

Using Eq(25). Under the conditions that some regularity conditions are satisfied, we can get confidence interval of the tree parameters of HPPB distribution as follows

$\hat{\alpha}\pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{11}}$,

$\hat{\beta}\pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{22}}$,

And

$\hat{\theta}\pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{33}}$.

Where $z_{\frac{\gamma}{2}}$ is the upper γ^{th} of the standard normal distribution.

3.2 Bayesian Estimation for HPPB Distribution

The Bayesian estimation method is an important method for estimating the distribution`s parameters and analyzing uncertainty of the estimates. The important property of Bayesian method that the parameters are random variables and have their own distribution that called the prior distribution. Assume that α,β and θ are independent and have gamma prior distributions $(\pi_1,\pi_2\ and\ \pi_3)$.

$$\pi_1(\alpha) = \frac{{m_1}^{q_1}\ \alpha^{q_1-1}e^{-m_1\alpha}}{\Gamma(q_1)} \qquad\qquad \alpha > 0,\ m_1 and\ q_1 > 0,$$

$$\pi_2(\beta) = \frac{{m_2}^{q_2}\ \beta^{q_2-1}e^{-m_2\beta}}{\Gamma(q_2)} \qquad\qquad \beta > 0,\ m_2 and\ q_2 > 0,$$

$$\pi_3(\theta) = \frac{{m_3}^{q_3}\ \theta^{q_3-1}e^{-m_3\theta}}{\Gamma(q_3)} \qquad\qquad \theta > 0,\ m_3 and\ q_3 > 0,$$

Here, the hyper-parameters $m_1,\ m_2,\ m_3, q_1, q_2, q_3$ are chosen to reflect the prior knowledge about the known parameters. Suppose that we have n number of samples available from HPPB distribution and the maximum likelihood estimates of (α, β, θ) are $\left(\hat{\alpha}^j,\ \hat{\beta}^j,\ \hat{\theta}^j\right),\ j = 1,2,3, \dots \dots, n.$

Therefore, the joint prior distribution of $\alpha, \beta\ and\ \theta$ can be written as

$$\pi(\alpha, \beta, \theta) = \pi_1(\alpha)\, \pi_2(\beta)\, \pi_3(\theta),$$

$$\pi(\alpha, \beta, \theta) = \frac{{m_1}^{q_1}\ \alpha^{q_1-1}e^{-m_1\alpha}}{\Gamma(q_1)}\ \frac{{m_2}^{q_2}\ \beta^{q_2-1}e^{-m_2\beta}}{\Gamma(q_2)}\ \frac{{m_3}^{q_3}\ \theta^{q_3-1}e^{-m_3\theta}}{\Gamma(q_3)}. \qquad (26)$$

The likelihood function of the HPPB distribution

$$L(\alpha, \beta, \theta) = \ (\alpha)^n\ (\beta)^{\alpha n}\theta^n \left(\prod_{i=1}^{n} x_i^{\theta-1}\right)\left[\prod_{i=1}^{n}\left(1 + x_i^{\theta}\right)^{-1}\right]\left\{\prod_{i=1}^{n}\left[log\left(1 + x_i^{\theta}\right)\right]^{\alpha-1}\right\}$$

$$.\, e^{-\beta^{\alpha}\sum_{i=1}^{n}\left[log\left(1+x_i^{\theta}\right)\right]^{\alpha}}, \qquad (27)$$

The joint posterior distribution of $\alpha, \beta\ and\ \theta$ can be written as

$$\pi(\alpha, \beta, \theta | x) = \frac{\pi(\alpha, \beta, \theta)\ L(\alpha, \beta, \theta)}{\int\int\int\left(\pi(\alpha, \beta, \theta)\ L(\alpha, \beta, \theta)\right) d\alpha\ d\beta\ d\theta}, \qquad (28)$$

By substituting Eq (26) and (27) into Eq (28), it is obtained that

$$\pi(\alpha,\beta,\theta|x) = k^{-1}\left[\frac{{m_1}^{q_1}\ \alpha^{q_1-1}e^{-m_1\alpha}}{\Gamma(q_1)}\ \frac{{m_2}^{q_2}\ \beta^{q_2-1}e^{-m_2\beta}}{\Gamma(q_2)}\ \frac{{m_3}^{q_3}\ \theta^{q_3-1}e^{-m_3\theta}}{\Gamma(q_3)}\right]$$

$$.(\alpha)^n\ (\beta)^{\alpha n}\theta^n\left(\prod_{i=1}^{n} x_i^{\theta-1}\right)\left[\prod_{i=1}^{n}\left(1+x_i^{\theta}\right)^{-1}\right]\left\{\prod_{i=1}^{n}\left[log\left(1+x_i^{\theta}\right)\right]^{\alpha-1}\right\}$$

$$.\quad e^{-\beta^{\alpha}\sum_{i=1}^{n}\left[log\left(1+x_i^{\theta}\right)\right]^{\alpha}} \tag{29}$$

Where

$$k = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left(\frac{{m_1}^{q_1}\ \alpha^{q_1-1}e^{-m_1\alpha}}{\Gamma(q_1)}\ \frac{{m_2}^{q_2}\ \beta^{q_2-1}e^{-m_2\beta}}{\Gamma(q_2)}\ \frac{{m_3}^{q_3}\ \theta^{q_3-1}e^{-m_3\theta}}{\Gamma(q_3)}\right)(\alpha)^n$$

$$.(\beta)^{\alpha n}\theta^n\left(\prod_{i=1}^{n} x_i^{\theta-1}\right)\left[\prod_{i=1}^{n}\left(1+x_i^{\theta}\right)^{-1}\right]\left\{\prod_{i=1}^{n}\left[log\left(1+x_i^{\theta}\right)\right]^{\alpha-1}\right\}$$

$$.\,e^{-\beta^{\alpha}\sum_{i=1}^{n}\left[log\left(1+x_i^{\theta}\right)\right]^{\alpha}}d\alpha\ d\beta\ d\theta,$$

Therefore, the Bayes estimate of any function of $\alpha,\beta\ and\ \theta$, say $g(\alpha,\beta,\theta)$, under the squared error loss function, is given by

$$\tilde{g}(\alpha,\beta,\theta) = E_{\alpha,\beta,\theta|x}(g(\alpha,\beta,\theta)) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} g(\alpha,\beta,\theta)\pi(\alpha,\beta,\theta|x)\,d\alpha\ d\beta\ d\theta,$$

It is clear from Eq (29) that there is no closed form for the estimators.

The Bayes estimates cannot be obtained in closed form. So, the MCMC is used for obtaining the Bayes estimates for $\alpha,\beta\ and\ \theta$.Gelfan and Smith (1990) proposed the MCMC algorithm for computing the approximate Bayes estimates of the parameters.

Then equating the mean and variance of $\left(\hat{\alpha}^j,\ \hat{\beta}^j\ ,\ \hat{\theta}^j\right)$ with the mean and the variance of the suggested priors (gamma priors), it is obtained [see Dey et al. (2016)]:

$$\frac{1}{n}\sum_{j=o}^{n}\hat{\alpha}^{j}=\frac{q_1}{m_1}, \qquad \frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\alpha}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\alpha}^{j}\right)^2=\frac{q_1}{{m_1}^2}, \tag{30}$$

$$\frac{1}{n}\sum_{j=o}^{n}\hat{\beta}^{j}=\frac{q_2}{m_2}, \qquad \frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\beta}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\beta}^{j}\right)^2=\frac{q_2}{{m_2}^2}, \tag{31}$$

$$\frac{1}{n}\sum_{j=o}^{n}\hat{\theta}^{j}=\frac{q_3}{m_3}, \qquad \frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\theta}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\theta}^{j}\right)^2=\frac{q_3}{{m_3}^2}, \tag{32}$$

Solving Eq (30), (31) and (32), it is obtained the estimated hyper-parameters as following:

$$m_1=\frac{\frac{1}{n}\sum_{j=o}^{n}\hat{\alpha}^{j}}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\alpha}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\alpha}^{j}\right)^2}, \quad q_1=\frac{(\frac{1}{n}\sum_{j=o}^{n}\hat{\alpha}^{j})^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\alpha}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\alpha}^{j}\right)^2},$$

$$m_2=\frac{\frac{1}{n}\sum_{j=o}^{n}\hat{\beta}^{j}}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\beta}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\beta}^{j}\right)^2}, q_2=\frac{(\frac{1}{n}\sum_{j=o}^{n}\hat{\beta}^{j})^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\beta}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\beta}^{j}\right)^2},$$

$$m_3=\frac{\frac{1}{n}\sum_{j=o}^{n}\hat{\theta}^{j}}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\theta}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\theta}^{j}\right)^2}, \quad q_3=\frac{(\frac{1}{n}\sum_{j=o}^{n}\hat{\theta}^{j})^2}{\frac{1}{n-1}\sum_{j=0}^{n}\left(\hat{\theta}^{j}-\frac{1}{n}\sum_{j=o}^{n}\hat{\theta}^{j}\right)^2}.$$

The Metropolis-Hastings algorithm (M-H) is used to generate samples from the conditional posterior distributions, and then it is computed the Bayes estimates.

It is considered the Metropolis-Hastings algorithm with a multivariate Normal proposal distribution to generate samples from the conditional posterior distributions. The following algorithm is used to generate the required samples.

M-H algorithm

1) Set initial values for the parameters $\varphi = (\alpha, \beta, \theta)$, where in this step $\varphi = \varphi^{(0)}$.
2) Generate a proposal φ^* that follows the multivariate normal $N(\varphi, S_\varphi)$, where S_φ is the standard deviation (it is suggested $S_\varphi = (0.001, 0.003, 0.002)$).
3) Calculate the acceptance probability $\tau = min\left(1, \frac{\pi(\varphi^*|x)}{\pi(\varphi|x)}\right)$.
4) Generate $U \sim U(0,1)$.
5) If $U \leq \tau$ $\varphi^{(i)} = \varphi^*$, otherwise set $\varphi^{(i)} = \varphi^{(i-1)}$.
6) Set $i = i + 1$.
7) Repeat steps 1 to 6, N times and obtain $\varphi^{(j)}, j = 1,2, \ldots \ldots N$.

After getting MCMC samples from the posterior distribution, it can be found the Bayes estimate for the parameters in the following way

$$\tilde{\alpha} = \frac{1}{N-B} \sum_{i=B+1}^{N} \hat{\alpha}^j,$$

$$\tilde{\beta} = \frac{1}{N-B} \sum_{i=B+1}^{N} \hat{\beta}^j,$$

And

$$\tilde{\theta} = \frac{1}{N-B} \sum_{i=B+1}^{N} \hat{\theta}^j.$$

Where, B is the number of burn-in number. Then it is calculated the highest posterior density (HPD) intervals for the known parameters of the HPPB distribution.

4. Simulation Study for the HPPB Distribution

In this section, It is performed a simulation study assess the performance of maximum likelihood estimation to estimate the parameters of the HPPB distribution, depending on the quantile function in Eq (17). Also, the Bayesian estimation is performed based on MCMC algorithm to find the Bayes estimates, it is simulated samples for the parameters true values $(\alpha = 10, \beta = 41, \theta = 8)$, $(\alpha = 10, \beta = 38, \theta = 8)$, $(\alpha = 15, \beta = 40, \theta = 14)$, $(\alpha = 10, \beta = 40, \theta = 8)$, $(\alpha = 10, \beta = 60, \theta = 50)$ and $(\alpha = 10, \beta = 50, \theta = 40)$ with different sample sizes n=30,50,100,200,500,1000. Using R package, the empirical results including the estimates of the parameters, mean square error (MSE) of estimates, bias of estimates, confidence interval for the estimates and confidence interval of parameters under the maximum likelihood are given in Table 2, Table 3, Table 4 and Table 5, the estimates of the Bayes parameters, mean square error (MSE) of estimates, bias of estimates, confidence interval for the highest posterior density (HPD) of the estimates and the coverage probability (CP) of estimates under the Bayesian estimation which are given in Table 6 and Table 7. It is evident that the estimates are quite stable and close to the true values of the parameters for theses sample sizes. Additionally, as the sample size increases, the biases and the mean square error of the MLEs and Bayes decrease.

Table 2: MLEs, MSE and Bias of Estimates of MLE for Various Parameter Values

	parameters	estimates			MSE of estimate			Bias of estimate		
n	(α, β, θ)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
30	(10,60,50)	11.175	61.129	50.201	3.6716	5.4402	0.236494	1.175683	1.12975	0.20171
	(10,50,40)	11.167	50.821	40.142	3.7608	7.3727	0.346356	1.167243	0.82154	0.14206
	(10,40,8)	11.185	40.552	8.0156	4.0119	12.512	0.042242	1.185353	0.55231	0.01561
	(10,40,14)	11.126	40.786	14.052	3.4515	13.423	0.115142	1.126326	0.78618	0.05283
	(15,40,14)	16.649	40.603	14.047	7.9288	2.7300	0.027149	1.649005	0.60395	0.04726
50	(10,60,50)	10.868	61.046	50.176	2.0309	3.3476	0.161442	0.868355	1.04659	0.17608
	(10,50,40)	10.958	50.872	40.159	2.2257	2.7651	0.118108	0.958200	0.87272	0.15973
	(10,40,8)	10.919	40.741	8.0312	2.1167	8.5594	0.025070	0.919187	0.74137	0.03124
	(10,40,14)	10.923	40.738	14.056	2.1477	7.3374	0.064111	0.923887	0.73880	0.05601
	(15,40,14)	16.257	40.598	14.051	4.2587	0.7794	0.006923	1.257867	0.59862	0.05160
100	(10,60,50)	10.810	60.972	50.184	1.2691	1.3502	0.058395	0.810087	0.97218	0.18456
	(10,50,40)	10.798	50.793	40.146	1.2331	0.9404	0.034353	0.798430	0.79354	0.14637
	(10,40,8)	10.806	40.650	8.0289	1.2676	3.5902	0.010848	0.806747	0.65026	0.02892
	(10,40,14)	10.850	40.592	14.051	1.3122	1.5792	0.017146	0.850315	0.59231	0.05119
	(15,40,14)	16.274	40.568	14.051	3.0472	0.4521	0.003851	1.274894	0.56890	0.05103

Table 3: Lower and Upper Value of CI and Length of CI for Various Parameter Values for Complete Samples Using MLE

	parameters	Lower value of CI			Upper value of CI			Length of CI		
n	(α,β,θ)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
30	(10,60,50)	8.2085	57.128	49.333	14.142	65.131	51.06943	5.934	8.003	1.73643
	(10,50,40)	8.1304	45.746	39.022	14.204	55.896	41.26199	6.074	10.15	2.24
	(10,40,8)	8.0192	33.701	7.6137	14.351	47.403	8.41748	6.332	13.702	0.80378
	(10,40,14)	8.2290	33.769	13.395	14.023	47.803	14.71012	5.794	14.034	1.31512
	(15,40,14)	12.173	37.588	13.737	21.124	43.619	14.35679	8.951	6.031	0.6198
50	(10,60,50)	8.6524	58.103	49.467	13.084	63.989	50.88430	4.4313	5.886	1.4173
	(10,50,40)	8.7158	48.097	39.563	13.200	53.648	40.75643	4.4842	5.551	1.19343
	(10,40,8)	8.7077	35.191	7.7268	13.130	46.291	8.335632	4.4223	11.1	0.608832
	(10,40,14)	8.6930	35.628	13.571	13.154	45.849	14.54023	4.461	10.221	0.96923
	(15,40,14)	13.049	39.326	13.923	19.465	41.871	14.17960	6.416	2.545	0.2566
100	(10,60,50)	9.2748	59.724	49.878	12.345	62.220	50.49044	3.0702	2.496	0.61244
	(10,50,40)	9.2850	49.700	39.923	12.311	51.886	40.36934	3.026	2.186	0.44634
	(10,40,8)	9.2666	37.160	7.8327	12.346	44.140	8.225142	3.0794	6.98	0.392442
	(10,40,14)	9.3450	38.418	13.814	12.355	42.765	14.28753	3.01	4.347	0.47353
	(15,40,14)	13.936	39.866	13.981	18.613	41.271	14.12027	4.677	1.405	0.13927

Table 4: MLEs, MSE and Bias of Estimates for Various Parameter Values

	parameters	estimates			MSE of estimate			Bias of estimate		
n	(α,β,θ)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
200	(10,60,50)	١٠.٧٤٧	٦٠.٨٨	٥٠.١٦	٠.٨٤١٥	١.٢٤٦	٠.٠٥٤	٠.٧٤٧	٠.٨٨	٠.١٦٤
	(10,50,40)	١٠.٧٠٥	٥٠.٧٢٤	٤٠.١٣٦	٠.٧٧٧	٠.٩٧٢	٠.٠٤٢٣	٠.٧٠٥٩	٠.٧٢٤٣	٠.١٣٦
	(10,40,8)	١٠.٧٤	٤٠.٥٤	٨.٠٢٥	٠.٨٥٢٤	١.٦٦١٧	٠.٠٠٦٣	٠.٧٤٠٩	٠.٥٤٢٧	٠.٠٢٥٦
	(10,40,14)	١٠.٧٣	٤٠.٥١	١٤.٠٤٢	٠.٨٢٤٦	٢.١٥٤١	٠.٠٢٤٣٥	٠.٧٣٤٦٩	٠.٥١١٢	٠.٠٤٢٣
	(15,40,14)	١٦.١٣	٤٠.٤٩٥	١٤.٠٤٣	١.٩٢٩٣	٠.٤٤٤١	٠.٠٠٣٩٣	١.١٣٠٧	٠.٤٩٥٩	٠.٠٤٣٧
500	(10,60,50)	١٠.٦٩٤	٦٠.٨٨٣	٥٠.١٦٤	٠.٥٩٣٧	٠.٨٨٦	٠.٠٣٣٢	٠.٦٩٤	٠.٨٨٣٢	٠.١٦٤٥
	(10,50,40)	١٠.٦٩٣	٥٠.٧١٩	٤٠.١٣٥	٠.٥٩٧٩	٠.٦٥٣١	٠.٠٢٥٣	٠.٦٩٣	٠.٧١٩	٠.١٣٥
	(10,40,8)	١٠.٦٨٩	٤٠.٦٥٠	٨.٠٣٣	٠.٥٩٠٧	٠.٥٥٢٨	٠.٠٠١٤٦٥	٠.٦٨٩٩	٠.٦٥٠	٠.٠٣٣
	(10,40,14)	١٠.٧٠١	٤٠.٦٢٨	١٤.٠٥٥	٠.٦١٢٥	٠.٧٣٨٣	٠.٠٠٧٠٥	٠.٧٠١٤	٠.٦٢٨٦	٠.٠٥٥٦
	(15,40,14)	١٦.٠٢٦	٤٠.٤٥٥	١٤.٠٤	١.٣٢٧	٠.٢٦٨٣	٠.٠٠٢٠٦	١.٠٢٦٥	٠.٤٥٥٩	٠.٠٤
1000	(10,60,50)	١٠.٦٧٩	٦٠.٨٨٧	٥٠.١٦٧	٠.٥١٩٣	٠.٨٧٧٧	٠.٠٣٢٤	٠.٦٧٩٣	٠.٨٨٧٧	٠.١٦٧١
	(10,50,40)	١٠.٦٧	٥٠.٧٠٨	٤٠.١٣٣	٠.٥٠٤٢	٠.٥٤٩٢	٠.٠٢٠٦	٠.٦٧٠٨	٠.٧٠٨٢	٠.١٣٣٥
	(10,40,8)	١٠.٦٦٣	٤٠.٧٠٢	٨.٠٣٥	٠.٤٩٥٠	٠.٦٢٨١	٠.٠٠١٦٧	٠.٦٦٣٦٩	٠.٧٠٢٤	٠.٠٣٥٧
	(10,40,14)	١٠.٦٧٦	٤٠.٧٠٤	١٤.٠٦٣	٠.٥١٠٨	٠.٥٦٩٤	٠.٠٠٤٦١	٠.٦٧٦٠	٠.٧٠٤٤	٠.٠٦٣٢
	(15,40,14)	١٦.٠٣٤	٤٠.٤٤	١٤.٠٣٩	١.٢٠٤	٠.٢٢٥٩	٠.٠٠١٨	١.٠٣٤٧	٠.٤٤٠٧	٠.٠٣٩١

Table 5: Lower and Upper Value of CI and Length of CI for Various Parameter Values for Complete Samples Using MLE

	parameters	Lower value of CI			Upper value of CI			Length of CI		
n	(α, β, θ)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
200	(10,60,50)	9.6846	60.046	49.961	11.785	61.737	50.37284	2.1004	1.691	0.41184
	(10,50,40)	9.6840	49.183	39.737	11.788	52.242	40.52868	2.104	3.059	0.79168
	(10,40,8)	9.6741	39.192	7.9414	11.795	42.021	8.118828	2.1209	2.829	0.1774
	(10,40,14)	9.669	39.044	13.88	11.8	42.17	14.221	2.131	3.126	0.341
	(15,40,14)	12.020	38.1	13.97	14.520	39.977	13.99242	2.5	1.877	0.02242
500	(10,60,50)	10.067	60.219	50.006	11.358	61.494	50.31780	0.688	1.275	0.3118
	(10,50,40)	10.032	49.925	39.948	11.366	51.471	40.31653	1.334	1.546	0.36853
	(10,40,8)	10.023	39.344	7.9527	11.371	41.882	8.109393	1.348	2.538	0.1567
	(10,40,14)	10.004	٣٩.٩١٢	13.986	11.360	41.445	14.13377	1.356	1.533	0.1477
	(15,40,14)	14.991	40.014	14.002	17.074	40.895	14.07798	2.083	0.881	0.07598
1000	(10,60,50)	10.235	60.327	50.037	11.147	61.432	50.29388	0.912	1.105	0.25688
	(10,50,40)	10.215	50.288	40.037	11.168	51.117	40.23123	0.953	0.829	0.19423
	(10,40,8)	10.225	39.886	7.9902	11.130	41.501	8.080452	0.905	1.615	0.0906
	(10,40,14)	10.217	40.175	14.014	11.138	41.217	14.11045	0.921	1.042	0.9645
	(15,40,14)	15.330	40.092	14.008	16.777	40.779	14.07034	1.447	0.687	0.06234

Table 6: Estimates, MSE and Bias of estimates of Bayesian Estimation for Various Parameter Values

	parameters	estimates			MSE of estimate			Bias of estimate		
n	(α,β,θ)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
30	(10,60,50)	9.9998	60.002	49.999	0.0001	0.0009	0.000377	-0.0001	.0016	-0.0001
	(10,50,40)	9.9998	50.0007	40.008	0.0001	0.0009	0.000383	-0.0002	.0007	.0008
	(10,40,8)	9.9997	39.999	7.9997	0.0009	0.0008	0.000431	-.0002	-0.0002	-0.0002
	(10,40,14)	9.9992	39.995	14.006	0.0002	0.0017	0.000684	-0.0007	-0.004	.0006
	(15,40,14)	15.0002	40.0011	14.0006	0.0001	0.0008	0.000397	.0002	.0011	.0006
50	(10,60,50)	9.9996	60.0008	49.999	0.0001	9.1760	0.000383	-0.0003	.0008	-0.0002
	(10,50,40)	9.9996	50.0011	40.0002	0.0001	0.0009	0.000402	-0.0003	.0001	.0001
	(10,40,8)	10.0003	39.998	80.001	0.0001	0.0009	0.000413	.00038	-0.0016	.0001
	(10,40,14)	9.9993	39.9939	14.010	0.0001	0.0018	0.000803	-0.0006	-0.006	.0102
	(15,40,14)	15.0002	40.0004	14.0004	0.0001	0.0008	0.000401	.0002	.0004	.0004
100	(10,60,50)	10.0004	59.9996	49.999	0.0001	0.0008	0.000366	.0004	-0.0003	-0.0002
	(10,50,40)	9.9999	50.0009	39.999	.00009	0.0008	0.000386	-.00008	.0009	-0.0007
	(10,40,8)	9.9998	40.001	7.9994	0.0001	0.0008	0.000370	-0.0001	0.001	-0.0005
	(10,40,14)	9.9999	39.9984	14.00003	.00009	0.0009	.0000004	-.0000001	-0.0015	.00003
	(15,40,14)	14.999	39.9998	14.000	0.0001	0.0008	.0003756	-0.0004	.0002	-0.0004
200	(10,60,50)	10.0001	59.9986	50.000	.00009	0.0009	.0003976	.00014	-0.0013	.0003
	(10,50,40)	14.999	40.0002	13.999	0.0001	.00094	0.000389	.00017	-.00003	.00037
	(10,40,8)	10.0007	39.9997	7.9999	.00009	0.0008	0.0004	.0007	-0.0002	-.00006
	(10,40,14)	9.9999	40.0019	13.999	.00009	.00095	0.000382	-.000087	.00197	-0.0009
	(15,40,14)	15.0001	40.0007	14.0004	.00009	.00088	.0004296	.0001	.0007	.0004
500	(10,60,50)	10.0001	60.0017	50.0002	.00010	.00091	.0003870	.0001	.0017	.0002
	(10,50,40)	9.9995	49.9989	39.999	0.0001	0.0009	0.000414	-0.0004	-0.001	-0.0006
	(10,40,8)	10.0000	39.9995	8.0001	.00010	0.0008	.0004084	.0001	-0.0004	.0001
	(10,40,14)	9.9996	40.0020	13.999	.00009	.00094	0.000397	-.0004	.00208	-0.0003
	(15,40,14)	15.0001	39.9995	14.000	.00010	0.0008	.0004084	.00015	-0.0004	.0002
1000	(10,60,50)	10.0001	60.0002	49.999	.00009	.00091	0.000388	-.000128	-0.0001	.000764
	(10,50,40)	10.000	50.0016	39.999	.00009	.00088	0.000391	.00015	.0016	-0.0002
	(10,40,8)	10.0001	40.0001	8.0010	.00010	.00084	.0000398	.00019	.000149	.00106
	(10,40,14)	9.9998	40.0004	14.00008	.00009	.00098	.0004050	.00019	.00014	.00106
	(15,40,14)	15.0002	40.0001	14.001	.00010	.00084	.0003980	-0.0003	-0.0002	.000066

Table 7: HPD interval and CP of Bayesian Estimation for Various Parameter Values

	parameters	Lower of HPD			Upper of HPD			CP		
n	(α,β,θ)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
30	(10,60,50)	9.9812	59.942	49.962	10.028	60.058	50.04	97.9	97. 1	98.3
	(10,50,40)	9.98	49.940	39.962	10.021	50.057	40.04	97.8	9.74	97.6 1
	(10,40,8)	9.9775	39.939	7.9606	10.017	40.054	8.04	96	97.5	97.8
	(10,40,14)	9.9721	39.911	13.958	10.028	40.072	14.06	98.1	96.5	98.2
	(15,40,14)	14.98	39.942	13.962	15.018	40.054	14.04	96.7	96.	97.3
50	(10,60,50)	9.9814	59.939	49.961	10.02	60.058	50.03	98.4	97.3	97.1
	(10,50,40)	9.9799	49.938	39.963	10.019	5.0059	40.04	97.5	97.6	98.3
	(10,40,8)	9.9802	39.937	7.9632	10.019	40.056	8.04	97.1	97	97.8
	(10,40,14)	9.973	39.91	13.960	10.027	40.075	14.06	98.2	97.4	97.8
	(15,40,14)	14.978	39.937	13.961	15.021	40.053	14.04	97.6	96.6	97
100	(10,60,50)	9.9804	59.944	49.963	10.02	60.059	50.04	97	98	97.8
	(10,50,40)	9.9801	4.9950	39.958	10.017	50.065	40.03	96.6	98.1	96.3
	(10,40,8)	9.9821	39.942	7.9626	10.023	40.055	8.04	98.6	97.7	98.1
	(10,40,14)	9.9812	39.936	13.959	10.019	40.053	14.04	97.2	97.1	97.3
	(15,40,14)	14.978	3.9942	13.962	15.019	40.06	14.04	96.2	97.1	98
200	(10,60,50)	9.9801	59.945	49.961	10.019	60.064	50.04	96.8	98.4	96.9
	(10,50,40)	9.9803	49.940	39.958	10.020	50.052	40.04	97.4	97.7	97.4
	(10,40,8)	9.9823	39.942	7.9617	10.020	40.054	8.04	98.1	97.9	97.5
	(10,40,14)	9.9806	39.947	13.961	10.02	40.065	14.04	97.6	98.4	97.4
	(15,40,14)	14.982	39.946	13.962	15.018	40.058	14.04	98.3	98	97.9
500	(10,60,50)	9.9796	59.941	49.960	10.021	60.059	50.04	97.7	96.8	97
	(10,50,40)	9.9769	49.938	39.959	10.018	50.057	40.04	96.1	97.6	97.1
	(10,40,8)	9.9815	39.941	7.9636	10.020	40.056	8.04	97.9	97.4	98.41
	(10,40,14)	9.9816	39.947	13.96	10.019	40.066	14.04	97.9	98.2	97
	(15,40,14)	14.498	39.941	13.964	15.020	40.056	14.04	97.9	97.4	98.4
1000	(10,60,50)	9.9793	59.937	49.965	10.018	60.054	50.04	98	97.8	96.6
	(10,50,40)	9.9824	49.943	39.958	10.021	50.057	40.03	98.5	97.4	96.6
	(10,40,8)	9.9827	59.944	49.958	10.020	60.063	50.04	97	97.2	96.8
	(10,40,14)	9.9825	39.943	13.958	10.018	40.063	14.04	97.9	98.2	97.3
	(15,40,14)	14.978	39.941	13.959	15.019	40.054	14.04	97	97.2	96.8

From Table 2 and Table 3, it is concluded that the MSE and bias of the estimates under MLE estimation are large values for the small sample sizes. From Table 4 and Table 5, it is concluded that the MSE and bias of the estimates under MLE estimation is somewhat small values for the small sample sizes. So, it is will be used large sample sizes to generate simulated data from the hazard power parameter Burr distribution.

From Table 6 and Table 7, it is concluded that the MSE and bias of the estimates under Bayesian estimation are small values for the small and large sample sizes. So, it is will be good use for the small and large sample sizes to generate simulated data from the hazard power parameter Burr distribution.

5. Application on Real Data

The application of real data illustrates the performance of HPPB distribution in comparison with other life time distribution. Using three sets which were data of prices, life time data and taxes revenues data is performed.

5.1 Data Set 1: Prices of children wooden toys

An application of the HPPB distribution on real data with the estimation of the parameters using the method of maximum likelihood estimation for a comparison of the HPPB distribution with some popular models is performed. These real data are given in Yusuf et al. (2015) that illustrate the prices of the 31 different children wooden toys on sale in a Suffolk craft shop in April 1991: 4.2, 1.12, 1.39, 2, 3.99, 2.15, 1.74, 5.81, 1.7, 2.85, 0.5, 0.99, 11.5, 5.12, 0.9, 1.99, 6.24, 2.6, 3, 12.2, 7.36, 4.75, 11.59, 8.69, 9.8, 1.85, 1.99, 1.35, 10, 0.65, 1.45.

The example has shown the flexibility of the HPPB distribution in comparison with other models including the Burr XII distribution with two parameters $urr(\theta,\beta)$, the three parameters generalized Burr XII distribution $GB(\alpha,\beta,\theta)$, Weibull distribution and Singh-Maddala distributions.

The PDF of the Burr distribution is given as follows

$$f(x;\theta,\beta) = \beta\theta\left(1+x^{\theta}\right)^{-(\beta+1)}x^{\theta-1} \qquad 0 \le x \le \infty \quad \theta\ ,\beta > 0,$$

And the corresponding CDF of the Burr distribution as following

$$F(x;\theta,\beta) = 1 - \left(\frac{1}{1+x^{\theta}}\right)^{\beta},$$

The PDF of GB distribution is

$$f(x;\alpha,\beta,\theta) = \alpha\beta\theta^{-\alpha}x^{\alpha-1}\left(1+\frac{x^{\alpha}}{\theta^{\alpha}}\right)^{-(\beta+1)} \qquad 0 \le x \le \infty \quad \theta,\alpha\ ,\beta > 0,$$

The CDF of GB distribution is

$$F(x;\alpha,\beta,\theta) = 1 - \left(\frac{1}{\left(1+\frac{x^{\alpha}}{\theta^{\alpha}}\right)}\right)^{\beta},$$

The PDF of HPPB distribution is

$$f_{HPPB}(x;\alpha,\beta,\theta) = \theta\alpha\beta\left(1+x^{\theta}\right)^{-1}x^{\theta-1}\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha-1} e^{-\left[\beta\, log\left(1+x^{\theta}\right)\right]^{\alpha}} \quad x \ge 0 \quad \beta,\alpha,\theta > 0,$$

The corresponding CDF of HPPB distribution is

$$F_{HPPB}(x;\alpha,\beta,\theta) = 1 - e^{-\beta\log\left(1+x^{\theta}\right)^{\alpha}}.$$

The PDF of Weibull distribution is

$$f(x;\theta,\beta) = \frac{\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1} e^{\left(\frac{x}{\beta}\right)^{\theta}} \qquad x > 0.$$

The corresponding CDF of Weibull distribution is

$$F(x;\theta,\beta) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\theta}} \qquad x > 0,$$

The PDF of S-M distribution is

$$f(x;\alpha,\theta,\beta)=\frac{\alpha\theta\left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta\left[1+\left(\frac{x}{\beta}\right)^{\alpha}\right]^{(\theta+1)}}\qquad x>0$$

The corresponding CDF of S-M distribution is

$$F(x;\alpha,\theta,\beta)=1-\left[1+\left(\frac{x}{\beta}\right)^{\alpha}\right]^{-\theta}\qquad x>0,$$

The MLEs of the HPPB, Burr XII, generalized Burr, Weibull and Singh-Maddala distributions were computed by maximizing the objective function using R software. The comparison of the three models is performed based on the estimated values of the parameters, log-likelihood statistic, Akaike information criterion (AIC) $AIC = 2p - 2log(L)$, Bayesian information criterion (BIC) $BIC = plog(n) - 2log(L)$ and consistent Akaike information criterion (CAIC) $CAIC = -2\,log\,(L) + p\,[(log\,n) + 1]$.

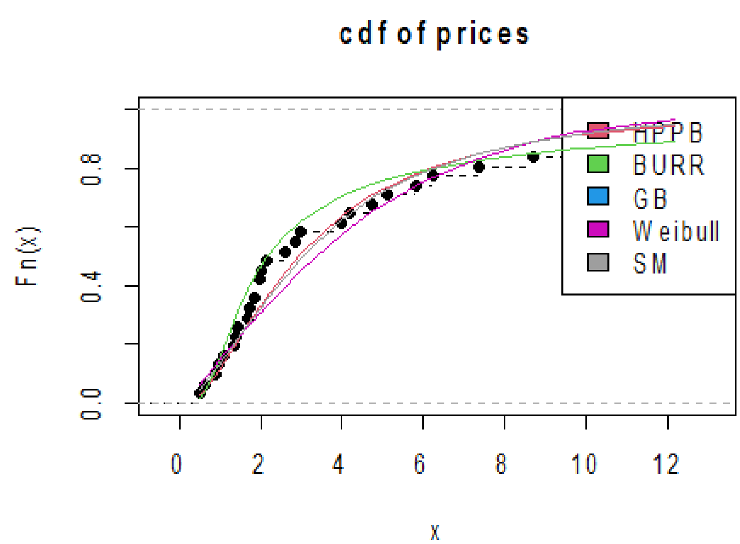

Figure (6): Estimated Distribution Functions Plotted over Empirical Distribution for Prices Data Set

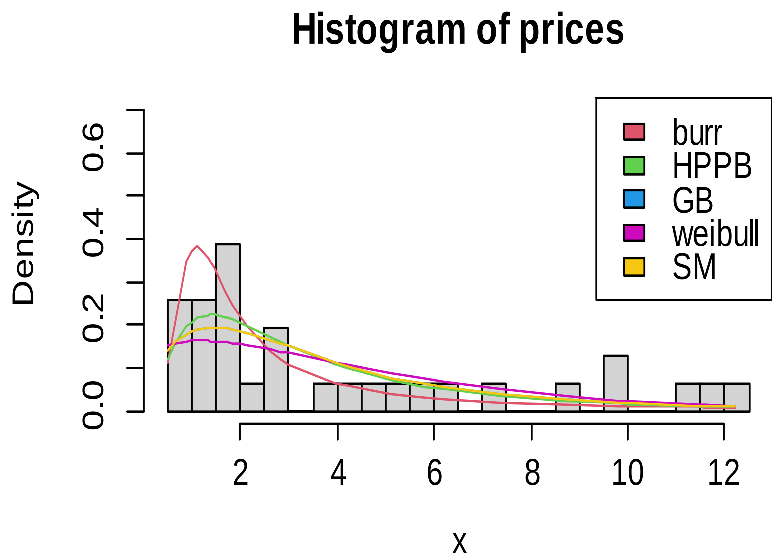

Figure 7: Histogram and Fitted Distributions for Prices Data Set

Table 8: The Statistic of the Kolmogorov-Smirnov Test and the P-Value of the Test of the Distributions with the Prices of the 31 Different Children Wooden Toys.

Distribution	K-S Statistic	P-value
$Burr(\beta, \theta)$	0.1312	0.66
$GB(\alpha, \beta, \theta)$	0.13499	0.6245
$HPPB(\alpha, \beta, \theta)$	0.12382	0.7288
$W(\beta, \theta)$	0.15584	0.4388
$SM(\alpha, \beta, \theta)$	0.13499	0.6244

Table 9: MLEs, AIC, BIC, and CAIC of the Distribution with the Prices of the 31 Different Children Wooden Toys.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$-log\ (L)$	AIC	BIC	$CAIC$
$Burr(\beta,\theta)$	----------	0.227696	3.8516	76.403	156.802	155.79	156.79
$GB(\alpha,\beta,\theta)$	1.579808	1.797359	4.9291	74.739	155.477	153.95	154.95
$HPPB(\alpha,\beta,\theta)$	1.9823403	0.491049	1.3817	73.832	153.663	152.14	153.14
$W(\beta,\theta)$	----------	4.5570	1.228	74.788	153.8	152.56	154.56
$SM(\alpha,\beta,\theta)$	1.579796	4.92928	1.7974	74.739	155.477	153.95	156.95

From Table 9, the values of AIC, CAIC and BIC are close values for all the models. It is clear that the HPPB distribution has the smallest value of Akaike information Criterion (AIC) and consistent Akaike information Criterion (CAIC) and the smallest value of Bayesian Information Criterion (BIC). This shows that the HPPB distribution is suitable model for fitting this data between the other models.

5.2 Data Set 2: Failure Times

An application of the HPPB distribution on real data with the estimation of the parameters using the method of maximum likelihood estimation for a comparison of the HPPB distribution with some popular models is performed. These real data are given in Murthy et al. (2004) that illustrate the failure times of 50 components:

0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 1.600, 2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 10.940, 11.020, 13.880, 14.730, 15.080.

The example has shown the flexibility of the HPPB distribution in comparison with other models including the Burr XII distribution with two parameters $Burr(\theta,\beta)$ and the three parameters generalized Burr XII distribution $GB(\alpha, \beta, \theta)$,the Weibull distribution $W(\beta, \theta)$ and Singh Maddala distribution $SM(\alpha, \beta, \theta)$.

The MLEs of the parameters of the five models are computed by maximizing the objective function using R software. The comparison of the five models is performed based on the estimated values of the parameters, log-likelihood statistic, Akaike information criterion (AIC), Bayesian information criterion (BIC) and consistent Akaike information criterion (CAIC).

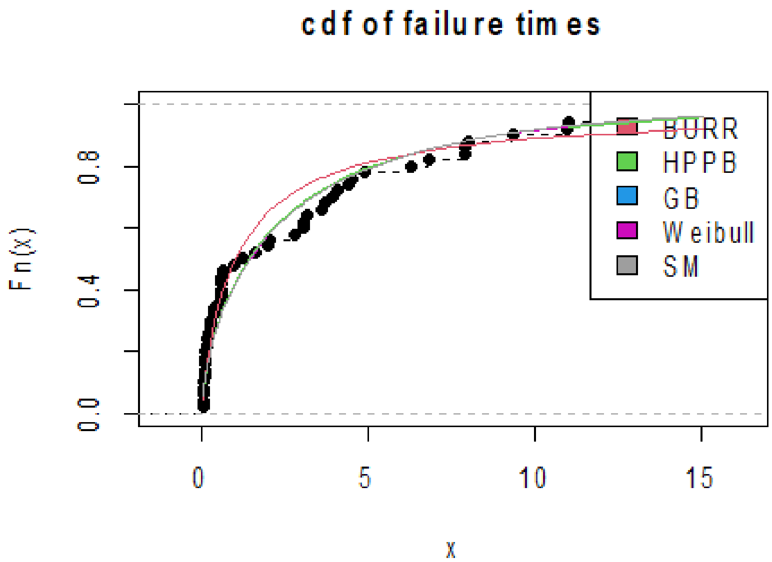

Figure 8: Estimated Distribution Functions Plotted over Empirical Distribution for Failure Times Data Set

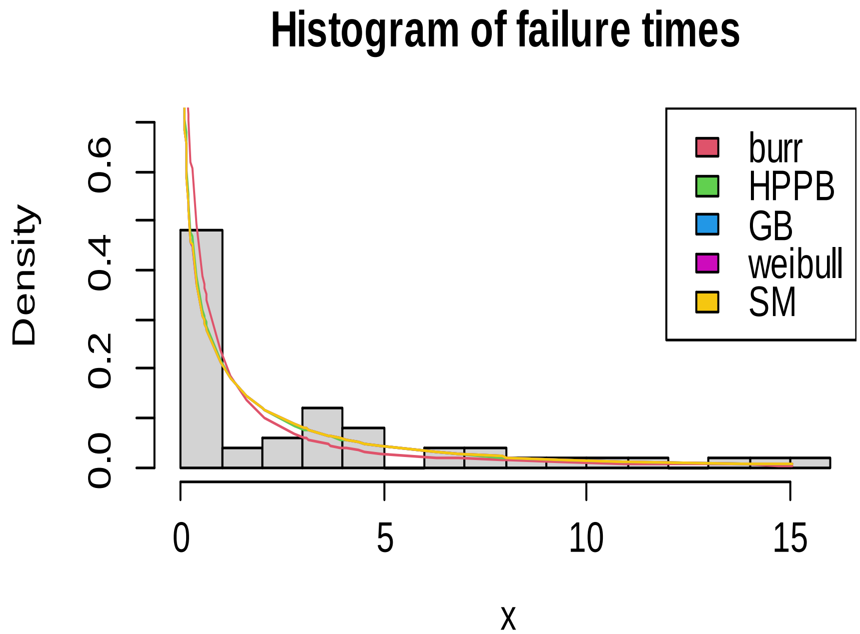

Figure 9: Histogram and Fitted Distributions for Failure Times Data

Table 10: The Statistic of the Kolmogorov-Smirnov Test and the P-Value of the Test of the Distributions with the Failure Times of 50 Components.

Distribution	Statistic of K-S	P-value
$Burr(\beta,\theta)$	0.15658	0.1547
$GB(\alpha,\beta,\theta)$	0.12678	0.3667
$HPPB(\alpha,\beta,\theta)$	0.12221	0.4111
$W(\beta,\theta)$	0.127	0.3646
$SM(\alpha,\beta,\theta)$	0.12671	0.3673

Table 11: MLEs, AIC, BIC, and CAIC of the Distribution with the Failure Times of 50 Components.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$-\ln(L)$	AIC	BIC	$CAIC$
$Burr(\beta,\theta)$	----------	0.994438	0.909175	105.583	215.167	214.57	215.57
$GB(\alpha,\beta,\theta)$	0.66131	0.00545	0.000001	102.37	210.74	209.84	210.84
$HPPB(\alpha,\beta,\theta)$	8.276117	1.34308	0.111765	102.29 0	210.58	209.68	210.68
$W(\beta,\theta)$	----------	2.530775	0.66126	102.364	208.729	208.13	209.13
$SM(\alpha,\beta,\theta)$	0.661	919131.2	4776.18	102.365	210.73	209.83	210.83

From Table 10, it is clear that the HPPB model is suitable for the failure times because the P-value of K-S test is greater than 0.01.

From Table 11, the values of AIC, CAIC and BIC for the five models are close values. So, the HPPB distribution is suitable for fitting the failure times data.

5.3 Data Set 3: Taxes Revenues

An application of the HPPB distribution on real data with the estimation of the parameters using the method of maximum likelihood estimation for a comparison of the HPPB distribution with some popular models is performed. These real data was analyzed by Bhatti et al. (2018) that illustrates the taxes revenues data:

3.7, 3.11, 4.42, 3.28, 3.75, 2.96,3.39,3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17,3.51,1.84,1.61,1.57,1.89, 2.74, 3.27, 2.41, 3.09, 2.43,2.53,2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19,1.87, 2.95, 2.67,4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50,1.47, 3.22, 3.15, 2.97, 2.93,3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61,2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22,2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85,1.80, 2.12, 3.65.

The example has shown the flexibility of the HPPB distribution in comparison with other models including: the three parameters generalized Burr XII distribution $GB(\alpha, \beta, \theta)$, the Weibull distribution $W(\beta, \theta)$ and Singh Maddala distribution $SM(\alpha, \beta, \theta)$.

The MLEs of the parameters of the four models are computed by maximizing the objective function using R software. The comparison of the four models is performed based on the estimated values of the parameters, log-likelihood statistic, Akaike information criterion (AIC), Bayesian information criterion (BIC) and consistent Akaike information criterion (CAIC).

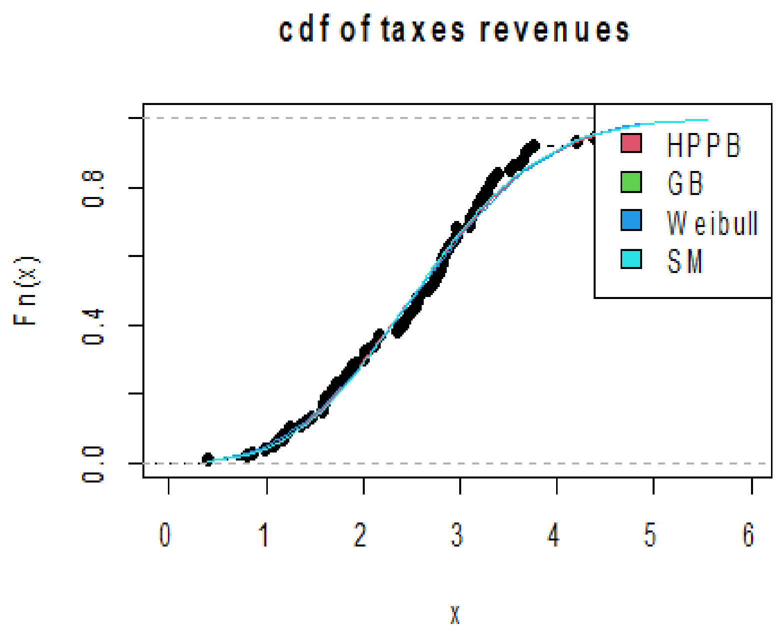

Figure 10: Estimated Distribution Functions Plotted over Empirical Distribution for Taxes Revenues Data Set

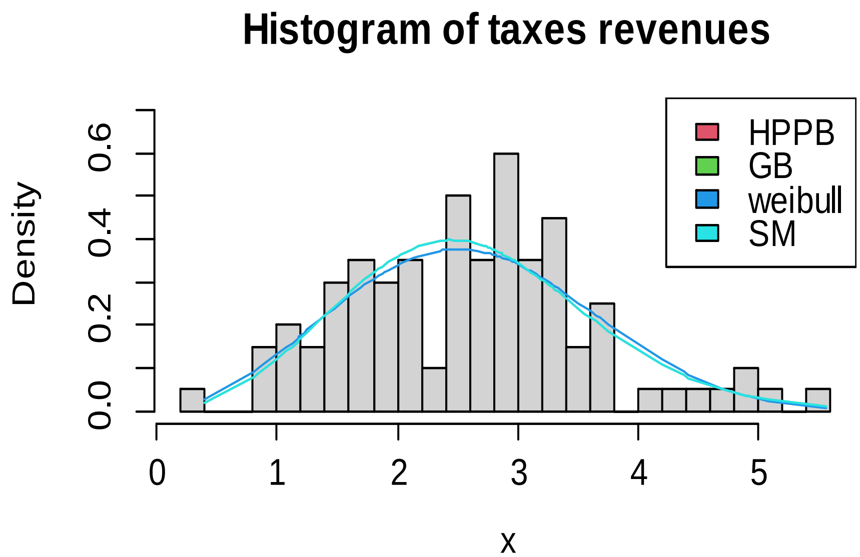

Figure 11: Histogram and Fitted Distributions for Taxes Revenues Data

Table 12: The Statistic of the Kolmogorov-Smirnov Test and the P-Value of the Test of the Distributions with the Taxes Revenues Data.

Distribution	Statistic of K-S	P-value
$GB(\alpha, \beta, \theta)$	0.062405	0.8309
$HPPB(\alpha, \beta, \theta)$	0.063909	0.8087
$W(\beta, \theta)$	0.063195	0.8194
$SM(\alpha, \beta, \theta)$	0.062405	0.8309

Table 13: MLEs, AIC, BIC, and CAIC of the Distribution with the Taxes Revenues Data

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$-\ln(L)$	AIC	BIC	CAIC
$HPPB(\alpha,\beta,\theta)$	8.42414	0.99021	0.52226	140.6	287.2	287.2	288.2
$GB(\alpha,\beta,\theta)$	3.06893	6.67235	5.24189	140.6	287.2	287.2	288.2
$W(\beta,\theta)$	----------	2.93215	2.79306	141	286	285.2	287
$SM(\alpha,\beta,\theta)$	3.06893	5.24188	6.67232	140.6	287.2	287.2	288.2

From Table 12, it is clear that the four models are suitable for the taxes revenues data because the P-value of K-S test is greater than 0.01.

From Table 13, the values of AIC, CAIC and BIC for the four models are close values. So, the HPPB distribution is suitable for fitting the taxes revenues data.

6. Conclusion

A new generalized Burr distribution was obtained by applying the hazard power parameter (HPP) family to the Burr Type XII distribution. The new distribution that called HPPB has two shape parameters $and\ \theta$, and a scale parameter namely β. It is found that the probability density function is unimodal and hazard rate function can be increasing, decreasing and upside-down failure rate. Also, it is obtained the mathematical properties.

In the other hand. to illustrate the flexibility of the hazard power parameter Burr distribution three sets of data is used using R package. The first set of data illustrates the prices of the 31 different children wooden toys on sale in a Suffolk craft shop in April 1991 that analyzed by Yusuf et al. (2015), the second set of data illustrates the failure times of 50 components that analyzed by Murthy et al. (2004), the third set of data illustrates the taxes revenues data that analyzed by Bhatti et al. (2018), using the maximum likelihood method for estimating the parameters and a comparison between HPPB, Burr XII and generalized Burr, Weibull, Singh-Maddala distributions. From the result, the HPPB distribution is suitable for fitting the data of prices, failure times and taxes revenues and it suitable for the data depending on AIC, BIC and CAIC.

A simulation study for the HPPB distribution is applied using the maximum likelihood estimation and the Bayesian estimation based on different values of the parameters and obtaining the MLEs, BEs, MSE of the three parameters, a confidence interval for the parameters. It is found that the estimates are quite stable and close to the true values of the parameters based on complete samples. In addition an application on real data with a comparison with three models including the Burr XII distribution with two parameters, the generalized Burr distribution with three parameters and the HPPB distribution with three parameters to know the best model for fitting the data and the results explained that the HPPB distribution is the best model for fitting the data.

7. References

1. Abozaid, A, H., Jamal, F., Khan, S. Nasir, M, A. and Tahir, M, H. (2021). New Modified Burr III Distribution, Properties and Applications. Mathematical And Computational Applications, 26(4), 82.
2. Afify, A.Z., Cordeiro, G.M., Ortega, E.M.M. and Yousof, H.M. (2018). The Four-Parameter Burr XII Distribution: Properties, Regression Model and Applications. Communications in Statistics-Theory and Methods, 47(11), 2605-2624.

3. Jamal, F., Reyad, H, M., Ahmed, O, S. and Muhammad, S. (2020). Mathematical Properties and Applications of Minimum Gumbel Burr Distribution. NED University Journal of Research, 17(2), 1-14.

4. Baharith, L. (2022). Alpha Power Kumaraswamy-Burr III Distribution with Application Of COVID-19 Data In Saudi Arabia. Results In Physics, 35(105353).
5. Bhatti, F., Hamedani, G. and Ahmed, M. (2018). On Modified Log Burr XII Distribution. Journal of the Iranian Statistical Society, 17(2), 57-89.
6. Burr, W, I. (1942). Cumulative Frequency Functions. The Annals of Mathematical Statistics, 13(2), 215-232.

7. Chiodo, E. and Pasquale, f. (2016). Inverse Burr Distribution For Extreme Wind Speed Prediction: Genesis, Identification And Estimation. Electric Power Systems Research, 141(4),549-561.
8. Dey, S., Day, T. and Luckett, D, J. (2016). Statistical Inference for The Generalized Inverted Exponential Distribution Based on Upper Record Values. Mathematics and Computers in simulation, 120(1), 64-78.
9. Gelfand, A, E. and Smith, A, F, M. (1990). Sampling-based Approaches to Calculating Marginal Densities. Journal of the American Statistical Association, 85, 398-409.

10. Muhammed, Z, H. (2020). A Class Of Bivariate Semi-parametric Families Of Distributions. Journal Of Data Science, 18(4), 761-781.
11. Muhammed, Z, H. (2022)[a]. On Some Bivariate Semi-parametric Families Of Distributions With a Singular component. Sankhya,1-47.
12. Muhammed, Z, H. (2022)[b]. A Generalized Gompertz Distribution with Hazard Power Parameter And Its Bivariate Extension: Properties And Application. Annals of Data Science, 1-26.
13. Murthy, D., Xie, M. and Jiang, R. (2004). *Weibull Models*. First Edition. John Wiley, New York.
14. Marshall and Olkin, I (2007). *Life Distributions: Structure of Non-Parametric, Semi-Parametric, and Parametric Families*. First Edition.Springer, New York.
15. Shaked, M. and Shanthikumar, J, D. (1994). *Stochastic Orders and Their Application*. Academic Press, New York. Doi: https://doi.org/10.1137/1037117
16. Yusuf, A., Mikail, B, B., Aliya, I, A. and Sulaiman, A, L. (2015). The Inverse Burr Negative Binomial Distribution with Application to Real Data. Journal of Statistics Application & probability, 5(1), 53-65.

Bayesian and Non-Bayesian Estimation of the Composed- Fréchet Exponential Distribution based on Censored Samples

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Abstract:

Statistical distributions are very useful in describing and predicting real data analysis. Although many distributions have been developed, there are always techniques for developing distributions which are either more flexible or for fitting specific real data analysis. In this paper we define and study a new model called the composed-Fréchet exponential (C-FE) distribution. We consider the estimation of the parameters of C-FE distribution based on Type II censored samples. The parameters are estimated by the maximum likelihood and Bayesian methods. The Fisher information matrix has been obtained and it can be used for constructing asymptotic confidence intervals. We prove empirically the importance and flexibility of the new model comparing it with other models.

1. Introduction

The Fréchet (Fr) distribution (Type II extreme value distribution) is one of the important distributions in extreme value theory and it has wide applicability in extreme value theory. It was developed by Maurice Fréchet in 1927. It is a continuous probability distribution and can also be called the inverse Weibull distribution. Fréchet distribution has several applications; some of which can be found in hydrology and finance for modeling extreme events. Some further details about the Fréchet distribution including its applications are available in Harlow (2002), Morales (2005), Nadarajah and Kotz (2008), Mubarak (2011), and Pehlivan and Vuong (2011). The probability density function (pdf) and the cumulative distribution function (cdf) of the Fréchet distribution (Fr) are given respectively by

$$g(x;\ a,b) = abx^{-a-1}e^{-bx^{-a}} \ ,x \geq 0, \tag{1}$$

$$\mathrm{G}(\mathrm{x}\ \mathrm{a},\mathrm{b}) = e^{-bx^{-a}}. \tag{2}$$

where a > 0 and b > 0 are shape and scale parameters, respectively.

Some extensions of the Fréchet distribution have previously been proposed in the literature, the Beta Fréchet Distribution (Barreto-Souza et al. (2011), Weibull Fréchet distribution (Afify et al. (2016)) and the Logistic Fréchet distribution (Tahir et al. (2016)) are remarkable examples. The generalization of the classical distributions is made by adding one or more shape parameter(s) to the existing probability distribution to improve the flexibility and goodness of fits of the distribution against the intuition of model parsimony. Several ways of generating new probability distributions from classic ones were developed and discussed. Some well-known generalized classes and generators are the beta generalized family of distributions introduced by Eugene et al. (2002), the general rank transmutation (GRT) defined by Shaw and Buckley (2007), Kumaraswamy-G family of distributions introduced by Cordeiro and de Castro (2011), transformed-transformer (T-X) by Alzaatreh et al. (2013).

In this paper, we introduce and study a new lifetime model called composed Fréchet-exponential (C-F E) distribution. Using composed−G Q family or shortly (C−G Q) family. We construct the three-parameter composed Fréchet- exponential model and derive some of its mathematical properties.

Let G and Q be two arbitrary continuous cumulative distribution functions of non-negative absolutely continuous random variables, G be strictly increasing on its support, and $G(0)=Q(0)$. Now define a cumulative distribution function (cdf), F, out of G and Q (called the composed- G Q family shortly (C- G Q)) as follows

$$F(x) = G\big(x.Q(x)\big), \quad \forall x \tag{3}$$

The corresponding probability density function (pdf) is given by

$$f(x) = g\big(x.Q(x)\big).\{x.q(x) + Q(x)\}. \tag{4}$$

Where g and q are the corresponding densities of G and Q, respectively.

In fact, the composed Fréchet- exponential distribution can provide better fits than many other models, each one having the same, less or more number of parameters. Furthermore, the composed Fréchet- exponential distribution due to its flexibility in accommodating all forms of the hazard rate function seems to be an important distribution that can be used to serve as an alternative model to other lifetime distributions for modeling positive real data in many areas.

In reliability or lifetime testing experiments, the mostly data are censored due to various reasons such as time limitation, cost or other resources. There are two common censoring samples known as Type-I and Type-II censoring samples. In Type-I censoring, we have a fixed time say X but the number of items fail during the experiment is random. Whereas, in Type-II censoring samples, the experiment is continued (i.e., time varies) until the specified number of failures occur. Here we discuss estimation based on Type-II censoring samples.

The main aim of this paper is to propose and study a new extention of the Fréchet distribution based on the method of composed-G Q family (C-G Q). The main purpose of the new model is to give several desirable properties and more flexibility. Estimate Parameters under complete and censoring samples based on bayesian and non-bayesian methods are proposed. The point and interval estimations will be proposed. The rest of the paper is organized as follows. In Section 2, The Composed-Fréchet Generated family and sub-model of the new generator called the composed Fréchet- exponential (C-F E) distribution are proposed. There is a demonstration of the graphs of the probability density function (pdf) and cumulative distribution function (cdf) of C-FE. In Section 3, the mathematical properties are obtained. In Section 4, we consider the estimation of parameters of C-FE distribution based on censored data Type II by using Bayesian and Non-Bayesian methods. In Section 5, simulation study under Type II censored data is proposed. While applications to real-life data sets are provided in Section 7. Finally, in Section 6, the paper is concluded.

2. The Composed-Fréchet Generated Family

Suppose G is the cdf of Fréchet distribution, which given in (1), a new Fréchet family can be introduced using (3) and (4). This family will be named the composed Fréchet Q family ($C -$ Fréchet Q ($C - F\ Q$)), and its cdf is given by

$$F(x; a, b, \lambda) = e^{-b\left(x.Q(x)\right)^{-a}}, \tag{5}$$

with the corresponding pdf

$$f(x; a, b, \lambda) = ab\big(x.Q(x)\big)^{-a-1} e^{-b(x.Q(x))^{-a}}.\{x.q(x) + x.Q(x)\}. \quad (6)$$

2.1 The Composed- Fréchet Exponential Distribution

Substituting $q(x;\ \lambda) = \lambda e^{-\lambda x}$, and accordingly $\mathrm{Q(x,\lambda)} = 1 - e^{-\lambda x}$ (where $x, \lambda > 0$) into (5) and (6), one gets the composed- Fréchet exponential (C-FE) distribution with cdf

$$F(x; a, b, \lambda) = e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}}, \quad (7)$$

and its corresponding pdf is

$$f(x; a, b, \lambda) = ab\big(x(1 - e^{-\lambda x})\big)^{-a-1} e^{-b\left(x(1-e^{-\lambda x})\right)^{-a}}$$

$$.\left\{x.\lambda e^{-\lambda x} + 1 - e^{-\lambda x}\right\}. \quad (8)$$

Figure 1 illustrates plots of the pdf and cdf of C-FE distribution for selected values of the parameters.

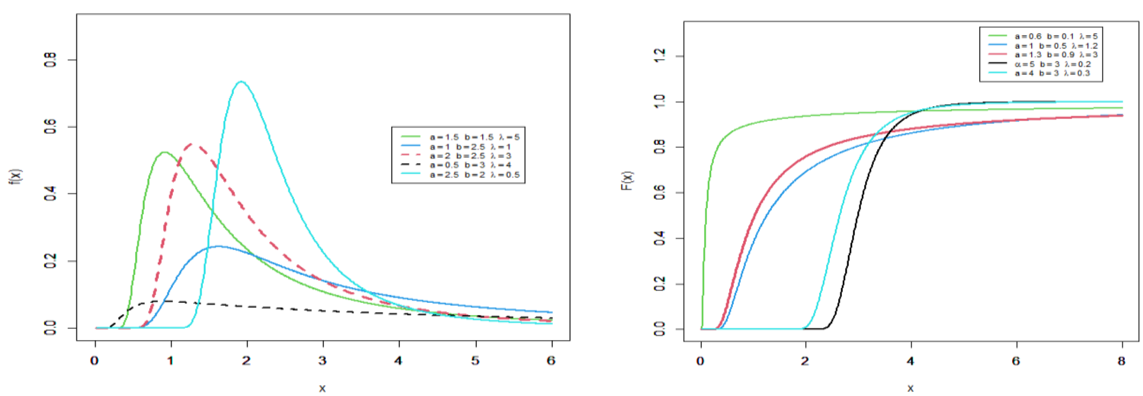


Figure 1: Plots of pdf and cdf of C-FE model for some parameters values.

3. Mathematical properties

Statistical properties like the quantile function, median, survival function, hazard function, reversed hazard function, distribution of order statistics, and estimation of parameters are provided in this section

3.1 Quantile Function

The quantile function of C-FE random variable X is given by

$$F^{-1}(.) = U$$

$$x_q\left(1-e^{-\lambda x_q}\right)+\left(\frac{b}{\log(u)}\right)^{\frac{1}{a}}=0. \tag{9}$$

Where, $x \sim U(0,1)$

Although (9) has no implicit form, it can be solved numerically. If we put q = 0.5 we will get the median of C-FE distribution.

3.2 Reliability analysis

The survival and hazard functions are important for lifetime modeling in reliability studies. These functions are used to measure failure distributions and predict reliability lifetimes. The survival, hazard, and reverse hazard functions of the C-FE distribution are defined respectively, by

$$S(x,a,b,\lambda)=1-F(x)\ =1-e^{-b\left(x\left(1-e^{-\lambda x}\right)\right)^{-a}}, \tag{10}$$

$$h(x,a,b,\lambda)=\frac{f(x)}{1-F(x)}=\frac{ab\left(x\left(1-e^{-\lambda x}\right)\right)^{-a-1}.\{x.\lambda e^{-\lambda x}+1-e^{-\lambda x}\}}{e^{-b\left(x\left(1-e^{-\lambda x}\right)\right)^{-a}}-1}, \tag{11}$$

and

$$r(x)=\frac{f(x)}{F(x)}=ab\left(x\left(1-e^{-\lambda x}\right)\right)^{-a-1}.\{x.\lambda e^{-\lambda x}+1-e^{-\lambda x}\}. \tag{12}$$

Figure 2 illustrates plots of the survival function of C-FE distribution for selected values of the parameters.

Figure 3 illustrates plots of the hazard function of C-FE distribution for selected values of the parameters.

The behavior of *h(x)* was studied for different values of parameters a and fixed b = 1 and $\lambda = 2$ by Glaser's lemma (Glaser (1980)) following:

- If $a \leq 1$, then *h(x)* has a decreasing failure rate which was satisfied in Figure 3c.
- If $a > 1$, then *h(x)* has an upside-down bathtub failure rate which was satisfied in Figure 3d.

Figure 4 illustrates plots of the reversed hazard function of C-FE distribution for selected values of the parameters.

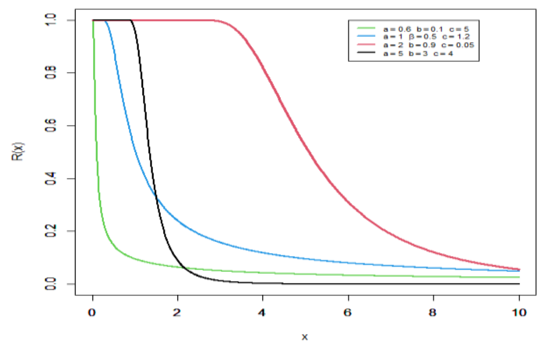
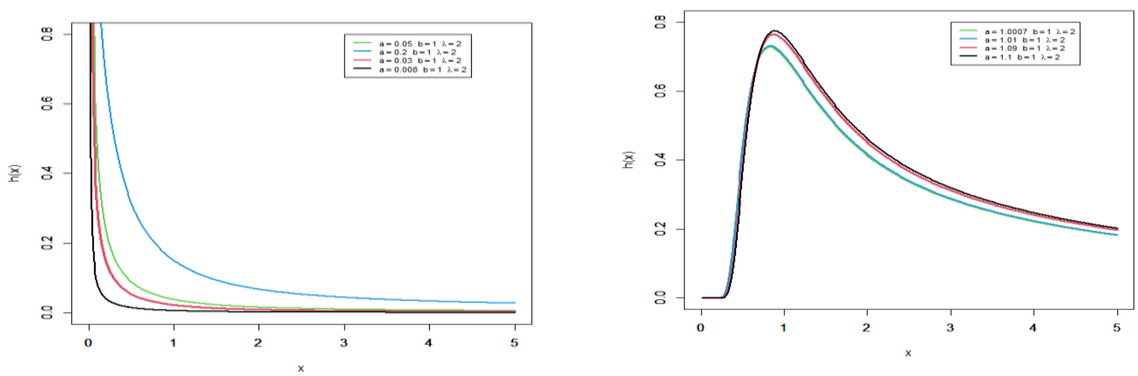


Figure 2: Plots of the survival function of the C-FE model for some parameters values.



(c) (d)

Figure 3: Plots of the hazard function of the C-FE model for some parameters values.

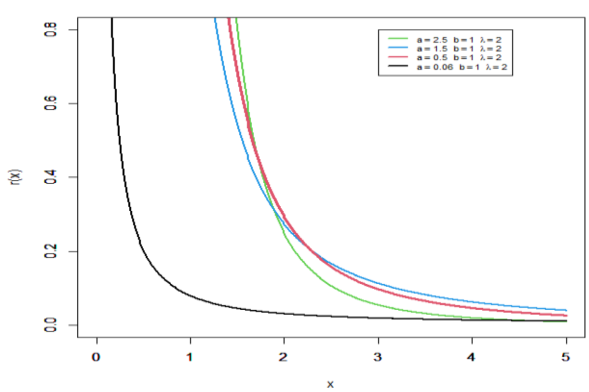


Figure 4: Plots of the reversed hazard function of the C-FE model for some parameters values.

3.3 Order Statistics

Let $X_{1;n},\ X_{2;n}, X_{3;n}, \dots, X_{i;n}$ be the order statistics of a random sample $X_1, X_2, X_3, \dots, X_n$ observed from C-FE distribution with cdf $F(x)$ and pdf $f(x)$ then the pdf of the j^{th} $X_{(i)}$ is given by

$$f_n\left(X_{(i)}\right) = \frac{n!}{(i-1)!(n-i)!} \sum_{r,j=0}^{\infty} k_{rj}\, ab^{r+1}\left[x\left(1-e^{-\lambda x}\right)\right]^{-a(1+r)-1}$$

$$.\{x.\lambda e^{-\lambda x} + 1 - e^{-\lambda x}\}. \tag{13}$$

Where,

$$k_{rj} = \frac{(-1)^{j+r}}{r!}\binom{n-i}{j}(i+j)^r.$$

where $x > 0$. Therefore, the pdf of the n^{th} order C-FE statistic $X_{(n)}$ is given by

$$f_n\left(X_{(n)}\right) = \sum_{r,j=0}^{\infty} \frac{(-1)^{j+r}}{r!} a(nb)^{r+1}\left(x\left(1-e^{-\lambda x}\right)\right)^{-a(r+1)-1}$$

$$.\{x.\lambda e^{-\lambda x} + 1 - e^{-\lambda x}\}.$$

and the pdf of the first order statistic of the C-FE statistic $X_{(1)}$is given

$$f_n\left(X_{(1)}\right) = n\sum_{r,j=0}^{\infty} k_{rj}\, ab^{r+1}\left(x\left(1-e^{-\lambda x}\right)\right)^{-a(1+r)-1}.\{x.\lambda e^{-\lambda x} + 1 - e^{-\lambda x}\}.$$

Where,

$$k_{rj} = \frac{(-1)^{j+r}}{r!}\binom{n-1}{j}(j+1)^r.$$

4. Estimation based on Censored samples

In this section, we will estimate the parameters of the C-FE distribution under Type-II censored samples by using MLE and Bayesian methods

4.1 MLE under Type-II Censored Samples

Suppose that $X_1, X_2, X_3, \dots, X_r$ be a Type-II censoring sample of size r observed from lifetime testing experiment on n items whose lifetime has the density expressed in (8). The likelihood function of Type-II censoring samples is

$$l = C_2 \left[1 - e^{-b\left[x_r(1-e^{-\lambda x_r})\right]^{-a}}\right]^{n-r} a^r b^r \prod_{i=1}^{r} \left[x_i(1-e^{-\lambda x_i})\right]^{-a-1} e^{-b\left[x_i(1-e^{-\lambda x_i})\right]^{-a}}$$
$$.\left\{x_i . \lambda e^{-\lambda x_i} + 1 - e^{-\lambda x_i}\right\}. \qquad (14)$$

Where, $C_2 = \frac{n!}{(n-r)!}$.

Then, the log-likelihood function $logl = logL(x; a, b, \lambda)$ can be written as;

$$logl = lnC2 + (n-r)\, log\left\{1 - e^{-b\left[x_r\left(1-e^{-\lambda x_r}\right)\right]^{-a}}\right\} + rlo\,g(\mathrm{a}) + \mathrm{r}\; log(\mathrm{b})$$
$$- (\mathrm{a}+1)\sum_{i=1}^{r} log\left[x_i\left(1-e^{-\lambda x_i}\right)\right] - b\sum_{i=1}^{r}\left[x_i\left(1-e^{-\lambda x_i}\right)\right]^{-a}$$
$$+ \sum_{i=1}^{r} log\left\{ x_i . \lambda e^{-\lambda x_i} + 1 - e^{-\lambda x_i}\right\}.$$

Differentiating the log-likelihood function $logl$ partially with respect to a, b and λ we get

$$\frac{\partial logl}{\partial a} = (n-r)\left\{\frac{be^{-b[v(x_r;a,\lambda)]}.[v(x_r;a,\lambda)].log[\mathrm{u}(x_r;\lambda)]}{1-e^{-b[v(x_r;a,\lambda)]}}\right\}$$
$$+ \frac{r}{a} - \sum_{i=1}^{r} log[\mathrm{u}(x_i;\lambda)] + \; b\sum_{i=1}^{n} \mathrm{v}(x_i;\mathrm{a},\lambda)\, log[\mathrm{u}(x_i;\lambda)],$$

$$\frac{\partial logl}{\partial b} = \frac{r}{b} - \sum_{i=1}^{n} v(x_i;a,\lambda) - (n-r)\left\{\frac{e^{-b[v(x_r;a,\lambda)]}.[v(x_r;a,\lambda)]}{1-e^{-b[v(x_r;a,\lambda)]}}\right\},$$

.

$$\frac{\partial logl}{\partial \lambda} = \; (n-r)\left\{\frac{abx_r{}^2 e^{-\lambda x_r}.[\mathrm{u}(x_r;\lambda)]^{-a-1}.e^{-b[v(x_r;a,\lambda)]}}{1-e^{-b[v(x_r;a,\lambda)]}}\right\} - (\mathrm{a}+1)\sum_{i=1}^{n}\frac{x_i{}^2 e^{-\lambda x_i}}{\mathrm{u}(x_i;\lambda)}$$
$$+ ab\sum_{i=1}^{n} x_i{}^2 e^{-\lambda x_i}\frac{v(x_i;a,\lambda)}{\mathrm{u}(x_i;\lambda)} + \sum_{i=1}^{n}\frac{2e^{-\lambda x_i} - \lambda x_i{}^2 e^{-\lambda x_i}}{c(x_i;\lambda)}\;.$$

where

$$\mathrm{u}(x_k;\lambda) = x_k \mathrm{d}(x_k;\lambda), k = i, r$$

and $v(x_k;a,\lambda) = u^{-a}(x_k;\lambda)) = \left[x_k\left(1-e^{-\lambda x_k}\right)\right]^{-a}, k = i, r.$

The MLEs can be determined numerically from the solution of a nonlinear system of equations after equating the three derivations to zero; subsequently, these solutions will yield the MLE estimators $\hat{a}, \hat{b}$ and $\hat{\lambda}$, as will be seen in Section 6.

4.2 Bayesian Estimation based on Type-II censoring

In this section, we consider the Bayesian estimators of the unknown parameters based on Type II censoring using MCMC. Assume that a, b and λ obey the gamma prior distribution independently. Then, the joint prior distribution of a, b and λ can be written as

$$\pi^*(a,b,\lambda) = \frac{{\beta_1}^{c_1} a^{c_1-1} e^{-\beta_1 a}}{\Gamma(c_1)} \frac{{\beta_2}^{c_2} b^{c_2-1} e^{-\beta_2 b}}{\Gamma(c_2)} \frac{{\beta_3}^{c_3} \lambda^{c_3-1} e^{-\beta_3 \lambda}}{\Gamma(c_3)} \quad a,b,\lambda > 0,\ \beta_i, c_i > 0. \quad (15)$$

where i = 1,2,3

From (14) and (15) the joint posterior density function of a, b and λ can be written as

$$\pi^*(a,b,\lambda|x) = k_1 \frac{{\beta_1}^{c_1} a^{c_1-1} e^{-\beta_1 a}}{\Gamma(c_1)} \frac{{\beta_2}^{c_2} b^{c_2-1} e^{-\beta_2 b}}{\Gamma(c_2)} \frac{{\beta_3}^{c_3} \lambda^{c_3-1} e^{-\beta_3 \lambda}}{\Gamma(c_3)}$$

$$. \prod_{i=1}^{r} \left(x_i\left(1 - e^{-\lambda x_i}\right)\right)^{-a-1} . e^{-b \sum_{i=1}^{r} \left(x_i\left(1-e^{-\lambda x_i}\right)\right)^{-a}}$$

$$. \prod_{i=1}^{r} \{x_i . \lambda e^{-\lambda x_i} + 1 - e^{-\lambda x_i}\}, \quad (16)$$

where, $k_1$is the normalizing constant which is given as

$$k^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty \frac{{\beta_1}^{c_1} a^{c_1-1} e^{-\beta_1 a}}{\Gamma(c_1)} \frac{{\beta_2}^{c_2} b^{c_2-1} e^{-\beta_2 b}}{\Gamma(c_2)} \frac{{\beta_3}^{c_3} \lambda^{c_3-1} e^{-\beta_3 \lambda}}{\Gamma(c_3)}$$

$$\prod_{i=1}^{r} \left(x_i(1 - e^{-\lambda x_i})\right)^{-a-1} . e^{-b \sum_{i=1}^{r} \left(x_i(1-e^{-\lambda x_i})\right)^{-a}}$$

$$. \prod_{i=1}^{r} \{x_i . \lambda e^{-\lambda x_i} + 1 - e^{-\lambda x_i}\}\, da db d\lambda \, . \quad (17)$$

Obviously, Equation (16) is complicated, and difficult to get the posterior conditional distribution of each parameter. Therefore, we consider the M-H algorithm, which is the simplest and most widely used MCMC method, as will be seen in Section 6.

6 Simulation Study

In this section, we compare the performances of these estimators numerically using Monte Carlo simulations. For comparison purposes, we simulate 1000 generating data from the C-FE distribution under Type-II censored samples for a different choice of n = 100, 150, 200, 250, 350, different percentages of failures r, and the values of the parameters a, b and λ is chosen as the following cases for the random variables generating:

Case 1: a = 8, b = 7.5, λ = 9
Case 2: a = 5.6, b = 4.5, λ = 6
Case 3: a = 4, b = 3.5, λ = 7.5

and we assume the informative gamma priors for a, b and λ that is when the hyper parameters are ($\mathrm{b}_1 = 12, \mathrm{m}_1 = 5.8, \mathrm{b}_2 = 14, \mathrm{m}_2 = 5.6, \mathrm{b}_3 = 1.5, \mathrm{m}_3 = 0.41$).

The Type-II censored samples are generated using the algorithm given by Balakrishanan and Shandhu (1995). The number of iterations for this algorithm is N = 10000 with the burn-in period B = 2000. The comparison is performed by calculating the Bias, the Mean Scare Error (MSE), and the Average Interval Lengths (AIL). All computations were performed using R programming language.

The results are reported in Tables 1-6 for comparison purposes. For the results of the simulation study, conclusions are drawn regarding the behavior of the estimators, which are summarized as follows:

i. Bayes estimates give more efficient results than the MLEs, whereas MSE, average bias and AIL in Bayesian method are less than MLEs method for all cases.
ii. Depending on Bayesian, for a fixed percentage of failure, as sample size increases, it is observed that the MSE of all estimators decrease, the prior has minimal effect on the posterior.

Tables 1-6: Simulation results based on Type-II censoring samples for the C-FE distribution are provided .

Table 1: Estimates Values and MSEs of C-FE distribution based on Type-II Censoring, case 1

N	r	Parameters	MLE			Bayesian		
			Parameters estimates	MSE	Bias	Bayes	MSE	Bias
100	20%	a	7.977	2.932	0.023	7.231	0.757	0.769
		b	10.281	30.411	2.781	6.768	1.191	0.732
		λ	8.350	4.983	0.651	8.889	0.757	0.111
	50%	a	7.285	1.670	0.715	7.170	0.837	0.831
		b	8.005	11.917	0.505	6.867	1.042	0.633
		λ	7.043	17.635	1.957	8.864	0.751	0.135
150	20%	a	7.730	1.823	0.271	7.174	0.829	0.826
		b	9.382	14.561	1.882	7.136	0.694	0.363
		λ	8.274	7.244	0.726	8.889	0.848	0.111
	50%	a	7.177	1.541	0.823	7.084	0.971	0.915
		b	7.702	8.827	0.202	7.160	0.685	0.339
		λ	7.078	17.822	1.921	8.916	0.739	0.083
250	20%	a	7.5493	1.288	0.451	7.083	0.987	0.916
		b	8.960	8.461	1.460	7.494	0.467	0.006
		λ	8.392	6.486	0.608	8.874	0.755	0.126
	50%	a	7.172	1.375	0.828	7.023	1.066	0.977
		b	7.732	7.769	0.232	7.472	0.493	0.028
		λ	7.314	15.735	1.686	8.880	0.888	0.121
350	20%	a	7.555	0.951	0.445	7.073	1.002	0.927
		b	8.955	6.663	1.455	7.732	0.502	0.232
		λ	8.540	5.295	0.461	8.864	0.827	0.136
	50%	a	7.137	1.380	0.863	7.042	1.029	0.958
		b	7.578	7.374	0.078	7.648	0.444	0.148
		λ	7.163	16.230	1.837	8.842	0.758	0.158

Table 2: CI, HPD interval, AILs and CPs of C-FE distribution based on Type-II Censoring, case 2

n	r	Parameters	MLE			Bayesian		
			CI	AIL	CP	HPD interval	AIL	CP
100	20%	a	(5.042,11.755)	6.713	96.4	(6.374, 7.964)	1.590	97
		b	(3.431, 21.242)	17.811	96.6	(5.065, 8.187)	3.122	95.7
		λ	(1.944, 10.552)	8.607	99.8	(7.238, 10.550)	3.312	97.7
	50%	a	(5.346, 9.371)	4.025	97.5	(6.422, 7.949)	1.527	97.8
		b	(2.966, 14.658)	11.692	97.6	(5.341, 8.467)	3.127	97.4
		λ	(1.774, 12.310)	10.536	100	(7.291, 10.460)	3.169	98.1
150	20%	a	(5.235, 10.676)	5.441	96.2	(6.295, 7.799)	1.503	96
		b	(3.476, 17.112)	13.636	96.7	(5.823, 8.687)	2.863	97.3
		λ	(1.938, 12.063)	10.125	99.9	(6.960, 10.622)	3.662	97.5
	50%	a	(5.378, 9.002)	3.623	97.4	(6.363, 7.781)	1.418	97.6
		b	(2.994, 13.010)	10.016	98.5	(5.846, 8.751)	2.905	98.2
		λ	(1.775, 12.575)	10.800	100	(7.187, 10.419)	3.232	96.4
250	20%	a	(5.459, 9.669)	4.210	97.1	(6.344, 7.815)	1.471	97.3
		b	(3.483, 13.940)	10.457	97.2	(6.101, 8.7001)	2.599	96.2
		λ	(1.9347, 12.538)	10.603	99.6	(7.368, 10.571)	3.203	98
	50%	a	(5.599, 8.561)	2.962	98.8	(6.426, 7.745)	1.319	98.8
		b	(3.066, 11.953)	8.887	99.5	(6.166, 8.829)	2.662	97.3
		λ	(1.779, 12.2763)	10.497	100	(7.084, 10.796)	3.713	97.6
350	20%	a	(5.519, 9.141)	3.622	98	(6.338, 7.796)	1.459	97.2
		b	(3.512, 12.703)	9.191	98.6	(6.462, 9.025)	2.563	97.3
		λ	(1.958, 12.129)	10.171	99.8	(7.155, 10.614)	3.461	97
	50%	a	(5.586, 8.392)	2.806	99.4	(6.362, 7.657)	1.295	96.8
		b	(3.055, 11.267)	8.212	99.7	(6.411, 8.875)	2.465	96.8
		λ	(1.768, 11.723)	9.955	100	(7.257, 10.531)	3.273	98.3

Table 3: Estimates Values and MSEs of C-FE distribution based on Type-II Censoring, case 2

N	r	Parameters	MLE			Bayesian		
			Parameters estimates	MSE	Bias	Bayes	MSE	Bias
100	20%	a	4.463	2.146	1.136	4.776	0.838	0.824
		b	5.341	4.508	0.841	5.404	1.315	0.904
		λ	4.974	11.760	1.026	5.815	0.771	0.185
	50%	a	4.274	2.160	1.325	4.566	1.199	1.033
		b	5.181	3.417	0.681	5.382	1.236	0.882
		λ	5.246	11.843	0.754	5.871	0.768	0.129
150	20%	a	4.334	2.220	1.266	4.699	0.959	0.901
		b	5.181	3.092	0.681	5.588	1.613	1.088
		λ	5.096	11.805	0.904	5.784	0.824	0.215
	50%	a	4.212	2.249	1.388	4.496	1.332	1.104
		b	5.012	2.963	0.512	5.557	1.545	1.057
		λ	5.155	12.516	0.845	5.881	0.807	0.119
250	20%	a	4.277	2.152	1.323	4.609	1.116	0.991
		b	5.130	2.433	0.630	5.787	2.021	1.287
		λ	5.347	12.154	0.652	5.846	0.785	0.154
	50%	a	4.222	2.154	1.378	4.430	1.457	1.171
		b	5.057	2.564	0.557	5.631	1.656	1.131
		λ	5.305	12.022	0.695	5.867	0.826	0.133
350	20%	a	4.264	2.129	1.335	4.582	1.154	1.018
		b	5.143	2.386	0.643	5.853	2.151	1.353
		λ	5.361	11.888	0.640	5.865	0.801	0.135
	50%	a	4.225	2.117	1.374	4.399	1.516	1.201
		b	5.067	2.299	0.567	5.638	1.595	1.138
		λ	5.517	11.883	0.483	5.888	0.797	0.112

Table 4: CI, HPD interval, AILs and CPs of C-FE distribution based on Type-II Censoring, case 2

n	r	Parameters	MLE			Bayesian		
			CI	AIL	CP	HPD interval	AIL	CP
100	20%	a	(3.038, 6.521)	3.483	95.6	(3.966, 5.514)	1.547	96.9
		b	(3.109, 9.774)	6.665	95.6	(4.004, 6.739)	2.735	96
		λ	(1.836, 10.564)	8.728	99.6	(4.251, 7.569)	3.317	98.9
	50%	a	(3.191, 5.653)	2.464	96.6	(3.841, 5.231)	1.392	96.7
		b	(2.658, 8.646)	5.987	97.3	(3.962, 6.665)	2.703	96.7
		λ	(1.564, 10.895)	9.331	99.4	(4.343, 7.667)	3.325	98
150	20%	a	(2.997, 6.137)	3.140	96	(3.966, 5.421)	1.455	97.5
		b	(3.118, 8.788)	5.669	95.6	(4.271, 6.841)	2.569	96.6
		λ	(1.832,10.710)	8.879	99.8	(4.079, 7.574)	3.495	97.7
	50%	a	(3.255, 5.301)	2.045	97.9	(3.891, 5.149)	1.259	97.9
		b	(2.594, 8.037)	5.444	98	(4.383, 6.871)	2.488	97.5
		λ	(1.455, 11.098)	9.643	99.2	(3.950, 7.423)	3.473	96.4
250	20%	a	(3.204, 5.588)	2.383	96.7	(3.869, 5.243)	1.374	96.8
		b	(3.183, 7.974)	4.791	97.3	(4.698, 7.017)	2.318	97.2
		λ	(1.819, 10.895)	9.075	99.4	(4.252, 7.576)	3.324	97.7
	50%	a	(3.257, 5.093)	1.837	98.9	(3.874, 5.012)	1.138	97.2
		b	(2.378, 7.522)	5.145	98.9	(4.485, 6.826)	2.340	96.9
		λ	(1.354, 11.005)	9.651	99.8	(3.921, 7.487)	3.567	97.2
350	20%	a	(3.245, 5.468)	2.223	96.6	(3.872, 5.202)	1.331	96.5
		b	(3.208, 7.720)	4.512	98	(4.722, 6.881)	2.159	96
		λ	(1.821, 10.960)	9.139	99.8	(4.155, 7.555)	3.411	98
	50%	a	(3.293, 5.044)	1.750	98.7	(3.841, 4.900)	1.061	96.6
		b	(2.504, 7.196)	4.692	99.7	(4.680, 6.831)	2.149	98.1
		λ	(1.439, 10.832)	9.393	100	(4.323, 7.634)	3.311	98.2

Table 5: Estimates Values and MSEs of C-FE distribution based on Type-II Censoring, case 3

N	r	Parameters	MLE			Bayesian		
			Parameters estimates	MSE	Bias	Bayes	MSE	Bias
100	20%	a	3.673	0.798	0.327	3.018	1.080	0.982
		b	8.215	35.318	4.715	5.417	4.147	1.916
		λ	5.685	10.781	1.814	7.395	0.759	0.105
	50%	a	3.392	0.601	0.608	3.011	1.064	0.990
		b	6.849	16.441	3.349	5.314	3.769	1.814
		λ	4.991	15.868	2.509	7.371	0.794	0.129
150	20%	a	3.534	0.646	0.466	3.001	1.093	0.999
		b	7.364	21.709	3.864	5.667	5.133	2.167
		λ	5.465	13.680	2.035	7.322	0.778	0.177
	50%	a	3.439	0.463	0.560	3.021	1.022	0.980
		b	7.113	16.365	3.613	5.561	4.672	2.061
		λ	5.994	10.438	1.506	7.352	0.798	0.148
250	20%	a	3.417	0.582	0.583	2.996	1.088	1.004
		b	6.991	15.945	3.491	5.944	6.392	2.444
		λ	5.335	14.121	2.165	7.315	0.725	0.185
	50%	a	3.447	0.421	0.553	3.082	0.893	0.918
		b	7.164	16.063	3.664	5.885	6.085	2.385
		λ	6.211	9.616	1.300	7.394	0.702	0.106
350	20%	a	4.241	2.185	1.359	4.548	1.225	1.052
		b	5.097	2.154	0.597	5.804	2.012	1.304
		λ	5.494	12.603	0.506	5.880	0.788	0.121
	50%	a	3.334	0.565	0.666	3.142	0.774	0.858
		b	6.646	12.793	3.146	6.159	7.421	2.659
		λ	5.079	15.357	2.421	7.415	0.687	0.085

Table 6: CI, HPD interval, AILs and CPs of C-FE distribution based on Type-II Censoring, case 3

n	r	Parameters	MLE			Bayesian		
			CI	AIL	CP	HPD interval	AIL	CP
100	20%	a	(2.359, 5.501)	3.143	96.2	(2.388, 3.709)	1.320	98.2
		b	(4.021, 16.441)	12.420	95.8	(4.072, 6.801)	2.728	96.7
		λ	(1.634, 9.511)	7.865	99.7	(5.636, 8.985)	3.349	97.3
	50%	a	(2.514, 4.367)	1.853	96.8	(2.420, 3.546)	1.125	97.1
		b	(3.519, 11.581)	8.060	97	(4.095, 6.771)	2.675	97.5
		λ	(1.362, 10.029)	8.667	99.9	(5.541, 8.905)	3.364	96.8
150	20%	a	(2.461, 5.085)	2.624	95.5	(2.400, 3.568)	1.168	97.5
		b	(4.139, 13.830)	9.691	95.4	(4.272, 6.862)	2.590	96.1
		λ	(1.618, 10.522)	8.904	99.8	(5.604, 8.913)	3.308	97.1
	50%	a	(2.621, 4.162)	1.542	97.9	(2.520, 3.485)	0.965	96.8
		b	(3.747, 10.627)	6.881	97.6	(4.397, 6.870)	2.474	96.8
		λ	(1.411, 10.423)	9.013	99.9	(5.591, 9.021)	3.429	97.6
250	20%	a	(2.561, 4.457)	1.896	96.7	(2.494, 3.571)	1.075	98.5
		b	(4.182, 11.152)	6.969	96.6	(4.789, 7.307)	2.518	97.1
		λ	(1.627, 10.292)	8.665	99.7	(5.671, 8.871)	3.211	97.6
	50%	a	(2.668, 4.031)	1.362	98.7	(2.674, 3.563)	0.889	97.5
		b	(3.825, 9.794)	5.969	99	(4.777, 7.140)	2.363	96.8
		λ	(1.409, 10.796)	9.387	99.8	(5.815, 9.065)	3.251	97.9
350	20%	a	(3.235, 5.376)	2.141\	97.5	(3.922, 5.232)	1.311	98
		b	(3.214, 7.404)	4.190	98.6	(4.786, 6.931)	2.144	97.2
		λ	(1.839, 11.211)	9.370	99.9	(4.185, 7.511)	3.327	97.6
	50%	a	(2.595, 3.904)	1.311	99.2	(2.798, 3.542)	0.744	98.3
		b	(3.537, 9.352)	5.815	99.1	(5.094, 7.403)	2.309	97.7
		λ	(1.275, 10.179)	8.904	99.3	(5.715, 8.975)	3.261	97.2

5. Applications for Real Data

This section illustrates the applicability and flexibility of the C-FE distribution with two real data sets, which will represents as follow:

7.1 Data set (1): Minimum Monthly Flows of Water on The Piracicaba River

Two real data sets are represented related to minimum monthly flows of water (m^3/s) during May and August on the Piracicaba River, located in S$\tilde{a}$o Paulo state, Brazil. The data have been provided by Ramos *et al.* (2020). The data is given by:

- May: 29.19, 18.47, 12.86, 151.11, 19.46, 19.46, 84.30, 19.30, 18.47, 34.12, 374.54, 19.72, 25.58, 45.74, 68.53, 36.04, 15.92, 21.89, 40.00, 44.10, 33.35, 35.49, 56.25, 24.29, 23.56, 50.85, 24.53, 13.74, 27.99, 59.27, 13.31, 41.63, 10.00, 33.62, 32.90, 27.55, 16.76, 47.00, 106.33, 21.03.
- August: 16.00, 9.52, 9.43, 53.72, 17.10, 8.52, 10.00, 15.23, 8.78, 28.97, 28.06, 18.26, 9.69, 51.43, 10.96, 13.74, 20.01, 10.00, 12.46, 10.40, 26.99, 7.72, 11.84, 18.39, 11.22, 13.10, 16.58, 12.46, 58.98, 7.11, 11.63, 8.24, 9.80, 15.51, 37.86, 30.20, 8.93, 14.29, 12.98, 12.01, 6.80.

The results obtained using the C-FE distribution are compared with the composed-Weibull exponential (C-WE), Weibull (W) and Fréchet (Fr). We consider certain discrimination criteria such as Kolmogorov-Smirnov (K-S) test statistic, Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), and corrected Akaike information criterion (CAIC). The preferred model is the one which provides the minimum values of the aforementioned statistics. Summary statistics of the data are mentioned in Table 7.

Table 7: Summary statistics of minimum flow of water during May and August at Piracicaba River in Brazil.

Month	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
May	10.00	19.46	28.59	44.96	44.51	374.54
Augus	6.80	9.80	12.46	17.44	18.26	58.98

Table 8: Estimates of the parameters and standard errors in parentheses for the models fitted to minimum flow of water during May and August at Piracicaba River in Brazil.

Month	Distributions	Estimates (standard errors)		
		$\hat{a}$	$\hat{b}$	$\hat{\lambda}$
May	**C-FE**	**1.7994 (0.7587)**	**21.6957 (13.4717)**	**5.7000 (0.0001)**
	Fr	1.4807 (0.1357)	105.1725 (42.6084)	
	C-WE	0.8962 (0.1294)	0.0422 (0.0271)	0.0369 (0.0177)
	W	1.1036 (0.1136)	0.0142 (0.0074)	
August	**C-FE**	**2.3915 (0.9098)**	**24.943 (14.301)**	**5.7000 (0.0001)**
	Fr	1.9327 (0.1926)	113.1832 (51.1638)	
	C-WE	1.5670 (0.6447)	0.4712 (0.2839)	8.7000 (0.0000000105)
	W	1.5763 (0.1717)	0.0091 (0.0054)	

Table 9: Analytical results for different probability distributions for the data sets related to the minimum flows of water (m^3/s) during May and August at Piracicaba River in Brazil.

Month	Measures	**C-FE**	Fr	C-WE	W
May	K_S (p-value)	**0.0633** **(0.9972)**	0.0964 (0.8511)	0.13454 (0.4639)	0.1875 (0.1202)
	-2log L	**354.0966**	357.4268	370.6919	383.5337
	AIC	**360.0966**	361.4268	376.6919	387.5337
	BIC	**358.9028**	360.6309	375.4981	386.7378
	CAIC	**361.9028**	362.6309	378.4981	388.7378
August	K_S (p-value)	**0.0754** **(0.9739)**	0.1149 (0.6514)	0.1841 (0.1242)	0.1868 (0.1144)
	-2log L	**276.6462**	280.4167	302.9114	302.9043
	AIC	**282.6462**	284.4167	308.9114	306.9043
	BIC	**281.4845**	283.6422	307.7497	306.1299
	CAIC	**384.4845**	285.6423	310. 7497	308.1299

Parameter estimation of C-FE, Fr, W and C-WE model are tabulated in Table 8. From Table 9, it will observe that C-FE distribution fits better than the chosen models. These results are confirmed from the K-S, AIC, BIC, and CAIC values, since C-FE distribution has the minimum values for the proposed data sets. Therefore, the proposed methodology can be used successfully to analyze the minimum flow of water during May and August at Piracicaba River using the C-FE distribution. The plots of the fitted C-FE, Fr, W and C-WE densities are shown in Figure 5.1 and 5.2.

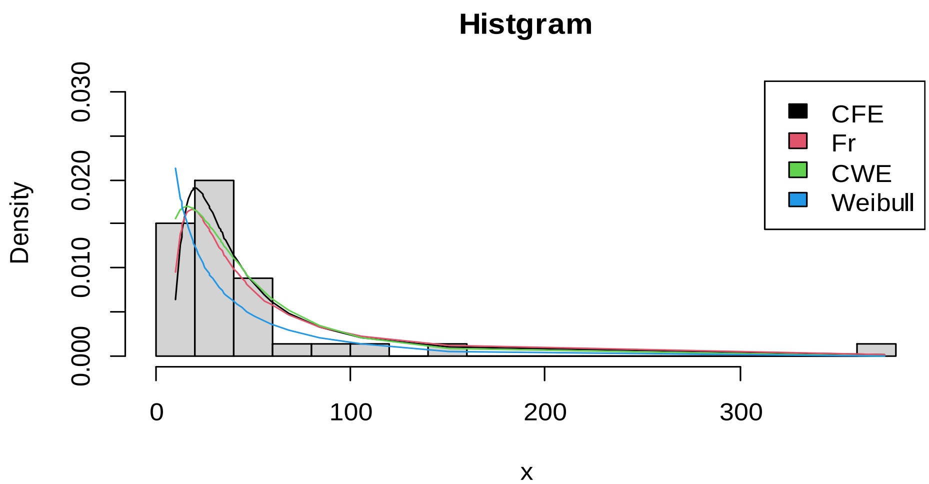


Figure 5.1: The fitted pdfs of the C-FE, Fr, W and C-WE models for minimum monthly flows of water (m^3/s) during May.

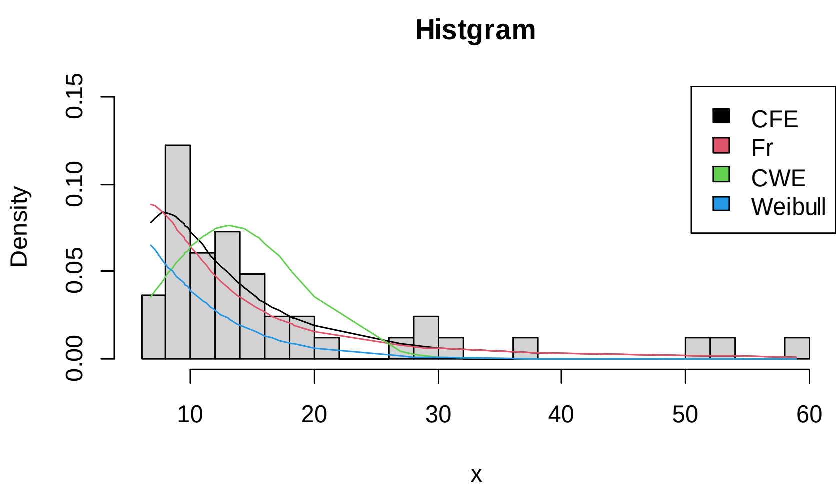


Figure 5.2: The fitted pdfs of the C-FE, Fr, W and C-WE models for minimum monthly flows of water (m^3/s) during August on the Piracicaba River,

7.2 Data Set (2): Active Repair Times for an Airborne Communication Transceiver

We provide data analysis to assess the goodness of fit of the C-FE model with respect to the active repair times (hours) for an airborne communication transceiver to see how C-EF distribution works in practice. The data is given by: 0.50,0.60,0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.5, and their source is Jorgensen (1982).

We fit a proposed model (C-FE) in comparison with six other well-known competing distributions. The cumulative functions of the competing models are:

I. Exponential (Ex) distribution

$$F(x) = 1 - e^{-\lambda x}, \quad x, \lambda \geq 0.$$

II. Exponentiated Fréchet (EFr) distribution introduced by Silva et al. (2013)

$$F(x) = 1 - \left(1 - exp^{-\left(\frac{b}{x}\right)^{a}}\right)^{\alpha}, \;\; x, \alpha, a, b \geq 0.$$

III. Exponentiated Generalized Fréchet (EGFr) distribution introduced by Abd-Elfattah et al. (2016)

$$F(x) = \left[1 - \left(1 - exp^{-\left(\frac{b}{x}\right)^{a}}\right)^{\alpha}\right]^{\beta}, \qquad x, \alpha, \beta, a, b \geq 0\,.$$

IV. Transmuted Exponentiated Fréchet (TEFr) distribution introduced by Elbatal et al. (2014)

$$F(x) = \left[1 - \left(1 - e^{-\left(\frac{b}{x}\right)^{a}}\right)^{\beta}\right]\left\{1 + \alpha\left[\left(1 - e^{-\left(\frac{b}{x}\right)^{a}}\right)\right]^{\beta}\right\}, x, \beta, a, b \geq 0, and\ |\alpha| \leq 1.$$

V. Topp Leone Fréchet (TLF) distribution introduced by Sapkota (2021)

$$F(x) = \left[1 - \left(1 - e^{-\left(\frac{b}{x}\right)^{a}}\right)^{2}\right]^{\alpha}, x, \alpha, a, b \geq 0.$$

VI. Alpha Power Transformed Fréchet (APTF) distribution introduced by Elbatal et al. (2019)

$$F(x) = \frac{e^{-\left(\frac{b}{x}\right)^{a}} \propto^{e^{-\left(\frac{b}{x}\right)^{a}}}}{\propto}, \qquad x, \alpha, a, b \geq 0 \text{ and } \alpha \neq 1.$$

The analytical measure such as Kolmogorov-Smirnov (KS) test statistic, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and corrected Akaike information criterion (CAIC) are considered for deciding the goodness of the fit results of the proposed model and other competing models. On considering these measures, it is shown that the newly proposed model provides greater distributional flexibility than the other well-known distributions. Summary statistics of the data are mentioned in Table 7.

Table 7: Summary statistics of active repair times (hours).

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.500	1.000	2.100	4.013	4.775	24.500

Table 8: Estimates of the parameters and standard errors in parentheses for the models fitted to active repair times (hours).

The model	Estimates (standard errors)				
	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
C-FE	1.135 (0.172)	1.442 (0.289)	3.374 (2.187)		
Ex			0.249 (0.039)		
EFr	1.241 (0.972)	1.405 (1.313)		0.956 (1.218)	
EGFr	6.591 (7.319)	0.387 (0.198)		0.156 (0.185)	3.085 (3.041)
TEFr	3.309 (21.947)	0.648 (1.767)		- 0.535 (2.622)	0.289 (2.103)
TLF	0.665 (0.158)	0.215 (0.926)		16.584 (76.313)	
APTF	1.256 (0.211)	1.321 (0.448)		1.246 (0.954)	

Table 9: Analytical results of C-FE and other competing model Data

The model	Measures				
	K_S (p-value)	-2log L	AIC	BIC	CAIC
C-FE	**0.091 (0.892)**	**177.819**	**183.819**	**188.886**	**184.486**
Ex	0.138 (0.430)	191.153	193.153	194.842	193.258
EFr	0.096 (0.857)	178.897	184.897	189.964	185.564
EGFr	0.095 (0.865)	177.949	185.949	192.704	186.792
TEFr	0.099 (0.831)	178.638	186.638	193.393	187.881
TLF	0.096 (0.856)	179.057	185.057	190.124	185.724
APTF	0.865 (0.095)	178.806	184.806	189.873	185.473

In Table 9, we compare the C-FE model with the Exponential (Ex), Exponentiated Fréchet (EFr), Exponentiated Generalized Fréchet (EGFr), Transmuted Exponentiated Fréchet (TEFr), the Topp Leone Fréchet (TLF),and Alpha power Transformed Fréchet (APTF) distributions. It is noted that the C-FE distribution has the lowest values for the AIC, BIC and CAIC statistics among all fitted models. So, the C-FE model can be chosen as the best model among all fitted models for this data.

Figure (5) illustrates plots of the fitted pdfs of the C-FE, Ex, EFr, EGFr, TEFr, APTF and TLF models for active repair times (hours) data.

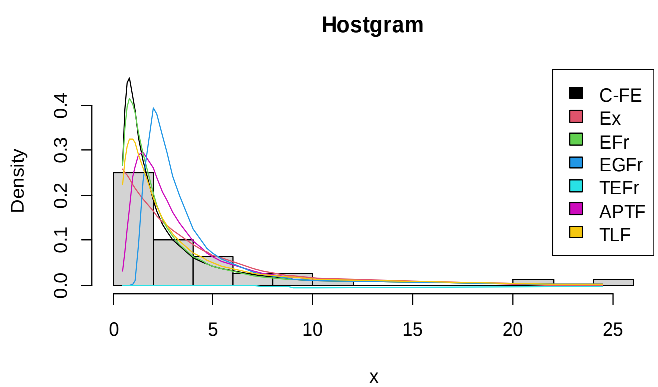


Figure 6: The fitted pdfs of the C-FE, Ex, EFr, EGFr, TEFr, APTF and TLF models for active repair times (hours) data.

6. Conclusions

In this paper, we proposed a new model, called the composed- Fréchet exponential (C-FE) distribution. An obvious reason for generalizing new distribution is the fact that the generated model can provide more flexibility to analyze real-life data. We provide some of its mathematical and statistical properties. We consider the estimation of parameters of C-FE distribution based on censored data type II. The parameters are estimated by the maximum likelihood and Bayesian methods. The proposed distribution is applied to a real data sets. It provides a better fit than several other competitive models.

References

Abd-Elfattah, A. M., Assar, S. M., and Abd-Elghaffar, H. I. (2016). Exponentiated Generalized Fréchet Distribution. International Journal of Mathematical Analysis and Applications, 3(5), 39-48.

Afify, A. Z., Yousof, H. M., Cordeiro, G. M., Ortega, E. M. and Nofal, Z. M. (2016). The Weibull Fréchet Distribution and its Applications. Journal of Applied Statistics, 43(14), 2608–2626.

Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new Method for Generating Families of Continuous Distribution. Merton, 7(1), 63-79.

Barreto-Souza, W., Cordeiro, G. M. and Simas, A. B. (2011). Some results for Beta Fréchet distribution. Communications in Statistics - Theory and Methods, 40(5), 798–811.

Cordeiro, G. M., and de Castro, M. (2011). A New Family of Generalized Distributions. Journal of Statistical Computation and Simulation, 81(1), 883-898.

Elbatal, I., Ahmad,Z., Elgarhy, M. and Almarashi, A. (2019). A New Alpha Power Transformed Family of Distributions: Properties and Applications to the Weibull Model. Journal of Nonlinear Science and Applications, 12(1), 120.

Elbatal, I. Asha, G. and Raja, V. (2014). Transmuted Exponentiated Fréchet Distribution: Properties and Applications. Journal of Statistics Applications and Probability, 3(3), 379-394.

Eugene, N., Lee, C. and Famoye, F. (2002). The Beta-Normal Distribution and Its Applications. Communications in Statistics-Theory and Methods, 31(4), 497–512.

Fattah, A. A. and Ahmed, A. N. (2018). A Unified Approach for Generalizing Some Families of Probability Distributions, With Applications to Reliability Theory. Pakistan Journal of Statistics and Operation Research, 14(2), 253-273.

Frechet, M. (1927). Sur La Loi de Probabilit de L" cart Maximum. Ann Soc Polon Math (Cracovie), 6,93–116.

Glaser, R. E. (1980). Bathtub and Related Failure Rate Characterizations. Journal of the American Statistical Association, 75 (371), 667-672.

Harlow, D. G. (2002). Applications of the Fréchet Distribution Function. International Journal of Material and product technology, 5(17),482-495.

Jorgensen, B. (1982). *Statistical Properties of the Generalized Inverse Gaussian Distribution*. Number 9 in Lecture Notes in Statistics. SpringerVerlag, New York, Heidelberg, Berlin.

Morales, F. (2005). Estimation of Max-Stable Processes Using Monte Carlo Methods with Applications to Financial Risk Assessment. PhD dissertation, University of North Carolina, USA.

Mubarak, M. (2011). Parameter Estimation Based on the Fréchet Progressive Type II Censored Data with Binomial Removals. International Journal of Quality, Statistics and Reliability, 1-5. http://doi:10.1155/2012/245910.

Nadarajah, S. and Kotz, S. (2008). Sociological Models Based on Fréchet Random Variables. Quality and Quantity, 42, 89-95. http://doi.org/10.1007/s12206-008-0426-5

Pehlivan, A. and Vuong, Q. (2011). International trade and supply function competition. Working paper. Turkey: Department of Economics, Bilkent University.

Ramos, E. Dey, S. Louzada, F. and Ramos, P. L. (2020). The Fr´echet distribution: Estimation and Application an Overview, Journal of Statistics and Management Systems, 23(3), 549-578.

Sapkota,L.P.(2021). Topp-Leone Fréchet Distribution with Theory and Application. Janapriya Journal of Interdisciplinary Studies,10(1), 66-80.

Shaw, W. and Buckley, I. (2007). The Alchemy of Probability Distributions: Beyond Gram- Charlier Expansions, and a Skew-Kurtotic-Normal Distribution From a Rank Transmutation Map. Research report at IMA First Conference on Computational Finance, UK.

Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M. and Zubair, M. (2016). The logistic-X family of distributions and its applications. Communications in Statistics-Theory and Methods, 45(24), 7326–7349.

Statistical Inference for the Length Biased Weighted Lomax Distribution Based on Progressive Type II Censoring

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Abstract

This study examines the estimates of the population parameters, the reliability function, and the hazard function of the length-biased weighted Lomax (LBWLo) distribution using progressively Type II censored samples. The maximum likelihood and Bayesian techniques are used to get the suggested estimators. The posterior distribution of the LBWLo distribution is derived using gamma and Jeffery's priors, which, respectively, act as informative and non-informative priors. To obtain the Bayesian estimates, the Metropolis-Hasting (MH) algorithm is also used. We derive asymptotic confidence intervals based on the Fisher information matrix. Using the sample produced by the MH approach, we construct the intervals with the highest posterior density. The approaches are evaluated for effectiveness through numerical simulation studies. We compare the offered estimates in terms of mean squared error using Monte Carlo simulation. 95% average interval lengths and coverage probabilities are achievable. The results of the investigation confirmed the hypothesis that Bayes estimates with an informative prior are generally preferable than alternative estimates. Furthermore, the study's conclusions were corroborated by one set of real data.

Keywords: *Length biased weighted Lomax, Maximum likelihood estimation, Squared error loss function, Informative prior, Markov chain Monte Carlo.*

1. INTRODUCTION

Among other areas, the field of life testing has given the Lomax (Pareto II) distribution a great deal of attention [1]. Data on income and wealth were modelled using the Lomax distribution (see [2] and [3]). In accordance with information in [4], it has also found use in biological science. In reliability and life testing investigations, it has been employed (see [5–9]).

The length biassed weighted Lomax (LBWLo) distribution was proposed as a more flexible option for modelling data in a range of domains, including lifetime analysis, engineering, and biomedical sciences. When observations from a sample are recorded with uneven probability, the LBWLo distribution manifests in practise and provides a unifying solution for the problems that arise when the observations fall into the non-experimental, non-replicated, and non-random categories. The LBWLo distribution was proposed in [10] and some of its statistical features were covered. The following is the formula for the probability density function (PDF) of the LBWLo distribution with shape parameter $\theta > 1$ and scale parameter $\lambda > 0$:

$$f(x) = \frac{\theta(\theta-1)x}{\lambda^2}\left(1+\frac{x}{\lambda}\right)^{-(\theta+1)} \qquad x, \lambda > 0, \theta > 1. \tag{1}$$

The cumulative distribution function (CDF) of the LBWLo distribution is given by:

$$F(x) = 1-\left(1+\frac{\theta x}{\lambda}\right)\left(1+\frac{x}{\lambda}\right)^{-\theta}, \qquad x, \lambda > 0, \theta > 1. \tag{2}$$

The LBWLo distribution's reliability function (RF) and hazard rate function (HRF) are described as follows:

$$S(x)=\left(1+\frac{\theta x}{\lambda}\right)\left(1+\frac{x}{\lambda}\right)^{-\theta}, \qquad x,\lambda>0, \theta>1,$$

and,

$$h(x)=\frac{\theta x(\theta-1)}{(\lambda+x)(\lambda+\theta x)}, \qquad x,\lambda>0, \theta>1.$$

Plots of the PDF and HRF are represented in Figure 1. As seen, the PDF and HRF of the LBWLo distribution take different shapes. There are several helpful forms for the PDF in Figure 1. The HRF of the LBWLo distribution can take on a number of shapes, including expanding, decreasing, and upside-down, as shown graphically in Figure 1.

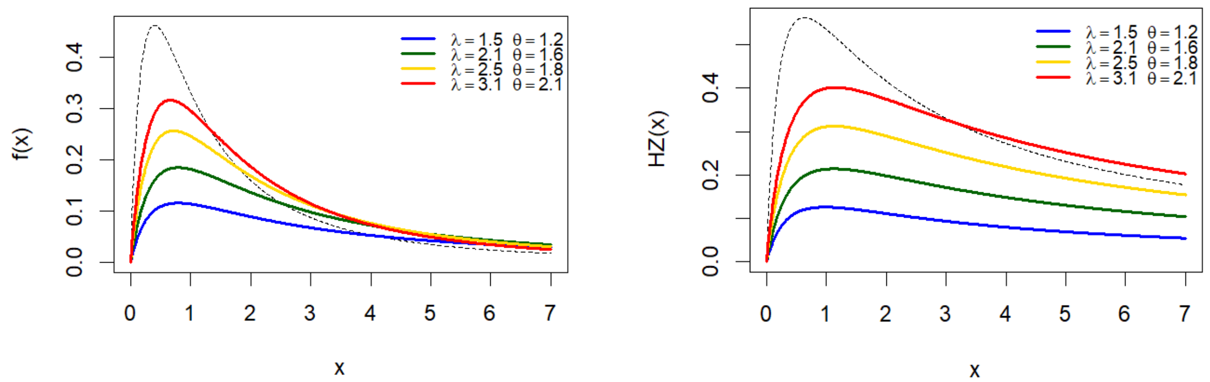


Figure 1. *The distribution's LBWLo PDF and HRF plots*

[11] examined the LBWLo distribution's stress strength reliability estimator in the presence of outliers. [12] discussed Bayesian estimator of accelerated life tests for LBWLo distribution. Therefore, this study was carried out considering that in [11] and [12], the LBWLo distribution is suitable for reliability and life tests and the progressive Type II censoring (PT2C) scheme is very important in these areas. Suppose n identical units are placed on a life testing experiment and the progressive censoring scheme $\underline{R}=(R_1,R_2,...,R_m)$ is pre-fixed such that after the first failure R_1 surviving items are extracted from remaining $(n-1)$ live items and after the second failure R_2 surviving items are eliminated from remaining $(n-R_1-2)$ live items, and so on. After m^{th} failure, this process is repeated until all $R_m=n-m-R_1-...R_{m-1}$ remaining items are eliminated (see [13]). Consequently, a PT2C scheme includes m and $R_1,R_2,...,R_m$; such that $\sum_{i=1}^{m}R_i+m=n$. Take note of the fact that the PT2C scheme is reduced to complete sampling scheme, for $R_1=R_2=...=R_m=0$. Additionally, PT2C offers the type II censoring method for $R_1=R_2=...=R_{m-1}=0$ and $R_m=n-m$ (see [14]). For more studies and application for progressive censoring, the reader can refer to [15–19].

The estimation of the LBWLo distribution is considered in the current work utilising PT2C data. The parameters, RF, and HRF are estimated using maximum likelihood (ML) and Bayesian methods. ACIs and BCIs, also known as approximate confidence intervals, are created. Here is a suggested structure for this essay. In section 2, you will find the ML estimators and ACIs. We present Bayesian estimation using uniform and gamma priors, together with their BCIs, in sections 3 and 4, respectively. Studies and findings related to numbers are covered in section 5. The study of one actual data set is discussed in section 6. A few concluding remarks are made in section 7.

2. MAXIMUM LIKELIHOOD ESTIMATION

The ML estimator of the population parameters, RF, and HRF of the LBWLo distribution are derived in this section.

Let $x_{(1)}, x_{(2)}, ..., x_{(m)}$ be an ordered PT2C sample, according to [20], the likelihood function of the observed sample is given by:

$$l(\underline{x}|\theta) = C\prod_{i=1}^{m} f(x_{(i)})\left[1-F(x_{(i)})\right]^{R_i}, \tag{3}$$

where $C = n(n-R_1-1)\,...\,(n-R_1-R_2-\cdots-R_{m-1}-m+1)$. Consider a random sample of size *n* from PDF (1) and CDF (2), the likelihood function of the LBWLo distribution under PT2C using (3) is as follows:

$$l(\underline{x}|\lambda,\theta) = C\prod_{i=1}^{m}\frac{\theta(\theta-1)x_i}{\lambda^2}\left(1+\frac{x_i}{\lambda}\right)^{-(\theta+1)}\left(\left(1+\frac{x_i}{\lambda}\right)^{-\theta}\left(1+\frac{\theta x_i}{\lambda}\right)\right)^{R_i}, \tag{4}$$

for simplicity write x_i instead of $x_{(i.)}$. The log likelihood function, denoted by $\ln l$, is given by:

$$\begin{aligned}\ln l &= \ln C + m\ln\theta + m\ln(\theta-1) - 2m\ln\lambda + \sum_{i=1}^{m}\ln x_i - (\theta+1)\sum_{i=1}^{m}\ln\left(1+\frac{x_i}{\lambda}\right)\\ &\quad + \sum_{i=1}^{m}R_i\ln\left(1+\frac{\theta x_i}{\lambda}\right) - \theta\sum_{i=1}^{m}R_i\ln\left(1+\frac{x_i}{\lambda}\right).\end{aligned} \tag{5}$$

The following non-linear equations must be simultaneously solved to provide the ML estimators, say $\hat{\lambda}$ and $\hat{\theta}$,

$$\frac{\partial\ln l}{\partial\lambda} = \frac{-2m}{\lambda} + (\theta+1)\sum_{i=1}^{m}\frac{x_i}{(\lambda^2+\lambda x_i)} - \sum_{i=1}^{m}\frac{\theta R_i x_i}{(\lambda^2+\lambda\theta x_i)} + \sum_{i=1}^{m}\frac{\theta R_i x_i}{(\lambda^2+\lambda x_i)},$$

and,

$$\frac{\partial\ln l}{\partial\theta} = \frac{m}{\theta} + \frac{m}{\theta-1} - \sum_{i=1}^{m}\ln\left(1+\frac{x_i}{\lambda}\right) - \sum_{i=1}^{m}R_i\ln\left(1+\frac{x_i}{\lambda}\right) + \sum_{i=1}^{m}R_i\frac{x_i}{(\lambda+\theta x_i)}.$$

By simultaneously solving the likelihood equations, the ML estimators $\hat{\lambda}$ and $\hat{\theta}$ for the PT2C sample are obtained:

$$\frac{\partial\ln l}{\partial\lambda}\Big|_{\lambda=\hat{\lambda}} = 0 \text{ and } \frac{\partial\ln l}{\partial\theta}\Big|_{\theta=\hat{\theta}} = 0.$$

The estimators will be numerically derived using non-linear optimization software because the equations lack a closed form solution. The RF and HRF estimators are assessed based on the invariance property of the ML estimators, as illustrated below:

$$\hat{S}(t) = \left(1+\frac{t}{\hat{\lambda}}\right)^{-\hat{\theta}}\left(1+\frac{\hat{\theta}t}{\hat{\lambda}}\right),$$

and,

$$\hat{h}(t) = \frac{\hat{\theta}t\,(\hat{\theta}-1)}{(\hat{\lambda}+t)(\hat{\lambda}+\hat{\theta}t)}.$$

Additionally, the type II censoring method is used to derive the ML estimators of $\lambda, \theta, S(t)$ and $h(t)$ for $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n - m$. The ML estimators of $\lambda, \theta, S(t)$ and $h(t)$ are produced from complete sample for $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = 0$ as well.

Additionally, the asymptotic variance-covariance matrix (VCM) for the estimators $\hat{\lambda}$ and $\hat{\theta}$ can be obtained by inverting Fisher information matrix (FIM) where its elements are the negative of second order derivatives of the natural logarithm of likelihood function (5)

$$\hat{I}(\hat{\lambda},\hat{\theta}) = -E\left.\begin{pmatrix} \dfrac{\partial^2 \ln l}{\partial \lambda^2} & \dfrac{\partial^2 \ln l}{\partial \lambda\, \partial \theta} \\ \dfrac{\partial^2 \ln l}{\partial \theta\, \partial \lambda} & \dfrac{\partial^2 \ln l}{\partial \theta^2} \end{pmatrix}\right|_{(\lambda=\hat{\lambda},\theta=\hat{\theta})}.$$

Unfortunately, it is challenging to find exact closed forms for the aforementioned requirements. Consequently, the observed FIM $\hat{I}(\hat{\lambda},\hat{\theta})$, which is derived by removing the expectation operator E, will be used to create confidence intervals of the parameters (see [21]). The second partial derivatives of the log-likelihood function, which are simple to construct, are the entries in the observed FIM. Consequently, the observed FIM is represented by:

$$\hat{I}(\hat{\lambda},\hat{\theta}) = -\left.\begin{pmatrix} \dfrac{\partial^2 \ln l}{\partial \lambda^2} & \dfrac{\partial^2 \ln l}{\partial \lambda\, \partial \theta} \\ \dfrac{\partial^2 \ln l}{\partial \theta\, \partial \lambda} & \dfrac{\partial^2 \ln l}{\partial \theta^2} \end{pmatrix}\right|_{(\lambda=\hat{\lambda},\theta=\hat{\theta})}.$$

The elements of the FIM are obtained as follows:

$$I_{11} = \frac{\partial^2 \ln l}{\partial \lambda^2} = \frac{2m}{\lambda^2} - (\theta+1)\sum_{i=1}^{m} \frac{2\lambda x_i + x_i^2}{(\lambda^2 + \lambda x_i)^2} + \sum_{i=1}^{m} \frac{\theta x_i R_i (2\lambda + \theta x_i)}{(\lambda^2 + \lambda\, \theta x_i)^2} - \sum_{i=1}^{m} \frac{\theta x_i R_i (2\lambda + x_i)}{(\lambda^2 + \lambda x_i)^2},$$

$$I_{22} = \frac{\partial^2 \ln l}{\partial \theta^2} = \frac{-m}{\theta^2} - \frac{m}{(\theta-1)^2} - \sum_{i=1}^{m} R_i \frac{x_i^2}{(\lambda + \theta x_i)^2},$$

and

$$I_{12} = I_{21} = \frac{\partial^2 \ln l}{\partial \theta\, \partial \lambda} = \sum_{i=1}^{m} \frac{x_i(1+R_i)}{(\lambda^2 + \lambda x_i)} - \sum_{i=1}^{m} \frac{R_i x_i}{(\lambda + \theta x_i)^2}.$$

In order to construct the asymptotic VCM $[\hat{V}]$, for the ML estimators, the observed FIM $\hat{I}(\hat{\lambda},\hat{\theta})$, is inverted as follows:

$$[\hat{V}] = \hat{I}^{-1}(\hat{\lambda},\hat{\theta}) = -\left.\begin{pmatrix} \operatorname{var}(\hat{\lambda}) & \operatorname{cov}(\hat{\lambda},\hat{\theta}) \\ \operatorname{cov}(\hat{\lambda},\hat{\theta}) & \operatorname{var}(\hat{\theta}) \end{pmatrix}\right|_{(\lambda=\hat{\lambda},\theta=\hat{\theta})}. \quad (6)$$

Again, a numerical technique using R and computer facilities are used to obtain the VCM. According to [22], under particular regularity conditions, $(\hat{\lambda},\hat{\theta})$ roughly follow a bivariate normal distribution with their mean of $(\hat{\lambda},\hat{\theta})$ and VCM $\hat{I}^{-1}(\lambda,\theta)$. Hence, the two-sided $100(1-\omega)\%$ ACI for λ and θ can be constructed based on the asymptotic normality conditions of the ML estimators as:

$$\text{ACI\_UL}(\hat{\lambda}) = \hat{\lambda} + Z_{\omega/2}\sqrt{\operatorname{var}(\hat{\lambda})}, \quad \text{ACI\_LL}(\hat{\lambda}) = \hat{\lambda} - Z_{\omega/2}\sqrt{\operatorname{var}(\hat{\lambda})},$$

$$\text{ACI\_UL}(\hat{\theta}) = \hat{\theta} + Z_{\omega/2}\sqrt{\operatorname{var}(\hat{\theta})}, \quad \text{ACI\_LL}(\hat{\theta}) = \hat{\theta} - Z_{\omega/2}\sqrt{\operatorname{var}(\hat{\theta})},$$

where AIL= ACI_UL – ACI_LL, and AIL is average width, the first and the second elements on the main diagonal of the asymptotic VCM (6) are $\mathrm{var}(\hat{\lambda})$ and $\mathrm{var}(\hat{\theta})$ respectively, and the right tail probability $\omega/2$ when $Z_{\omega/2}$ is the percentile of the standard normal distribution.

Additionally, we must ascertain their variations in order to derive the ACI for RF and HRF. We employ the delta approach described in [23] to derive a rough estimate of $S(t)$ and $h(t)$, This methodology allows us to approximate the variance of $S(t)$ and $h(t)$, respectively, as follows:

$\mathrm{var}(\hat{S}(t)) = [\nabla_1 \hat{S}(t)]^T [\hat{V}][\nabla_1 \hat{S}(t)]$, and $\mathrm{var}(\hat{h}(t)) = [\nabla_2 \hat{h}(t)]^T [\hat{V}][\nabla_2 \hat{h}(t)]$,

where $\nabla_1 \hat{S}(t) = \left(\frac{\partial S(t)}{\partial \lambda}, \frac{\partial S(t)}{\partial \theta}\right)$, and $\nabla_2 \hat{h}(t) = \left(\frac{\partial h(t)}{\partial \lambda}, \frac{\partial h(t)}{\partial \theta}\right)$. Thus, the two-sided $100(1-\omega)\%$ ACI of $\hat{S}(t)$ and $\hat{h}(t)$ can be constructed as follows:

$$\mathrm{ACI\_UL}(\hat{S}(t)) = \hat{S}(t) + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{S}(t))}, \qquad \mathrm{ACI\_UL}(\hat{S}(t)) = \hat{S}(t) + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{S}(t))}$$

$$\mathrm{ACI\_UL}(\hat{h}(t)) = \hat{h}(t) + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{h}(t))}, \qquad \mathrm{ACI\_UL}(\hat{h}(t)) = \hat{h}(t) + Z_{\omega/2}\sqrt{\mathrm{var}(\hat{h}(t))}$$

3. BAYESIAN ESTIMATORS USING DIFFERENT PRIORS

The squared error loss function (SELF) will be used in this section to analyses Bayesian estimators (BAEs) under the assumption that λ and θ have informative prior (IFP) and non-informative prior (NIFP) distributions.

First, the gamma prior distributions of the LBWLo distribution parameters based on PT2C are assumed to be true in order to obtain the BAEs. We assumed that the gamma distribution of λ and θ exists. If λ and θ are separately distributed, then the following formula gives the combined prior distribution of parameters:

$$g_1(\lambda,\theta) \propto \lambda^{a_1-1}\theta^{a_2-1}e^{-b_1\lambda-b_2\theta}. \tag{7}$$

Given the data $\underline{x}$ based on the likelihood function (4) and the joint prior distribution (7), the joint posterior density of λ and θ is calculated as follows:

$$\pi(\lambda,\theta|\underline{x}) = \frac{l(\underline{x}|\lambda,\theta)\, g_1(\lambda,\theta)}{\int_\theta\int_\lambda l(\underline{x}|\lambda,\theta)\, g_1(\lambda,\theta)\, d\lambda\, d\theta}$$

$$\propto \lambda^{a_1-2m-1}\theta^{a_2+m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i},$$

where $\eta_i = \left(1+\frac{x_i}{\lambda}\right)$ and $\tau_i = \left(1+\frac{\theta x_i}{\lambda}\right)$. The following are the forms of λ and θ marginal posterior distributions:

$$g_{11}(\lambda|\underline{x}) \propto \lambda^{a_1-2m-1}e^{-b_1\lambda}\int_0^\infty \theta^{a_2+m-1}(\theta-1)^m e^{-b_2\theta}\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta,$$

and

$$g_{12}(\theta|\underline{x}) \propto \theta^{a_2+m-1}(\theta-1)^m e^{-b_2\theta}\int_0^\infty \lambda^{a_1-2m-1}e^{-b_1\lambda}\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\lambda.$$

The following steps are taken to obtain the BAEs for λ and θ, under SELF, symbolised by $\hat{\lambda}_1$ and $\hat{\theta}_1$,

$$\hat{\lambda}_1 = D_1^{-1}\int_0^\infty\int_0^\infty \lambda^{a_1-2m}\theta^{a_2+m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\,\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\lambda d\theta,$$

and

$$\hat{\theta}_1 = D_1^{-1}\int_0^\infty\int_0^\infty \theta^{a_2+m}\lambda^{a_1-2m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

where, $D_1 = \int_0^\infty\int_0^\infty \lambda^{a_1-2m-1}\theta^{a_2+m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\lambda d\theta.$

The BAEs of $S(t)$ and $h(t)$ are given by:

$$\hat{S}_1(t) = D_1^{-1}\int_0^\infty\int_0^\infty \lambda^{a_1-2m-1}\theta^{a_2+m-1}(\theta-1)^m e^{-b_1\lambda-b_2\theta}\left(1+\frac{\theta t}{\lambda}\right)\left(1+\frac{t}{\lambda}\right)^{-\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

and

$$\hat{h}_1(t) = D_1^{-1}\int_0^\infty\int_0^\infty \frac{\theta^{a_2+m}\, t\,(\theta-1)^{m+1}}{(\lambda+t)(\lambda+\theta t)}\lambda^{a_1-2m-1}e^{-b_1\lambda-b_2\theta}\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda .$$

The above BAEs $\hat{\lambda}_1, \hat{\theta}_1, \hat{S}_1(t)$ and $\hat{h}_1(t)$ can be quantitatively assessed for the provided values of $a_1, b_1, a_2, b_2, n, m, \underline{x}$ and $\underline{R}$ even though they are not in closed forms.

Second, assuming the hyper-parameters to be zero on Equation (7), the joint prior distribution for λ and θ in the case of NIPF is defined as follows:

$$g_2(\lambda,\theta) \propto \frac{1}{\lambda\theta} \qquad 0<\lambda,\ \theta<1.$$

The joint posterior distribution of parameters λ and θ in this case is given by:

$$\pi_1(\lambda,\theta|\underline{x}) \propto \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i}. \tag{8}$$

As a result, the λ and θ marginal posterior distributions have the following shapes:

$$g_{21}(\lambda|\underline{x}) \propto \int_0^\infty \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta,$$

and

$$g_{22}(\theta|\underline{x}) \propto \int_0^\infty \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\lambda.$$

The BAEs of λ and θ denoted by $\hat{\lambda}_1$ and $\hat{\theta}_1$, under SELF are obtained as follows:

$$\hat{\lambda}_2 = D_2^{-1}\int_0^\infty\int_0^\infty \theta^{m-1}\lambda^{-2m}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

and,

$$\hat{\theta}_2 = D_2^{-1}\int_0^\infty\int_0^\infty \theta^{m}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

where $D_2 = \int_0^\infty\int_0^\infty \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\prod_{i=1}^{m} x_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda.$

The BAEs of $S(t)$ and $h(t)$ are given by:

$$\hat{S}_2(t) = D_2^{-1}\int_0^\infty\int_0^\infty \theta^{m-1}\lambda^{-2m-1}(\theta-1)^m\left(1+\frac{\theta t}{\lambda}\right)\left(1+\frac{t}{\lambda}\right)^{-\theta}\prod_{i=1}^{m} t_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda,$$

and,

$$\hat{h}_2(t) = D_2^{-1}\int_0^\infty\int_0^\infty \frac{\theta^{m}\, t\,(\theta-1)^{m+1}}{(\lambda+t)(\lambda+\theta t)}\lambda^{-2m-1}\prod_{i=1}^{m} t_i\eta_i^{-(\theta+1)}\left[\tau_i\eta_i^{-\theta}\right]^{R_i} d\theta d\lambda .$$

The above BAEs $\hat{\lambda}_2, \hat{\theta}_2, \hat{S}_2(t)$, and $\hat{h}_2(t)$ have no closed forms but can be evaluated numerically for the given values of $n, m, \underline{x}$ and $\underline{R}$.

4. BAYESIAN CREDIBLE INTERVALS

The BCI is an interval with a specific subjective probability that an unobserved parameter value will fall within. It is a range inside the scope of either a predictive or a posterior probability distribution. The BCIs of λ and θ denoted by, $\dddot{\lambda}_1$ and $\dddot{\theta}_1$ are acquired under IFP as follows:

$$\dddot{\lambda}_1 = D_1^{-1} \int_L^U \int_0^\infty \lambda^{a_1-2m-1}\, \theta^{a_2+m-1}\, (\theta-1)^m\, e^{-b_1\lambda - b_2\theta} \prod_{i=1}^m x_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\theta\, d\lambda = 0.95,$$

and,

$$\dddot{\theta}_1 = D_1^{-1} \int_L^U \int_0^\infty \lambda^{a_1-2m-1}\, \theta^{a_2+m-1}\, (\theta-1)^m\, e^{-b_1\lambda - b_2\theta} \prod_{i=1}^m x_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\lambda\, d\theta = 0.95.$$

These integrals are extremely difficult to resolve analytically, hence the Metropolis-Hasting (MH) technique will be used to do so. Additionally, the NIFP is used to obtain the BCIs of λ and θ symbolized by $\dddot{\lambda}_2$ and $\dddot{\theta}_2$ as follows:

$$\dddot{\lambda}_2 = D_2^{-1} \int_L^U \int_0^\infty \theta^{m-1}\, \lambda^{-2m-1}\, (\theta-1)^m \prod_{i=1}^m x_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\theta\, d\lambda = 0.95,$$

and,

$$\dddot{\theta}_2 = D_2^{-1} \int_L^U \int_0^\infty \theta^{m-1}\, \lambda^{-2m-1}\, (\theta-1)^m \prod_{i=1}^m x_i\, \eta_i^{-(\theta+1)} \left[\tau_i \eta_i^{-\theta}\right]^{R_i} d\lambda\, d\theta = 0.95\,.$$

Since it is quite challenging to evaluate these integrals analytically, the MH technique will be used to solve them.

5. NUMERICAL STUDIES & RESULTS

In this section, a simulation is run to get the maximum likelihood estimates (MLEs) and Bayesian estimates (BEs) of $\lambda, \theta, S(t)$ and $h(t)$ for the LBWLo distribution under two PT2C-based schemes. We see the strategies as follows:

- Scheme 1 (Sc. 1): $R_1 = n - m,\ R_2 = R_3 = ... = R_m = 0$.
- Scheme 2 (Sc. 2): $R_1 = R_2 = (n-m)/2,\ R_3 = R_4 = ... = R_m = 0$.

The following steps are done via R 3.6.1 program.

Step 1: Using the same algorithm offered in [24], a random sample $x_1, x_2, ..., x_n$ is generated from the LBWLo distribution with the following steps:

1. Generate *m* independent and identically (*iid*) random numbers $P_1, P_2, ..., P_m$ from uniform distribution *U* (0,1).

2. Set $V_i = P_i^{(1/i + R_m + R_{m-1} + ... + R_{m-i+1})}$ for *i* = 1, 2,…, *m*.

3. Set $U_i = 1 - V_m V_{m-1} ... V_{m-i+1}$ and for *i* = 1, 2,…, *m*. Then $U_1, U_2, ..., U_m$ is the PT2C sample from uniform (0,1) distribution.

4. Finally, set $x_i = F^{-1}(U_i)$ for *i* = 1, 2,…, *m* , where $F^{-1}(.)$ is the inverse LBWLo CDF. Then $x_1, x_2, ..., x_n$ is the required PT2C from LBWLo distribution with censoring scheme $\underline{R} = (R_1, R_2, ..., R_m)$.

The MH is one of the Markov chain Monte Carlo methods that is most frequently used in real-world applications today to simulate deviations from the posterior density and generate accurate approximations. The MH algorithm's proposed distribution $q(\alpha'|\alpha)$ and beginning values $\alpha^{(0)}$ for the unidentified parameters were defined. Consider a bivariate normal distribution for the proposal distribution. To do this, choose $q(\alpha'|\alpha)=N_2(\alpha,S_\alpha^\bullet)$ where $\alpha=(\lambda,\theta)$ and $S_\alpha^\bullet$ denotes the VCM. It should be highlighted that the bivariate normal distribution could result in undesirable negative findings (see [25]). The MH algorithm's steps for selecting a sample from the posterior density are as follows:

a. Set beginning values of α as $\alpha=\alpha^{(0)}$.

b. For i = 1, 2,…, M repeat the following steps:

a) Set $\alpha=\alpha^{(i-1)}$.

b) Generate a new candidate parameter value δ from $N_2(\ln\alpha, I_0^{-1}(\lambda,\theta))$.

c) Set $\alpha'=\exp(\delta)$.

d) Calculate $\varepsilon=\min\left[1,\dfrac{\pi(\alpha'|x)\,\alpha'}{\pi(\alpha|x)\,\alpha}\right]$.

e) Update $\alpha^{(i)}=\alpha'$ with probability ε; otherwise set $\alpha^{(i)}=\alpha$.

The choice of $I^{-1}(\lambda,\theta)$ in the MH algorithm, where the acceptance rate depends on it, is a crucial decision. In the case where $I(.)$ is the FIM, it is thought to be the asymptotic VCM $\hat{I}^{-1}(\alpha)$. The MLE of α is regarded as the beginning value for α.

Finally, some of the initial samples (burn-in) can be removed from the random samples of size M derived from the posterior density, and remaining samples can be used to calculate BEs. The BE is given by the SELF and depends on:

$$\tilde{g}_{MH}(\alpha)=\frac{1}{M-\iota_B}\sum_{i=\iota_B}^{M} g(\alpha),$$

where ι_B reflects the quantity of samples used for burn-in.

Step 2: For the propose of generating random samples, the following parameter values are chosen:

(i) $\lambda=0.5$, $\theta=1.5$ and (ii) $\lambda=2$, $\theta=1.5$.

Step 3: For each set of parameters, the MLEs and BEs of λ and θ are calculated using different censoring schemes $\underline{R}=(R_1,R_2,...,R_m)$ and stage counts (m) for each sample size of n = 50, 100, and 150. We use t = 0.8 for the estimates of the HRF and RF.

Step 4: According to [26], the mean and variance of $\hat{\lambda}^i$ and $\hat{\theta}^i$ are equated with the mean and variance of the gamma distributions to determine the hyper-parameters for gamma priors,

$$a_1=\frac{\left(N^{-1}\sum_{i=1}^{N}\hat{\lambda}^i\right)^2}{(N-1)^{-1}\left(\sum_{i=1}^{N}\left(\hat{\lambda}^i-N^{-1}\sum_{i=1}^{N}\hat{\lambda}^i\right)^2\right)},\quad b_1=\frac{N^{-1}\sum_{i=1}^{N}\hat{\lambda}^i}{(N-1)^{-1}\left(\sum_{i=1}^{N}\left(\hat{\lambda}^i-N^{-1}\sum_{i=1}^{N}\hat{\lambda}^i\right)^2\right)},$$

and,

$$a_2=\frac{\left(N^{-1}\sum_{i=1}^{N}\hat{\theta}^i\right)^2}{(N-1)^{-1}\left(\sum_{i=1}^{N}\left(\hat{\theta}^i-N^{-1}\sum_{i=1}^{N}\hat{\theta}^i\right)^2\right)},\quad b_2=\frac{N^{-1}\sum_{i=1}^{N}\hat{\theta}^i}{(N-1)^{-1}\left(\sum_{i=1}^{N}\left(\hat{\theta}^i-N^{-1}\sum_{i=1}^{N}\hat{\theta}^i\right)^2\right)},$$

where N is the number of samples and $i=1,2,...,N$.

Step 5: Using the MH approach, the deviations from the posterior density are simulated (see [26]).

Step 6: The bias, mean squared error (MSE), AIL, and coverage probability (CP) for two schemes are used to compare results for different sample sizes, with 10000 repeated samples**.**

The numerical outcomes are displayed in Figures 2–7 and reported in Tables A1–A4 in the Appendix. The effectiveness of different estimates is revealed in the following observations.

- In the majority of the cases, the MSEs, biases, and AILs for all estimates (ML and Bayesian) decrease with *n* and *m*.
- The CPs of HRF estimates increase as *n* and *m* increase.
- The MSEs of parameter estimates at true value $\lambda = 0.5,\ \theta = 1.5$ get the largest values in Sc.1and Sc.2, while the MSEs of $\hat{\lambda}_1$ and $\hat{\theta}_1$ get the smallest value under Sc.1and Sc.2 (see Figure 2).

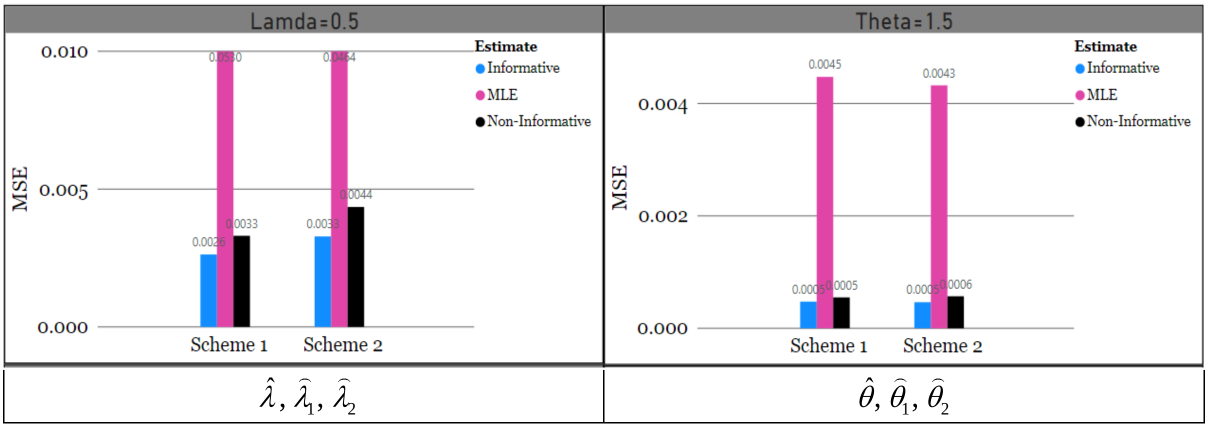


Figure 2. *MSEs for parameter estimates at* $\lambda = 0.5,\ \theta = 1.5$ *for all values of m*

- For both schemes, the MSEs of $\hat{\lambda}$ and $\hat{\theta}$ at $\lambda = 2,\ \theta = 1.5$ have the biggest value, whereas the MSEs of $\hat{\lambda}_1$ and $\hat{\theta}_1$ take the least value (see Figure 3).

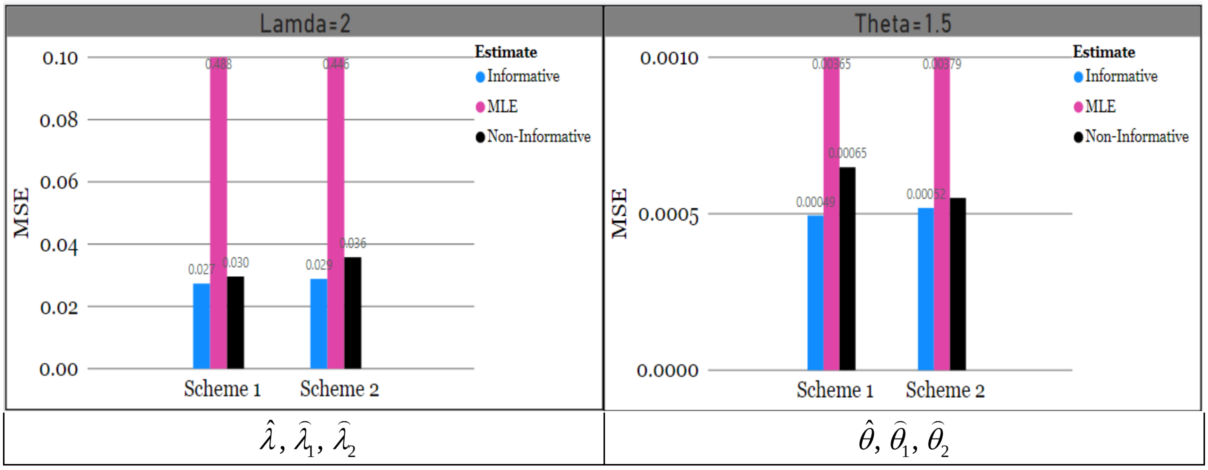


Figure 3. *MSEs for parameter estimates at* $\lambda = 2,\ \theta = 1.5$ *for all values of m*

- As shown in Figure 4, the MSEs of $\hat{S}_1(t)$ and $\hat{h}_1(t)$ have the least values for the two schemes, where as $\hat{S}(t)$ and $h(t)$ have the biggest values for Sc. 1 and Sc. 2 at $\lambda = 0.5,\ \theta = 1.5$.

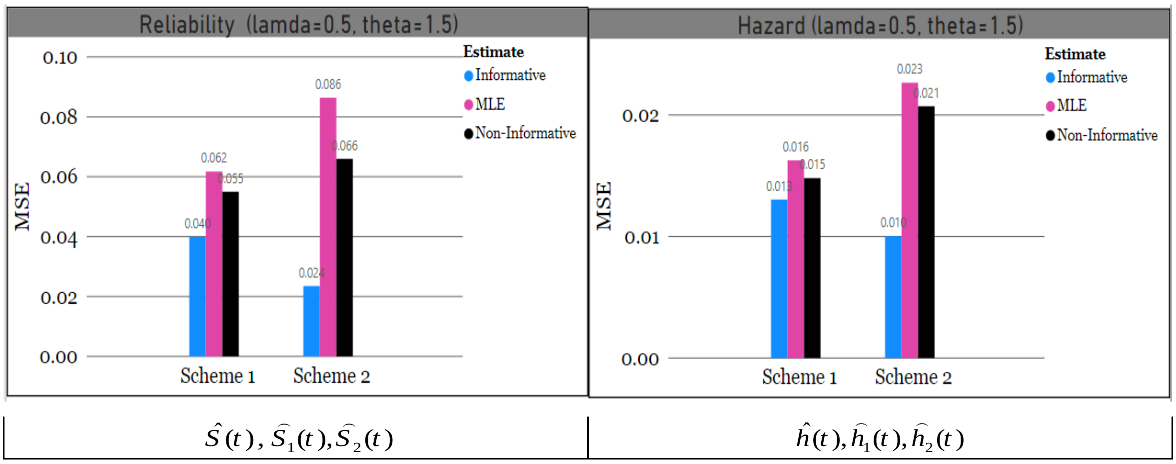


Figure 4. *MSEs at* $\lambda = 0.5$, $\theta = 1.5$ *for all values of* m *for RF and HRF estimates*

- The values of $\hat{S}_1(t)$ and $\hat{h}_2(t)$ MSEs are the least in Sc.1 and Sc.2, respectively. The biggest values in both schemes' MSEs are found for $\hat{S}(t)$ and $\hat{h}(t)$ (see Figure 5).

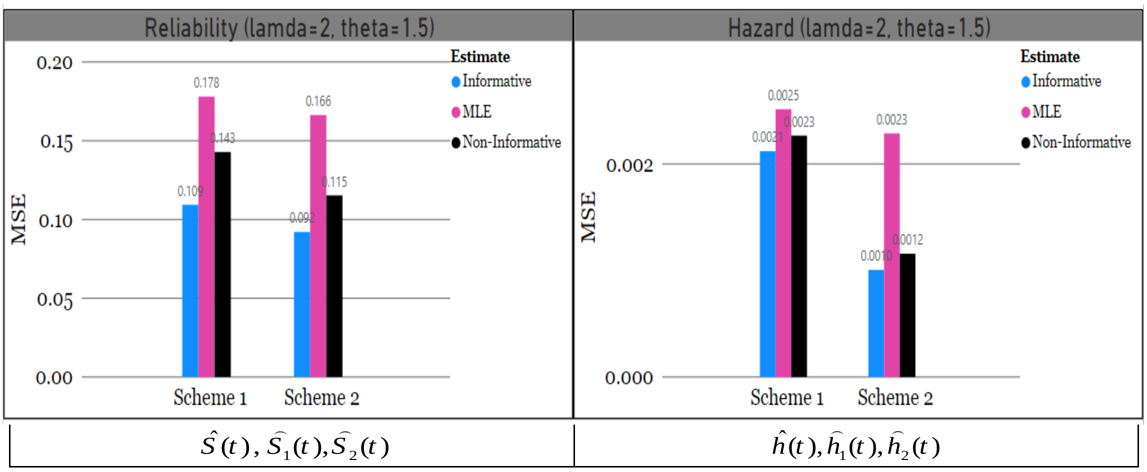
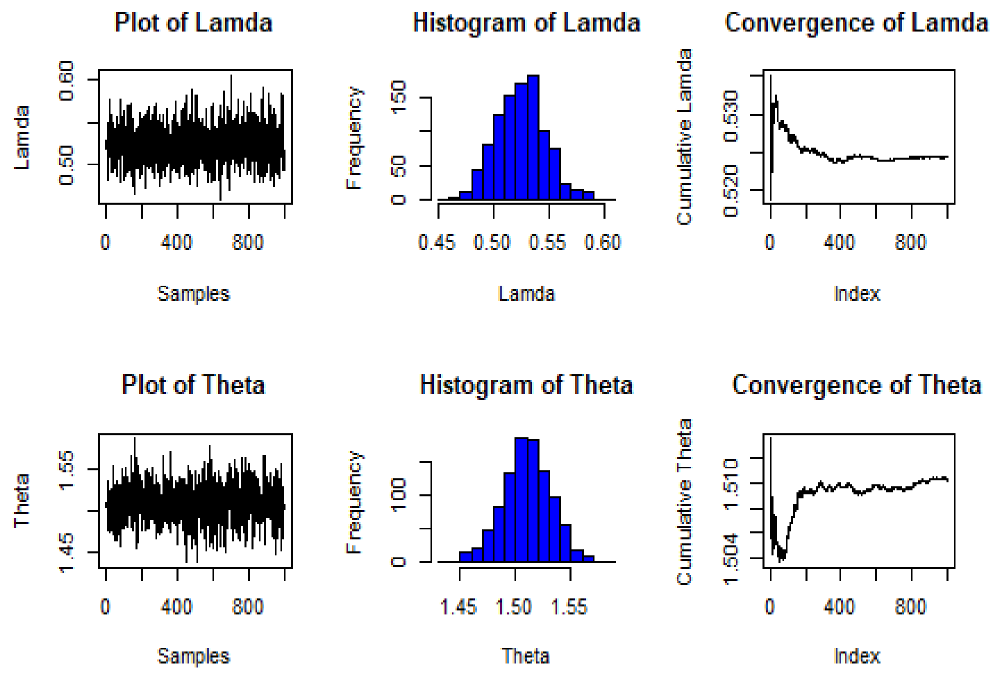
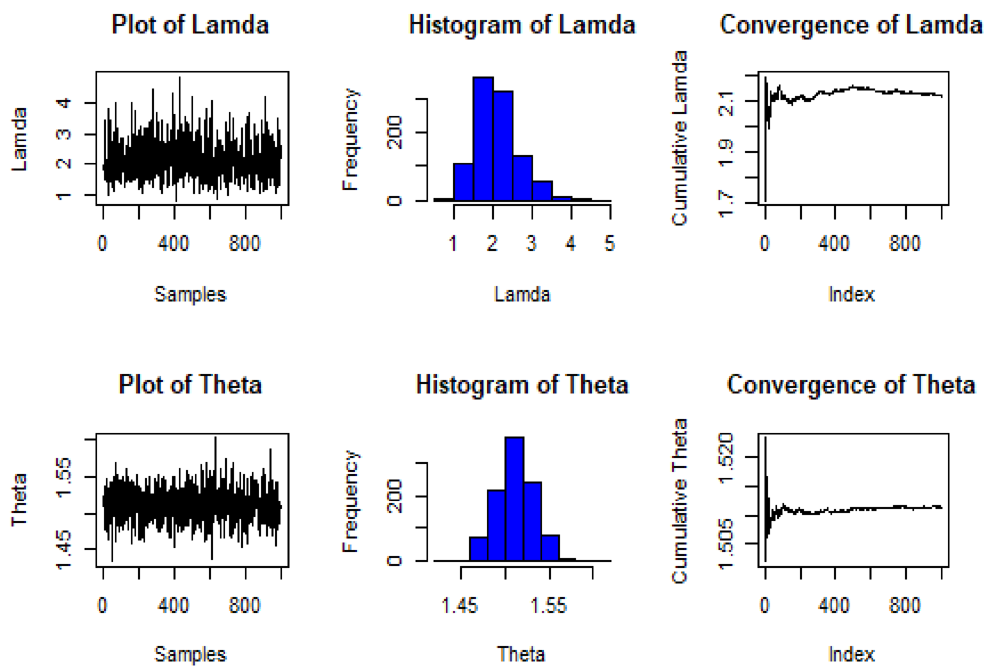


Figure 5. *MSEs at* $\lambda = 2$, $\theta = 1.5$ *for all values of* m *for RF and HRF estimates*

- The MSEs of parameters under IFP generally take the smallest values in all schemes in roughly the majority of circumstances, it may be said.
- The biases of $\hat{\lambda}_1$ and $\hat{\theta}_1$ for LBWLo distribution are smaller than that the corresponding of $\hat{\lambda}$, $\hat{\theta}$, $\hat{\lambda}_2$ and $\hat{\theta}_2$.
- The AILs for BCI estimates of the LBWLo distribution under IFP are smaller than that the corresponding of the MLEs and BEs under NIFP.
- The CPs of the BEs for the LBWLo distribution under gamma priors are greater than that the corresponding of the MLEs and BEs under IFPs.
- Figure 6 shows history graphs for various λ and θ estimaties in NIFP situations. The plots of the parameters of the chains λ and θ resemble a horizontal band without any obvious extended upward or downward trends, which are evidence of convergence.



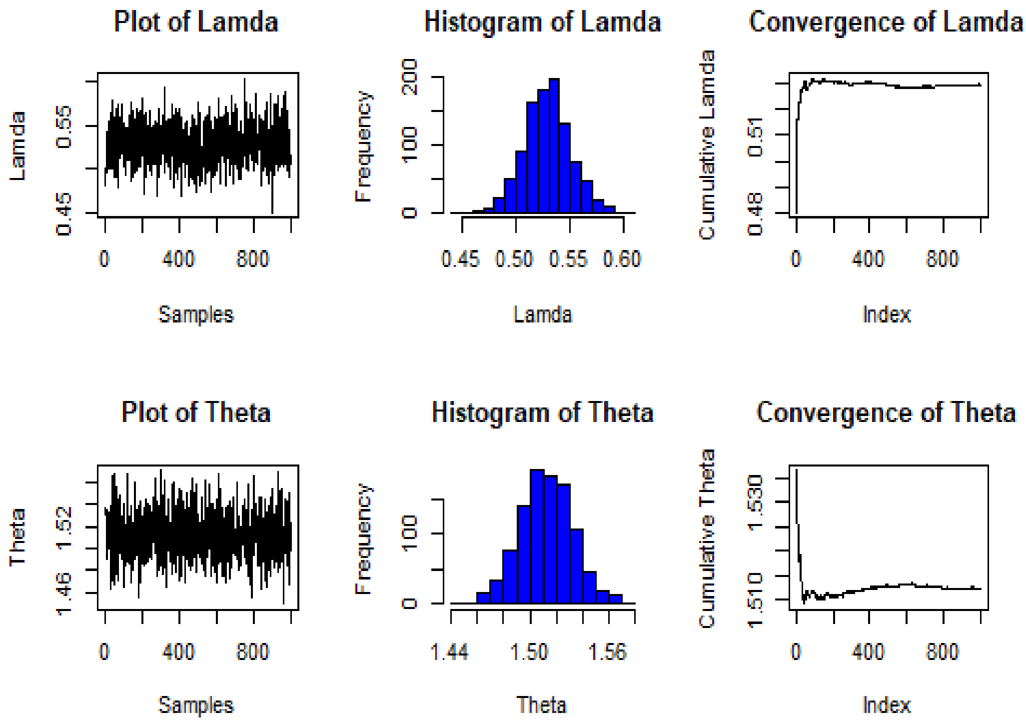
(*a*) $\hat{\lambda}_2$ and $\hat{\theta}_2$ at n=100, m=50 for $\lambda = 0.5$, θ=1.5



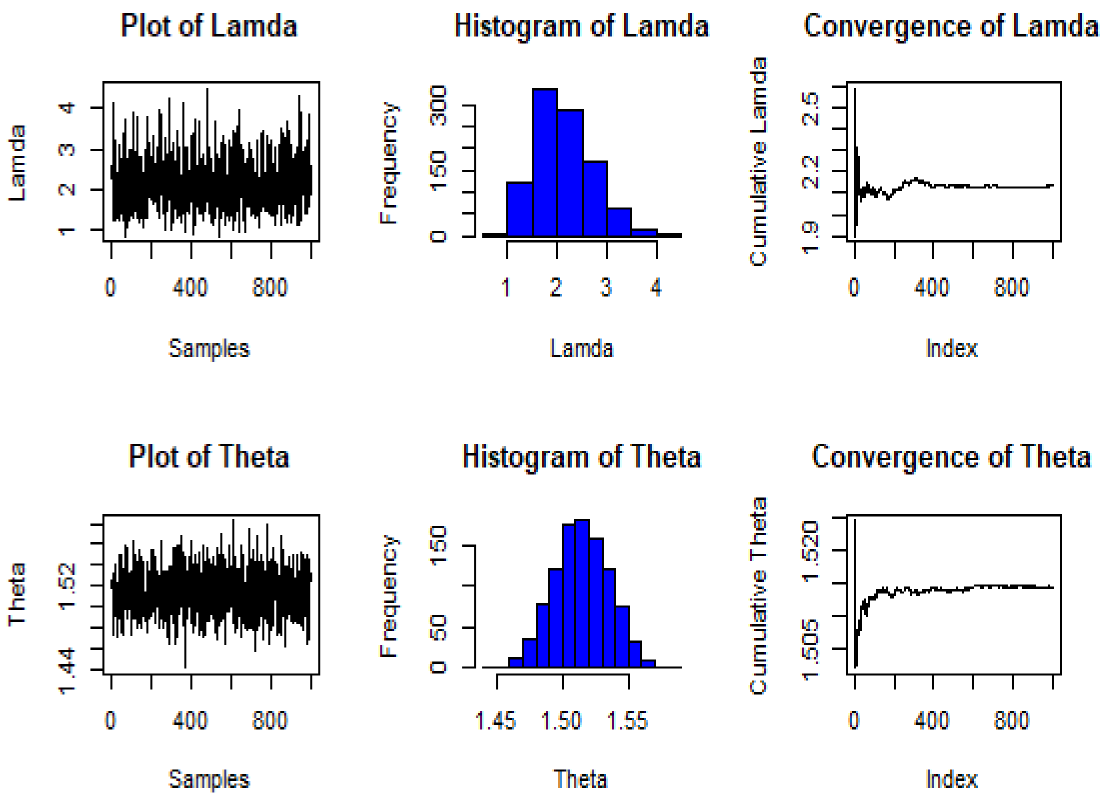
(*b*) $\hat{\lambda}_2$ and $\hat{\theta}_2$ at n=100, m=50 for λ =2, θ=1.5

Figure 6. *Different BEs for λ and θ under uniform priors*

- For IFPs, Figure 7 displays history plots for several estimates of λ and θ. Without any discernible long-term rising or falling trends, which are indications of convergence, the plots of the chains for the parameters create a horizontal band.



(a) *$\hat{\lambda}_1$ and $\hat{\theta}_1$ at n=100, m=50 for* $\lambda = 0.5$, $\theta = 1.5$



(b) *$\hat{\lambda}_1$ and $\hat{\theta}_1$ at n=100, m=50 for* $\lambda = 2$, $\theta = 1.5$

Figure 7. *Different BEs for λ and θ under gamma priors*

6. DATA ANALYSIS

The data set is connected to the period between failures for three repairable objects and is presented below. The data set are provided in [27] and explored later by [28].

1.43	0.11	0.71	0.77	2.63	1.49	3.46	2.46	0.59	0.74
1.23	0.94	4.36	0.40	1.74	4.73	2.23	0.45	0.70	1.06
1.46	0.30	1.82	2.37	0.63	1.23	1.24	1.97	1.86	1.17

To assess if the data distribution is suitable for the LBWLo distribution or not, the Kolmogorov-Smirnov (K-S) test was applied. The K-S distance is calculated to be 0.065072, and the P-value is 0.996. The estimated PDF, CDF, and probability-probability (PP) plots are displayed in Figure 8.

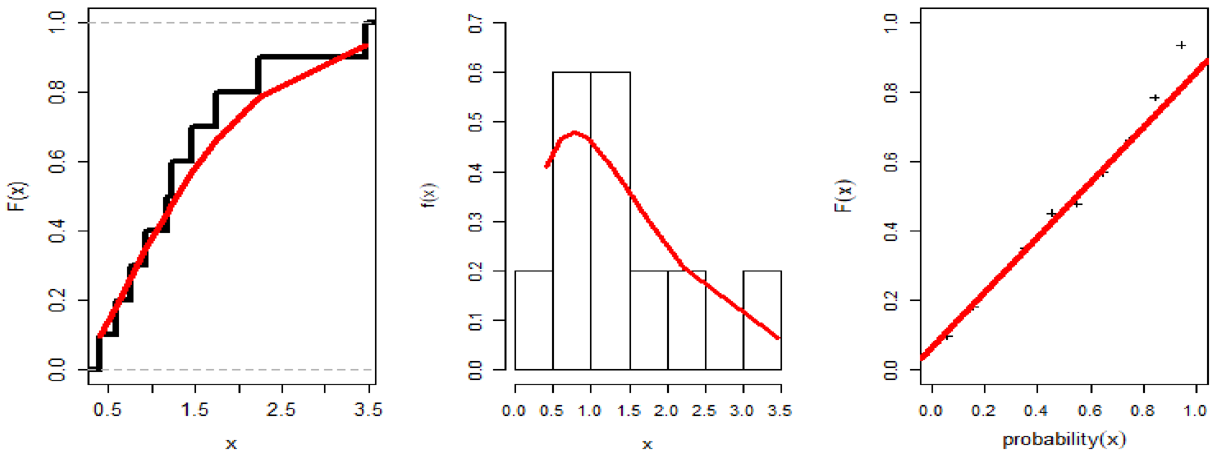


Figure 8. *The estimated PDF, CDF and PP plots for the LBWLo distribution*

Now, let us examine what occurs if the data set is censored. Using the uncensored data set, we produce two artificial PT2C sets in the ways described below (see Table 1):

- Sc.1: $R_1 = n - m, R_2 = R_3 = ... = R_m = 0$.
- Sc.2: $R_1 = R_2 = (n - m)/2, R_3 = R_4 = ... = R_m = 0$.

Table 1 discusses MLEs and BEs along with their standard errors (SEs) based on PT2C samples. We employed a NIFP to calculate the BEs because we have no knowledge of the priors; thus, we chose.

Table 1: The MLEs and BEs under the PT2C

Scheme	***m***		**ML**		**Bayesian**	
			Estimate	**SE**	**Estimate**	**SE**
1	**10**	λ	33.116	73.435	131.651	34.607
		θ	21.801	50.746	87.265	41.757
		$S(t)$	0.753	0.050	0.739	0.047
		$h(t)$	0.266	0.030	0.290	0.027
	20	λ	31.120	27.108	104.367	24.179
		θ	34.400	29.793	107.602	21.628
		$S(t)$	0.578	0.053	0.593	0.051
		$h(t)$	0.569	0.032	0.548	0.029

Scheme	m		ML		Bayesian	
			Estimate	**SE**	**Estimate**	**SE**
2	**10**	λ	27.116	50.650	145.010	48.942
		θ	30.935	59.721	127.989	45.024
		$S(t)$	0.569	0.048	0.647	0.040
		$h(t)$	0.588	0.019	0.443	0.018
	20	λ	75.000	56.737	112.992	47.187
		θ	80.199	64.818	128.974	54.747
		$S(t)$	0.582	0.050	0.556	0.049
		$h(t)$	0.571	0.029	0.633	0.027

The Bayesian estimation strategy, which has the lowest SE values, is the best choice for estimating the parameters, RF, and HRF for the LBWLo distribution based on PT2C. Additionally, it is noted that Scheme 2 fits the data better than other schemes because it has the lowest value among SE.

7. SUMMARY AND CONCLUSION

This paper investigates parameter estimators, reliability function estimators, and hazard rate function estimators for the LBWLo distribution under PT2C samples using ML and Bayesian methodologies. The SELF is used to derive the BAEs while accounting for gamma and uniform priors. ACIs and BCIs are built using IFP and NIFP as a foundation. Simulation study is carried out to compare the performance of estimates. According to simulation study, BEs perform better than MLEs in the majority of situations. When adopting gamma and uniform priors, respectively, the MSEs of BEs typically take the largest value for Scheme 1 and the smallest value for Scheme 2. The BEs under gamma priors have a higher coverage probability than the ML and BEs with uniform priors. Finally, one set of actual data supports the suggested estimations.

REFERENCES

[1] Lomax, K. S., "Business failures: Another example of the analysis of failure data", Journal of the American Statistical Association, 49 (268): 847–852, (1954).

[2] Harris, C. M., "The Pareto distribution as a queue service discipline", Operations Research, 16 (2): 307–313, (1968).

[3] Atkinson, A.B., and Harrison, A.J., "Distribution of Personal Wealth in Britain", Cambridge University Cambridge, (1987).

[4] Holland, O., Golaup, A., and Aghvami, A. H., "Traffic characteristics of aggregated module downloads for mobile terminal reconfiguration", IEE Proceedings- Communications, 153(5): 683–690, (2006). http: //dx.doi.org/10.1049/ip-com:20045155.

[5] Hassan, A. S., and Al-Ghamdi, A. S., "Optimum step stress accelerated life testing for Lomax distribution", Journal of Applied Sciences Research, 5: 2153–2164, (2009).

[6] Hassan, A. S., Assar, S. M., and Shelbaia, A., "Optimum step stress accelerated life test plan for Lomax distribution with an adaptive type-II progressive hybrid censoring", British Journal of Mathematics and Computer Science, 13(2): 1–19, (2016).

[7] Hassan, A.S., and Mohamed, R.E, " Parameter estimation of inverse exponentiated Lomax with right censored data", Gazi University Journal of Science, 32(4): 1370–1386,(2019).

[8] Muhammad, IJAZ, " Bayesian estimation of the shape parameter of Lomax distribution under uniform and Jeffery prior with engineering applications", Gazi University Journal of Science, 34(2): 562–577, (2021).

[9] Hassan, A.S., and Ismail, D, "Estimation of parameters of Topp-Leone inverse Lomax distribution in presence of right censored samples", Gazi University Journal of Science, 34(4): 1193–1208, (2021).

[10] Ahmad, A., Ahmad, S.P., and Ahmed, A., Length-biased weighted Lomax distribution: statistical properties and application", Pakistan Journal of Statistics and Operation Research, 12: 245-255, (2016).

[11] Karimi, H., and Nasiri, P., "Estimation parameter of R = P(Y < X) for length-biased weighted Lomax distributions in the presence of outliers," Mathematical and Computational Applications, 23(9): 1–9, (2018).

[12] Bantan, R., Hassan, A.S., Almetwally, E., Elgarhy, M. Jamal, F., Chesneau, C., and Elsehetry, M., "Bayesian analysis in partially accelerated life tests for weighted Lomax distribution", Computers, Materials & Continua, 68 (3): 2859–2875, (2021).

[13] Hofmann, G., Cramer, E., Balakrishnan, N., and Kunert, G., "An asymptotic approach to progressive censoring", Journal of Statistical Planning and Inference, 130: 207–227, (2005).

[14] Krishna, H., and Kumar, K., “Reliability estimation in Lindley distribution with progressively type II right censored sample”, Mathematics and Computers in Simulation, 82, 281–294, (2011).

[15] Kohansal, A.,“ Statistical analysis of two-parameter bathtub-shaped lifetime distribution under progressive censoring with binomial removals”, Gazi University Journal of Science, 29(4): 783-792, (2016).

[16] Cetinkaya, C., “Estimation in step-stress partially accelerated life tests for the power Lindley distribution under progressive censoring”, Gazi University Journal of Science, 34(2): 579–590, (2021).

[17] Shrahili, M., El-Saeed, A.R., Hassan, A.S., Elbatal, I., and Elgarhy, M., “Estimation of entropy for log-Logistic distribution under progressive type II censoring”, Journal of Nanomaterials, (2022) doi.org/10.1155/2022/2739606.

[18] Hassan, A. S., Mousa, R. M., and Abu-Moussa, M. H., “Analysis of progressive type-II competing risks data with applications”, Lobachevskii Journal of Mathematics, 43 (9): 2479–2492, (2022).

[19] Akdogan, Y., Kus, C., and Wu,S-J.,“ Planning life tests for Burr XII distributed products under progressive group-censoring with cost considerations”, Gazi University Journal of Science, 25(2):425-434 (2012).

[20] Balakrishnan, N., and Aggarwala, R., “Progressive Censoring Theory, Methods and Applications”, Birkhauser Boston, MA, (2000).

[21] Cohen, A. C., “Maximum likelihood estimation in the Weibull distribution based on complete and censored samples”, Technometrics, 7: 579−588, (1965).

[22] Lawless, J.F., “Statistical models and methods for lifetime data”, Wiley, New York, (1982)

[23] Greene, W.H., “Econometric analysis”, 4th edn. Prentice-Hall, New York, (2000)

[24] Balakrishnan, N., and Sandhu, R.A., “A simple simulation algorithm for generating progressively type II censored samples”, American Statistical Association, 49 (2): 229–230, (1995).

[25] Dey, S., and Pradhan, B., “Generalized inverted exponential distribution under hybrid censoring”, Statistical Methodology, 18: 101−114, (2014).

[26] Dey, S., Singh, S., Tripathi, Y.M., and Asgharzadeh, A., “Estimation and prediction for a progressively censored generalized inverted exponential distribution”, Statistical Methodology, 32: 185−202, (2016).

[27] Hassan, A. S., and Assar, S. M., “The exponentiated Weibull-power function distribution. Journal of Data Sciences”, 15(4): 589-614, (2017).

[28] Gadde, S.R., and Al-Omari, A.I., “Attribute control charts based on TLT for length-biased weighted Lomax distribution” Journal of Mathematics,(2022).

Appendix

Table A1. *MSE and Bias for different Estimates of parameters, RF, and HRF at* $\lambda = 0.5$ *and* $\theta = 1.5$

			Scheme I					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
50	**20**	λ	0.153	0.308	0.152	0.024	0.153	0.024
		θ	0.085	0.078	0.084	0.008	0.085	0.008
		$S(t)$	0.260	0.067	0.259	0.067	0.260	0.068
		$h(t)$	0.125	0.016	0.125	0.016	0.125	0.016
	30	λ	0.109	0.143	0.109	0.012	0.108	0.012
		θ	0.055	0.035	0.055	0.004	0.054	0.003
		$S(t)$	0.273	0.075	0.273	0.075	0.273	0.075
		$h(t)$	0.134	0.018	0.134	0.018	0.134	0.018
100	**20**	λ	0.129	0.167	0.128	0.017	0.128	0.017
		θ	0.068	0.055	0.067	0.005	0.067	0.005
		$S(t)$	0.304	0.093	0.304	0.092	0.304	0.093
		$h(t)$	0.150	0.022	0.150	0.022	0.150	0.022
	50	λ	0.065	0.059	0.062	0.004	0.052	0.003
		θ	0.032	0.014	0.029	0.001	0.027	0.001
		$S(t)$	0.266	0.071	0.302	0.091	0.334	0.112
		$h(t)$	0.135	0.018	0.144	0.021	0.146	0.021
	70	λ	0.038	0.034	0.037	0.002	0.038	0.002
		θ	0.015	0.008	0.015	0.001	0.015	0.001
		$S(t)$	0.247	0.061	0.247	0.061	0.247	0.061
		$h(t)$	0.122	0.015	0.122	0.015	0.122	0.015
150	**50**	λ	0.060	0.053	0.064	0.005	0.062	0.004
		θ	0.028	0.014	0.032	0.001	0.030	0.001
		$S(t)$	0.248	0.062	0.308	0.095	0.346	0.119
		$h(t)$	0.128	0.016	0.147	0.022	0.160	0.026
	70	λ	0.031	0.027	0.031	0.001	0.029	0.001
		θ	0.016	0.008	0.015	0.001	0.017	0.001
		$S(t)$	0.287	0.082	0.286	0.082	0.287	0.083
		$h(t)$	0.139	0.019	0.139	0.019	0.138	0.019
	100	λ	0.025	0.021	0.038	0.002	0.012	0.002
		θ	0.009	0.005	0.016	0.001	0.015	0.001
		$S(t)$	0.325	0.105	0.339	0.115	0.357	0.128
		$h(t)$	0.151	0.023	0.149	0.022	0.157	0.025
	130	λ	0.021	0.017	0.021	0.001	0.003	0.001
		θ	0.012	0.004	0.008	0.000	0.011	0.001
		$S(t)$	0.275	0.076	0.264	0.070	0.310	0.096
		$h(t)$	0.131	0.017	0.127	0.016	0.144	0.021

Continued Table A1.

		Scheme 2						
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
50	**20**	λ	0.162	0.288	0.161	0.026	0.161	0.026
		θ	0.080	0.068	0.079	0.007	0.080	0.007
		$S(t)$	0.431	0.186	0.431	0.185	0.431	0.186
		$h(t)$	0.183	0.033	0.183	0.033	0.183	0.033
	30	λ	0.104	0.114	0.104	0.011	0.103	0.011
		θ	0.050	0.027	0.050	0.003	0.050	0.003
		$S(t)$	0.261	0.068	0.261	0.068	0.261	0.068
		$h(t)$	0.128	0.016	0.128	0.016	0.128	0.016
100	**20**	λ	0.098	0.168	0.097	0.010	0.098	0.010
		θ	0.060	0.058	0.060	0.004	0.059	0.004
		$S(t)$	0.149	0.022	0.149	0.022	0.148	0.022
		$h(t)$	0.109	0.012	0.108	0.012	0.109	0.012
	50	λ	0.057	0.051	0.067	0.005	0.062	0.004
		θ	0.029	0.013	0.027	0.001	0.028	0.001
		$S(t)$	0.273	0.074	0.273	0.075	0.291	0.085
		$h(t)$	0.135	0.018	0.139	0.019	0.137	0.019
	70	λ	0.044	0.034	0.044	0.002	0.043	0.002
		θ	0.018	0.008	0.017	0.001	0.018	0.001
		$S(t)$	0.333	0.111	0.332	0.110	0.333	0.111
		$h(t)$	0.163	0.027	0.163	0.027	0.163	0.027
150	**50**	λ	0.058	0.046	0.053	0.003	0.050	0.003
		θ	0.033	0.015	0.026	0.001	0.024	0.001
		$S(t)$	0.327	0.107	0.306	0.094	0.349	0.122
		$h(t)$	0.151	0.023	0.143	0.021	0.158	0.025
	70	λ	0.041	0.032	0.042	0.002	0.041	0.002
		θ	0.018	0.008	0.018	0.001	0.018	0.001
		$S(t)$	0.283	0.080	0.283	0.080	0.284	0.080
		$h(t)$	0.134	0.018	0.134	0.018	0.134	0.018
	100	λ	0.023	0.019	0.029	0.001	0.029	0.001
		θ	0.012	0.005	0.015	0.001	0.009	0.001
		$S(t)$	0.294	0.086	0.273	0.075	0.255	0.065
		$h(t)$	0.139	0.019	0.127	0.016	0.122	0.015
	130	λ	0.012	0.017	0.025	0.001	0.028	0.001
		θ	0.012	0.004	0.009	0.000	0.010	0.001
		$S(t)$	0.333	0.111	0.307	0.094	0.320	0.102
		$h(t)$	0.162	0.026	0.144	0.021	0.148	0.022

Table A2. *MSE and Bias for different Estimates of parameters, RF, and HRF at* $\lambda=2$ *and* $\theta=1.5$

		Scheme 1						
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
50	**20**	λ	0.376	2.225	0.153	0.024	0.452	0.205
		θ	0.050	0.049	0.085	0.008	0.068	0.005
		$S(t)$	0.328	0.108	0.260	0.068	0.433	0.187
		$h(t)$	0.042	0.002	0.125	0.016	0.057	0.003
	30	λ	0.302	0.897	0.108	0.012	0.322	0.104
		θ	0.045	0.022	0.054	0.003	0.050	0.003
		$S(t)$	0.506	0.256	0.273	0.075	0.459	0.211
		$h(t)$	0.060	0.004	0.134	0.018	0.056	0.003
100	**20**	λ	0.389	1.456	0.128	0.017	0.356	0.127
		θ	0.070	0.049	0.067	0.005	0.056	0.004
		$S(t)$	0.442	0.195	0.304	0.093	0.500	0.250
		$h(t)$	0.053	0.003	0.150	0.022	0.063	0.004
	50	λ	0.179	0.446	0.052	0.003	0.169	0.029
		θ	0.029	0.013	0.027	0.001	0.026	0.001
		$S(t)$	0.425	0.181	0.334	0.112	0.429	0.184
		$h(t)$	0.051	0.003	0.146	0.021	0.051	0.003
	70	λ	0.107	0.268	0.038	0.002	0.128	0.017
		θ	0.017	0.007	0.015	0.001	0.023	0.001
		$S(t)$	0.403	0.163	0.247	0.061	0.481	0.231
		$h(t)$	0.049	0.002	0.122	0.015	0.059	0.003
150	**50**	λ	0.188	0.488	0.062	0.004	0.140	0.020
		θ	0.028	0.013	0.030	0.001	0.022	0.001
		$S(t)$	0.422	0.178	0.346	0.119	0.458	0.210
		$h(t)$	0.055	0.003	0.160	0.026	0.056	0.003
	70	λ	0.130	0.298	0.029	0.001	0.115	0.014
		θ	0.020	0.008	0.017	0.001	0.018	0.001
		$S(t)$	0.484	0.234	0.287	0.083	0.378	0.143
		$h(t)$	0.059	0.003	0.138	0.019	0.048	0.002
	100	λ	0.094	0.193	0.012	0.002	0.098	0.010
		θ	0.015	0.006	0.015	0.001	0.016	0.001
		$S(t)$	0.431	0.186	0.357	0.128	0.434	0.188
		$h(t)$	0.050	0.003	0.157	0.025	0.053	0.003
	130	λ	0.046	0.132	0.003	0.001	0.091	0.009
		θ	0.007	0.004	0.011	0.001	0.014	0.001
		$S(t)$	0.426	0.181	0.310	0.096	0.412	0.170
		$h(t)$	0.050	0.003	0.144	0.021	0.050	0.002

Continued Table A2.

			Scheme 2					
n	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
50	**20**	λ	0.369	1.371	0.161	0.026	0.368	0.136
		θ	0.054	0.041	0.080	0.007	0.056	0.004
		$S(t)$	0.500	0.250	0.431	0.186	0.496	0.246
		$h(t)$	0.059	0.004	0.183	0.033	0.063	0.004
	30	λ	0.249	0.718	0.103	0.011	0.285	0.082
		θ	0.038	0.019	0.050	0.003	0.042	0.002
		$S(t)$	0.491	0.241	0.261	0.068	0.433	0.188
		$h(t)$	0.058	0.003	0.128	0.016	0.054	0.003
100	**20**	λ	0.377	1.356	0.098	0.010	0.329	0.109
		θ	0.069	0.054	0.059	0.004	0.065	0.005
		$S(t)$	0.469	0.220	0.148	0.022	0.502	0.252
		$h(t)$	0.059	0.004	0.109	0.012	0.066	0.004
	50	λ	0.155	0.407	0.062	0.004	0.158	0.026
		θ	0.028	0.012	0.028	0.001	0.025	0.001
		$S(t)$	0.453	0.206	0.291	0.085	0.417	0.174
		$h(t)$	0.056	0.003	0.137	0.019	0.052	0.003
	70	λ	0.086	0.244	0.043	0.002	0.162	0.027
		θ	0.012	0.007	0.018	0.001	0.020	0.001
		$S(t)$	0.392	0.154	0.333	0.111	0.451	0.203
		$h(t)$	0.047	0.002	0.163	0.027	0.057	0.003
150	**50**	λ	0.181	0.446	0.050	0.003	0.142	0.021
		θ	0.033	0.014	0.024	0.001	0.025	0.001
		$S(t)$	0.482	0.233	0.349	0.122	0.408	0.166
		$h(t)$	0.060	0.004	0.158	0.025	0.050	0.003
	70	λ	0.105	0.252	0.041	0.002	0.124	0.016
		θ	0.013	0.007	0.018	0.001	0.018	0.001
		$S(t)$	0.422	0.178	0.284	0.080	0.391	0.153
		$h(t)$	0.052	0.003	0.134	0.018	0.046	0.002
	100	λ	0.094	0.197	0.029	0.001	0.099	0.010
		θ	0.014	0.005	0.009	0.001	0.014	0.001
		$S(t)$	0.408	0.166	0.255	0.065	0.436	0.190
		$h(t)$	0.048	0.002	0.122	0.015	0.053	0.003
	130	λ	0.051	0.128	0.028	0.001	0.072	0.006
		θ	0.010	0.004	0.010	0.001	0.010	0.001
		$S(t)$	0.429	0.184	0.320	0.102	0.464	0.215
		$h(t)$	0.051	0.003	0.148	0.022	0.056	0.003

Table A3. *AIL and CP for different estimates for parameters, RF, and HRF at* $\lambda = 0.5$ *and* $\theta = 1.5$

			Scheme 1					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
50	**20**	λ	2.094	97.9	0.082	97.3	0.081	97.4
		θ	1.040	96.0	0.086	97.9	0.086	97.5
		$S(t)$	0.922	95.0	0.977	100.0	0.977	100.0
		$h(t)$	0.269	95.0	0.275	100.0	0.276	100.0
	30	λ	1.417	96.2	0.082	97.2	0.076	97.7
		θ	0.696	96.0	0.086	96.8	0.085	98.3
		$S(t)$	0.855	96.7	0.856	100.0	0.855	100.0
		$h(t)$	0.272	96.7	0.271	100.0	0.272	100.0
100	**20**	λ	1.524	96.3	0.087	97.3	0.079	97.0
		θ	0.877	96.3	0.083	97.3	0.088	97.1
		$S(t)$	0.916	95.0	0.928	100.0	0.928	100.0
		$h(t)$	0.268	95.0	0.277	100.0	0.277	100.0
	50	λ	0.914	95.4	0.080	98.1	0.084	97.4
		θ	0.451	96.0	0.081	97.5	0.084	99.4
		$S(t)$	0.949	96.0	0.903	96.0	0.903	96.0
		$h(t)$	0.286	96.0	0.284	100.0	0.278	96.0
	70	λ	0.703	95.4	0.083	98.2	0.083	97.8
		θ	0.341	95.5	0.076	97.7	0.085	9.8
		$S(t)$	0.977	97.1	0.955	100.0	0.955	100.0
		$h(t)$	0.289	97.1	0.289	95.7	0.289	95.7
150	**50**	λ	0.871	95.8	0.086	96.8	0.086	96.7
		θ	0.456	96.3	0.080	98.0	0.085	97.3
		$S(t)$	0.951	96.0	0.959	96.0	0.954	96.0
		$h(t)$	0.286	96.0	0.288	98.0	0.281	96.0
	70	λ	0.637	95.8	0.082	97.9	0.086	96.3
		θ	0.347	95.4	0.078	9.8	0.087	97.5
		$S(t)$	0.945	97.1	0.939	98.6	0.939	98.6
		$h(t)$	0.294	97.1	0.287	95.7	0.289	95.7
	100	λ	0.558	95.0	0.084	98.5	0.085	97.9
		θ	0.283	96.4	0.080	97.6	0.080	98.3
		$S(t)$	0.935	97.0	0.921	97.0	0.844	97.0
		$h(t)$	0.293	97.0	0.283	96.0	0.279	96.0
	130	λ	0.507	95.5	0.085	98.1	0.085	97.5
		θ	0.258	96.0	0.078	97.5	0.083	96.4
		$S(t)$	0.933	96.9	0.929	99.2	0.944	100.0
		$h(t)$	0.298	96.9	0.294	99.2	0.292	95.4

Continued Table A3.

			Scheme 2					
n	***m***	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
50	**20**	λ	2.008	96.2	0.083	97.7	0.085	98.5
		θ	0.976	95.9	0.084	96.5	0.083	97.9
		$S(t)$	0.923	95.0	0.944	100.0	0.944	100.0
		$h(t)$	0.255	95.0	0.265	100.0	0.265	100.0
	30	λ	1.260	95.9	0.079	97.9	0.083	97.6
		θ	0.617	95.9	0.084	98.5	0.081	98.1
		$S(t)$	0.956	96.7	0.953	100.0	0.953	100.0
		$h(t)$	0.273	96.7	0.271	96.7	0.271	96.7
100	**20**	λ	1.559	96.8	0.084	98.3	0.085	99.3
		θ	0.913	97.0	0.076	97.6	0.085	97.9
		$S(t)$	0.977	95.0	0.983	100.0	0.983	100.0
		$h(t)$	0.281	95.0	0.284	100.0	0.283	100.0
	50	λ	0.860	95.0	0.085	96.4	0.083	97.9
		θ	0.431	96.3	0.082	98.0	0.085	97.8
		$S(t)$	0.949	96.0	0.865	100.0	0.942	100.0
		$h(t)$	0.283	96.0	0.275	100.0	0.284	96.0
	70	λ	0.702	95.5	0.085	97.2	0.083	98.2
		θ	0.344	95.7	0.082	97.7	0.085	97.9
		$S(t)$	0.957	97.1	0.938	100.0	0.938	100.0
		$h(t)$	0.287	97.1	0.285	95.7	0.287	95.7
150	**50**	λ	0.814	95.6	0.087	97.3	0.083	97.0
		θ	0.455	95.7	0.078	98.2	0.091	98.3
		$S(t)$	0.937	96.0	0.951	100.0	0.941	96.0
		$h(t)$	0.288	96.0	0.288	96.0	0.277	96.0
	70	λ	0.687	95.8	0.080	97.7	0.087	97.2
		θ	0.340	96.2	0.085	97.7	0.084	99.2
		$S(t)$	0.915	97.1	0.912	100.0	0.912	100.0
		$h(t)$	0.285	97.1	0.282	95.7	0.282	95.7
	100	λ	0.527	96.2	0.084	98.8	0.085	97.3
		θ	0.272	96.1	0.083	98.1	0.081	98.6
		$S(t)$	0.931	97.0	0.916	96.0	0.941	96.0
		$h(t)$	0.291	97.0	0.294	97.0	0.288	96.0
	130	λ	0.498	95.8	0.086	97.2	0.088	97.4
		θ	0.253	96.3	0.077	97.0	0.081	98.0
		$S(t)$	0.955	96.9	0.948	95.4	0.908	95.4
		$h(t)$	0.287	96.9	0.288	95.4	0.293	95.4

Table A4. *AIL and CP for different estimates for parameters, RF, and HRF at* $\lambda=2$ *and* $\theta=1.5$

			Scheme 1					
N	**m**	**Estimate**	**ML**		**IFP**		**NIFP**	
			AIL	**CP**	**AIL**	**CP**	**AIL**	**CP**
50	**20**	λ	5.664	97.4	0.081	97.4	0.087	97.7
		θ	0.859	96.9	0.086	97.5	0.083	97.9
		$S(t)$	0.807	96.0	0.977	100.0	0.943	100.0
		$h(t)$	0.085	96.0	0.276	100.0	0.088	96.0
	30	λ	3.520	94.8	0.076	97.7	0.081	97.3
		θ	0.560	95.8	0.085	98.3	0.084	96.8
		$S(t)$	0.908	97.1	0.855	100.0	0.915	97.1
		$h(t)$	0.091	97.1	0.272	100.0	0.091	97.1
100	**20**	λ	4.480	95.4	0.079	97.0	0.083	98.3
		θ	0.823	95.5	0.088	97.1	0.083	98.2
		$S(t)$	0.895	96.0	0.928	100.0	0.949	100.0
		$h(t)$	0.092	96.0	0.277	100.0	0.087	96.0
	50	λ	2.524	96.2	0.084	97.4	0.084	96.9
		θ	0.424	95.9	0.084	99.4	0.084	97.5
		$S(t)$	0.954	96.0	0.903	96.0	0.940	96.0
		$h(t)$	0.096	96.0	0.278	96.0	0.094	96.0
	70	λ	1.985	96.2	0.083	97.8	0.087	97.0
		θ	0.329	96.1	0.085	9.8	0.085	97.9
		$S(t)$	0.869	97.3	0.955	100.0	0.937	98.7
		$h(t)$	0.097	97.3	0.289	95.7	0.096	96.0
150	**50**	λ	2.639	95.6	0.086	96.7	0.084	97.5
		θ	0.427	95.8	0.085	97.3	0.081	98.2
		$S(t)$	0.918	96.0	0.954	96.0	0.965	100.0
		$h(t)$	0.094	96.0	0.281	96.0	0.096	96.0
	70	λ	2.080	95.2	0.086	96.3	0.085	97.2
		θ	0.352	95.6	0.087	97.5	0.081	97.5
		$S(t)$	0.949	97.3	0.939	98.6	0.866	100.0
		$h(t)$	0.096	97.3	0.289	95.7	0.095	100.0
	100	λ	1.684	95.7	0.085	97.9	0.084	97.2
		θ	0.286	96.2	0.080	98.3	0.082	98.6
		$S(t)$	0.907	97.0	0.844	97.0	0.935	100.0
		$h(t)$	0.098	97.0	0.279	96.0	0.097	96.0
	130	λ	1.415	95.6	0.085	97.5	0.082	96.5
		θ	0.235	96.4	0.083	96.4	0.085	97.0
		$S(t)$	0.934	96.9	0.944	100.0	0.941	97.7
		$h(t)$	0.099	96.9	0.292	95.4	0.098	100.0

Continued Table A4.

n	m	Estimate	Scheme 2					
			ML		IFP		NIFP	
			AIL	CP	AIL	CP	AIL	CP
50	**20**	λ	4.332	95.5	0.085	98.5	0.084	96.9
		θ	0.750	96.3	0.083	97.9	0.084	97.2
		$S(t)$	0.964	96.0	0.944	100.0	0.929	96.0
		$h(t)$	0.091	96.0	0.265	100.0	0.089	100.0
	30	λ	3.172	94.3	0.083	97.6	0.081	98.0
		θ	0.517	94.9	0.081	98.1	0.090	97.2
		$S(t)$	0.879	97.1	0.953	100.0	0.975	97.1
		$h(t)$	0.093	97.1	0.271	96.7	0.093	100.0
100	**20**	λ	4.321	96.1	0.085	99.3	0.081	97.8
		θ	0.868	96.6	0.085	97.9	0.083	96.7
		$S(t)$	0.889	96.0	0.983	100.0	0.915	100.0
		$h(t)$	0.090	96.0	0.283	100.0	0.089	100.0
	50	λ	2.429	95.6	0.083	97.9	0.085	98.1
		θ	0.420	96.4	0.085	97.8	0.086	98.0
		$S(t)$	0.935	96.0	0.942	100.0	0.946	96.0
		$h(t)$	0.096	96.0	0.284	96.0	0.096	96.0
	70	λ	1.908	95.4	0.083	98.2	0.081	97.3
		θ	0.332	96.7	0.085	97.9	0.083	99.0
		$S(t)$	0.914	97.3	0.938	100.0	0.940	97.3
		$h(t)$	0.097	97.3	0.287	95.7	0.095	97.3
150	**50**	λ	2.520	95.7	0.083	97.0	0.085	98.4
		θ	0.450	95.3	0.091	98.3	0.087	99.2
		$S(t)$	0.969	96.0	0.941	96.0	0.927	100.0
		$h(t)$	0.097	96.0	0.277	96.0	0.097	100.0
	70	λ	1.924	95.8	0.087	97.2	0.085	97.7
		θ	0.331	96.1	0.084	99.2	0.085	97.4
		$S(t)$	0.914	97.3	0.912	100.0	0.933	100.0
		$h(t)$	0.096	97.3	0.282	95.7	0.097	100.0
	100	λ	1.701	96.0	0.085	97.3	0.087	98.0
		θ	0.283	96.1	0.081	98.6	0.079	97.5
		$S(t)$	0.965	97.0	0.941	96.0	0.946	100.0
		$h(t)$	0.098	97.0	0.288	96.0	0.097	96.0
	130	λ	1.390	96.4	0.088	97.4	0.084	98.2
		θ	0.238	95.8	0.081	98.0	0.084	96.7
		$S(t)$	0.921	96.9	0.908	95.4	0.939	98.5
		$h(t)$	0.099	96.9	0.293	95.4	0.097	95.4

Bayesian Analysis of a Non-Identical Component-Strengths System Using Upper Record Ranked Set Sampling from the Bathtub-Shaped Distribution

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Abstract

To build more flexible models in the reliability theory field, the statistical inference of stress-strength reliability is investigated in a multicomponent system with non-identical component strengths based on a bathtub-shaped distribution. In the context of upper record ranked set sampling, classical and Bayesian methodologies are used to assess the system's reliability. In classical estimation, the reliability's maximum likelihood estimator is developed, and the behavior of various estimates is evaluated using a simulation study based on absolute biases and mean squared errors. The Bayesian estimators of reliability under squared error and linear exponential (LINEX) loss functions are computed. Since these estimators are incapable of being reduced to simple closed forms, we employ the Markov Chain Monte Carlo approach. Through simulated research, we discovered that as the number of records increases, measurement accuracy decreases. The Bayesian estimates via the LINEX loss function generally hold the lowest values in the majority of cases. An illustration drawn from real data sets is used to verify the theoretical findings.

Keywords: *Bathtub-shaped model, Upper record ranked set sampling, Bayesian inference, Stress-strength model.*

1. Introduction

In many practical fields, including hydrology, sports, medicine, life testing, etc., record values (RVs) have recently become an important area of research. Chandler (1952) proposed the fundamental concept of record values. In statistics, records are defined as the extremes that follow one another in a series of random variables. The greatest (smallest) number derived from a series of random variables is referred to as an upper (lower) RV. Record values have the advantage of requiring fewer measures than a whole sample, which is particularly advantageous for damaging studies when expensive measurements must be conducted.

A new sampling strategy for generating record-breaking data was introduced and called record ranked set sampling (**RRSS**). Salehi and Ahmadi (2014) introduced the RRSS system to aid scientists in instances when the only observations that are going to be used are the last record-breaking data, such as athletic data, weather data, and Olympic data. Assuming there are *n* independent sequential sequences of continuous random variables, the i^{th} sequence sampling is stopped when the i^{th} RV is achieved. Only the last RV in each sequence is used as an observation for analysis. The last RV of the i^{th} sequence in this plane is denoted by $U_{i,i}$, then the available observations are $\underline{U}_{i,i} = (U_{1,1}, U_{2,2}, \ldots, U_{n,n})^T$. The following diagram can be used to illustrate this observational process:

$$
\begin{array}{lllllll}
1: & \boldsymbol{U}_{(1)1} & \boldsymbol{U}_{(2)1} & \boldsymbol{U}_{(3)1} & \cdots & \boldsymbol{U}_{(m)1} & \rightarrow & \boldsymbol{U}_{(1)1}=\boldsymbol{U}_{1,1} \\
2: & \boldsymbol{U}_{(1)2} & \boldsymbol{U}_{(2)2} & \boldsymbol{U}_{(3)2} & \cdots & \boldsymbol{U}_{(m)2} & \rightarrow & \boldsymbol{U}_{(2)2}=\boldsymbol{U}_{2,2} \\
\vdots & \vdots & \vdots & \ddots & \ldots & \vdots & \rightarrow & \vdots \\
\mathrm{n}: & \boldsymbol{U}_{(1)n} & \boldsymbol{U}_{(2)n} & \boldsymbol{U}_{(3)n} & \cdots & \boldsymbol{U}_{(m)n} & \rightarrow & \boldsymbol{U}_{(m)n}=\boldsymbol{U}_{\mathrm{n,n}}
\end{array}
$$

where, $U_{(i)j}$ is the i^{th} ordinary upper (lower) record in the j^{th} sequence. Unlike ordinary records, these $U_{i,i}$'s are independent random variables that are not necessarily ordered.

In reliability testing, the stress-strength (SS) model is essential. When a stress *Y* is greater than a strength *X*, $\Re = P(X > Y)$ it is a measure of component reliability; to put it another way, the system continues to work so long as the stress does not exceed its capacity. This concept was first presented by Birnbaum (1956), and it was later developed by Birnbaum and McCarty (1958). Inferences based on various techniques and distributional presumptions have been the subject of numerous studies; for some recent works, see Sharma (2018), Wang *et al.* (2018), Hassan *et al.* (2018), Sadeghpour *et al.* (2020), and Hassan *et al.* (2021).

The fundamental concept of $\Re = P(X > Y)$ can be altered to create a system with two or extra components. Bhattacharyya and Johnson (1974) investigated the multi-component SS (MSS) model under the concept that *q* out of *p* system components, where $(1 \le q \le p)$ components resist a common random stress *Y*. The MSS is relevant to a variety of industries, including military, manufacturing, and logistical operations. An electrical power plant, for instance, can only generate the right amount of energy if at least six out of its eight generating units are running.

Assume that a system with *p* similar components will operate if $q\,(1 \le q \le p)$ or more of the components work together. A stress *Y* that is a random variable with a cumulative distribution function (cdf) *G*(*y*) is applied to the system in its operational environment. The component strengths, or the minimal stresses necessary to manufacture failure, are identically independent, distributed (iid) random variables with cdf *F*(*x*), then the reliability of the *q*-out-of-*p* system, denoted by $\Re_{q,p}$, that advanced by Bhattacharyya and Johnson (1974), is given by

$$
\begin{aligned}
\Re_{q,p} &= \Pr\left[at\, least\, q\, of\ (X_1, X_2, ..., X_p)\ \ exceed\, Y \right] \\
&= \sum_{a=q}^{p} \binom{p}{a} \int_0^{\infty} [1 - F(x)]^a F(x)^{p-a}\, dG_Y(x).
\end{aligned}
$$

The evaluation of MSS models' reliability using different SS distributions and sampling techniques was a topic of study for many academics, for instance, Hassan and Basheikh (2012a) Rao *et al.* (2015), Rao *et al.* (2017), Dey *et al.* (2017), Pak *et al.* (2019),Kayal *et al.* (2020); Kızılaslan (2018), Hassan *et al.* (2020), Kotb and Raqab (2021), Jana and Bera (2022), Azhad *et al.* (2022) and Hassan *et al.* (2022).

Due to the varied structures of system components, many real-world situations may invalidate the assumption of identical strength distributions. This is often the case with systems that contain backup components. Even when formed of the same material, different things might have different strengths. For example, a metal that has been heat treated to provide acceptable mechanical properties may break in a variety of ways when it is quenched or cooled. The strengths of the components differ as a result. Another example, if two distinct ropes are used to consolidate a rope, the distribution of the tensile strengths of the two ropes could not be equal. It appears that a more realistic model is one that at least includes non-identical random strengths for system components (see Kotz and Pensky (2003)).

Assume a system has p components, of which p_1 are of kind 1, p_2 are of kind 2,…, and the remainder $p_n = p - \sum_{i=1}^{n-1} p_i$ components are of kind n. Let $F_i(.)$, be the cdf of the random strengths for components of the i^{th} kind with i =1, 2,…, n. Assume that Y is a common stress with cdf $Q(.)$ that all components are subjected to. The system will function as long as the q-out-of-the-p components can resist the stress. Johnson (1988) presented the system reliability $\Re_{q_1,\ldots,q_n,p_1,\ldots,p_n}$ with non-identical component strengths as follows:

$$\Re_{c_1,\ldots,c_n,k_1,\ldots,k_n} = \sum_{j_1=c_1}^{k_1} \cdots \sum_{j_n=c_n}^{k_n} \left(\prod_{i=1}^{n} \binom{k_i}{j_i} \right) \int_0^{\infty} \prod_{i=1}^{n} (1-F_i(x))^{j_i} (F_i(x))^{k_i - j_i} dQ_Y(x), \qquad (1)$$

where summation is applied to all combinations (j_1, j_2,…, j_n) with $0 \le j_i \le p_i$ for i = 1, 2, ... , n such $q \le \sum_{i=1}^{n} j_i \le p$. Each q_i indicates the minimal number of components of the i^{th} type that the system needs to function.

Considering the investigation of a system with two different sorts of components, model (1) can be expressed as follows:

$$\Re_{c_1,c_2,k_1,k_2} = \sum_{j_1=c_1}^{k_1} \sum_{j_2=c_2}^{k_2} \binom{k_1}{j_1} \binom{k_2}{j_2} \int_0^{\infty} \prod_{i=1}^{n} (1-F_1(x))^{j_1} (F_1(x))^{k_1 - j_1} (1-F_2(x))^{j_2} (F_2(x))^{k_2 - j_2} dQ_Y(x). \qquad (2)$$

Building more realistic models requires, the Bayesian estimation of $\Re_{q_1,q_2,p_1,p_2}$ implying the Weibull and exponential distributions on the strength and stress variates, respectively, which was taken into consideration by Pandey *et al.* (1992) and Paul and Uddin (1997). Hassan and Basheikh (2012b) estimated $\Re_{q_1,q_2,p_1,p_2}$ for non-identical MSS using the exponentiated Pareto distribution. The situation of non-identical component strengths in the family of Kumaraswamy generalized distributions under upper record data was recently studied by Rasethuntsa and Nadar (2018). Karam *et al.* (2020) examined the estimation of $\Re_{q_1,q_2,p_1,p_2}$ when component strengths and stress follow the inverse Lomax distribution based on complete samples. A non-identical-MSS system's estimation under adaptive hybrid progressive censoring samples was examined by Kohansal *et al.* (2021) who considered the estimation of $\Re_{q_1,q_2,p_1,p_2}$ when component strengths and stress follow bathtub-shaped distribution (**BShD**). Çetinkaya (2021) proposed estimation of $\Re_{q_1,q_2,p_1,p_2}$ when component strengths and stress follow Weibull distributions in a generalized progressive hybrid censoring system. Under adaptive Type-II hybrid progressive censoring samples, Arshad *et al.* (2022) created an estimation of the p-type non-identical MSS from the PRHR family using records.

It is crucial to remember that much of the MSS reliability estimation research done so far has focused on complete or censored samples, and record values have not been used frequently. There is currently no evidence of an RRSS scheme, particularly in the estimate of MSS systems with non-identical component strengths. We are interested in developing MSS models within the RRSS scheme in the case of non-identical component strengths, where component strengths and stress follow a BShD. A maximum likelihood estimator (MLE) of $\Re_{q_1,q_2,p_1,p_2}$ is created for the upper RRSS (URRSS), and a simulation study is investigated. Linear exponential (LINEX) and squared error (SE) loss functions are used to build the Bayesian estimator of $\Re_{q_1,q_2,p_1,p_2}$. For Bayesian estimates of $\Re_{q_1,q_2,p_1,p_2}$, these estimators cannot be reduced to simple closed forms, thus we employ the Markov Chain Monte Carlo (MCMC) approach. We also looked at actual data sets to demonstrate the applicability of our research.

The structure of the paper is organized as follows. A formulation of $\Re_{q_1,q_2,p_1,p_2}$ and its MLE under URRSS, as well as a numerical analysis, are presented in Section 2. Section 3 discusses the Bayesian estimators of $\Re_{q_1,q_2,p_1,p_2}$, under SE and LINEX loss functions. Section 4 of this article presents the MCMC method. For demonstration reasons, real data sets are provided in Section 5. Final remarks are included in Section 6.

2. Model Description & Classical Estimation of $\Re_{q_1,q_2,p_1,p_2}$

In this section, a model description of $\Re_{q_1,q_2,p_1,p_2}$ is provided. The MLE of $\Re_{q_1,q_2,p_1,p_2}$ is obtained in the presence of URRSS. Additionally, numerical analysis is done.

2.1 Expression of $\Re_{q_1,q_2,p_1,p_2}$

Here, formula of MSS reliability is provided when the strength and stress random variables follow the BShD.

The BShD developed by Chen (2000) has been regarded as one of the most significant distributions among the others in lifetime data analysis. It is a significant model that may be applied to research a wide range of issues in experiments on reliability and life testing. Several researchers discussed the BShD's studies and applications; for instance, see (Wu (2008), Rastogi *et al.* (2012), Sarhan *et al.* (2012), and Ahmed (2014)). The BShD 's cdf with scale and shape parameters θ and λ is provided by:

$$G(x) = 1 - \exp\left[\theta\left(1 - e^{x^{\lambda}}\right)\right],\ x,\lambda,\theta > 0,$$

The probability density function (pdf) of BShD is:

$$g(x) = \lambda\theta x^{\lambda-1}\exp\left[\theta\left(1 - e^{x^{\lambda}}\right)\right],\ x,\lambda,\theta > 0.$$

From all of the p system components in the model (2), let the first p_1 of first kind component strengths follow BShD (λ,θ_1), while the remaining $p_2 = p - p_1$ of kind 2 component strengths follow BShD (λ,θ_2). Also suppose that Y follows BShD (λ,θ_3) independently. The respective distribution functions are as below:

$$F_i(x) = 1 - \exp\left[\theta_i\left(1 - e^{x^{\lambda}}\right)\right],\ x,\lambda,\theta_i > 0, i = 1,2, \tag{3}$$

$$Q(y) = 1 - \exp\left[\theta_3\left(1 - e^{y^{\lambda}}\right)\right],\ y,\lambda > 0, \theta_3 > 0. \tag{4}$$

By replacing $F_1(.)$, $F_2(.)$, and $Q(.)$ given in (2) by (3) and (4), the formula of $\Re_{q_1,q_2,p_1,p_2}$ for such a system is as follows:

$$\Re_{q_1,q_2,p_1,p_2} = \sum_{j_1=q_1}^{p_1}\sum_{j_2=q_2}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\int_0^{\infty}\left[\exp\left(\theta_1(1-e^{x^{\lambda}})\right)\right]^{j_1}\left[1-\exp\left(\theta_1(1-e^{x^{\lambda}})\right)\right]^{(p_1-j_1)}\left[\exp\left(\theta_2(1-e^{x^{\lambda}})\right)\right]^{j_2}$$
$$\times\left[1-\exp\left(\theta_2(1-e^{x^{\lambda}})\right)\right]^{(p_2-j_2)}\theta_3\lambda x^{\lambda-1}\exp\left(\theta_3(1-e^{x^{\lambda}})+x^{\lambda}\right)dx.$$

Let $z = \exp\left(1 - e^{\lambda^{x}}\right)$, $dz = -\lambda x^{\lambda-1}e^{x^{\lambda}}\exp\left(1-e^{x^{\lambda}}\right)dx,$ and then inserting in previous equation

$$\Re_{q_1,q_2,p_1,p_2} = \sum_{j_1=q_1}^{p_1}\sum_{j_2=q_2}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\theta_3\int_0^1 z^{\theta_1 j_1}(1-z^{\theta_1})^{(p_1-j_1)}z^{\theta_2 j_2}(1-z^{\theta_2})^{(p_2-j_2)}z^{\theta_3-1}dz.$$

Using the binomial expansion for $(1-z^{\theta_1})^{p_1-j_1}$ and $(1-z^{\theta 2})^{p_2-j_2}$ leads to the following

$$\Re_{q_1,q_2,p_1,p_2} = M_{j_1,j_2,i_1,i_2}\,\theta_3 \int_0^1 z^{\theta_1(i_1+j_1)+\theta_2(i_2+j_2)+\theta_3-1} dz = \frac{M_{j_1,j_2,i_i,i_2}\,\theta_3}{\theta_1(i_i+j_1)+\theta_2(i_2+j_2)+\theta_3}, \qquad (5)$$

where $M_{j_1,j_2,i_1,i_2} = \sum_{j_1=q_1}^{p_1}\sum_{j_2=q_2}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\sum_{i_1=0}^{p_1-j_1}\sum_{i_2=0}^{p_2-j_2}\binom{p_1-j_1}{i_1}\binom{p_2-j_2}{i_2}(-1)^{i_1+i_2}$. Note that expression (5) depends on the parameters θ_1, θ_2 and θ_3.

2.2 MLE of $\Re_{q_1,q_2,p_1,p_2}$ Via URRSS

In this subsection, we obtain MLE of $\Re_{q_1,q_2,p_1,p_2}$ using URRSS data from the BShD. To compute the MLE of $\Re_{q_1,q_2,p_1,p_2}$, denoted by $\hat{\Re}_{q_1,q_2,p_1,p_2}$, the MLE of $\theta_1,\theta_2,\theta_3$ and λ, denoted by $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\lambda}$, must be computed first.

Assume that $\underline{r_{i,i}} = \left(r_{1,1},...,r_{n,n}\right)^T$ represents the n observed random vector $\underline{R}_{i,i} = \left(R_{1,1},...,R_{\mathrm{n,n}}\right)^T$ of the URRSS from BShD(λ,θ_1). Let $\underline{t_{j,j}} = \left(t_{1,1},...,t_{m,m}\right)^T$ be the observed m random vector size-m URRSS $\underline{T}_{j,j} = \left(T_{1,1},...,T_{m,m}\right)^T$ from BShD(λ,θ_2). Also, assume that $\underline{s_{u,u}} = \left(s_{1,1},...,s_{w,w}\right)^T$ is the observation of the random vector $\underline{S}_{u,u} = \left(S_{1,1},...,S_{w,w}\right)^T$ of size w from URRSS obtained from BShD (λ,θ_3). Note that $\underline{R}_{i,i}$, $\underline{T}_{j,j}$ and $\underline{S}_{u,u}$ are independent.

The joint density of URRSS of size n according to Arnold *et al.* (2011), is defined by:

$$L\left(\underline{u_{i,i}}\,|\,\eta\right) = \prod_{i=1}^{n}\frac{\left[-\ln(1-H\,(u_{i,i};\eta))\right]^{i-1}}{(i-1)!}h(u_{i,i};\eta) \qquad ;\eta\in\Theta, \qquad (6)$$

where, $\underline{u_{i,i}} = \left(u_{1,1},u_{2,2},...,u_{\mathrm{n,n}}\right)^T$ is the observed values of $\underline{U_{i,i}}$, η is real valued parameter and Θ is the parameter space, $h(u_{i,i};\eta)$ is the pdf of URRSS and $H\,(u_{i,i};\eta)$ is cdf of URRSS. Hence, the observed URRSS data $\underline{r_{i,i}}, \underline{t_{j,j}}$ and $\underline{s_{u,u}}$, given η, based on (6), are given as below:

$$L_1\left(\underline{r_{i,i}}\,|\,\lambda,\theta_1\right) = (\lambda\theta_1)^n\prod_{i=1}^{n}\frac{\left[-\theta_1\tau_{i,i}\right]^{i-1}}{(i-1)!}\exp\left(\theta_1\tau_{i,i}+(r_{i,i})^{\lambda}\right)(r_{i,i})^{\lambda-1},$$

$$L_2\left(\underline{t_{j,j}}\,|\,\lambda,\theta_2\right) = (\lambda\theta_2)^m\prod_{j=1}^{m}\frac{\left[-\theta_2\varphi_{j,j}\right]^{j-1}}{(j-1)!}\exp\left(\theta_2\varphi_{j,j}+(t_{j,j})^{\lambda}\right)(t_{j,j})^{\lambda-1},$$

$$L_3\left(\underline{s_{u,u}}\,|\,\lambda,\theta_3\right) = (\lambda\theta_3)^w\prod_{u=1}^{w}\frac{\left[-\theta_3\delta_{u,u}\right]^{u-1}}{(u-1)!}\exp\left(\theta_3\delta_{u,u}+(s_{u,u})^{\lambda}\right)(s_{u,u})^{\lambda-1},$$

where $\tau_{i,i} = 1-e^{(r_{i,i})^{\lambda}}$, $\varphi_{j,j} = 1-e^{(t_{j,j})^{\lambda}}$, $\delta_{u,u} = 1-e^{(s_{u,u})^{\lambda}}$, $i = 1,...n,\ j = 1,...m, u = 1,..w$.

The joint likelihood function (LF) of $\eta = (\theta_1,\theta_2,\theta_3)$, based on URRSS, is given by:

$$L\left(\underline{r_{i,i}},\underline{t_{j,j}},\underline{s_{u,u}}\mid\eta\right)=\theta_1^{\ n}\theta_2^{\ m}\theta_3^{\ w}\lambda^{n+m+w}\prod_{i=1}^{n}\frac{\left[-\theta_1\tau_{i,i}\right]^{i-1}}{(i-1)!}\exp\left(\theta_1\tau_{i,i}+(r_{i,i})^{\lambda}\right)(r_{i,i})^{\lambda-1}$$
$$\times\prod_{j=1}^{m}\frac{\left[-\theta_2\varphi_{j,j}\right]^{j-1}}{(j-1)!}\exp\left(\theta_2\varphi_{j,j}+(t_{j,j})^{\lambda}\right)(t_{j,j})^{\lambda-1}$$
$$\times\prod_{u=1}^{w}\frac{\left[-\theta_3\delta_{u,u}\right]^{u-1}}{(u-1)!}\exp\left(\theta_3\delta_{u,u}+(s_{u,u})^{\lambda}\right)(s_{u,u})^{\lambda-1},$$

Consequently, the joint log-LF, denoted by $\ln\ell$, is derived as:

$$\begin{aligned}\ell\propto\ & n\ln\theta_1+m\ln\theta_2+w\ln\theta_3+(n+m+w)\ln\lambda+\sum_{i=1}^{n}\left((i-1)\ln(-\theta_1\tau_{i,i})+(\lambda-1)\ln r_{i,i}\right)\\ &+\sum_{i=1}^{n}\theta_1\tau_{i,i}+r_{i,i}^{\ \lambda}+\sum_{j=1}^{m}\left((j-1)\ln(-\theta_2\varphi_{j,j})+(\lambda-1)\ln t_{j,j}+\theta_2\varphi_{j,j}+t_{j,j}^{\ \lambda}\right)\\ &+\sum_{u=1}^{w}\left((u-1)\ln\left[-\theta_3\delta_{u,u}\right]+(\lambda-1)\ln s_{u,u}+\theta_2\delta_{u,u}+s_{u,u}^{\ \lambda}\right).\end{aligned}\tag{7}$$

Given that λ is known, the partial derivatives of log-LF, with respect to θ_1,θ_2 and θ_3, are as follows:

$$\frac{\partial\ell}{\partial\theta_1}=\frac{n}{\theta_1}+\sum_{i=1}^{n}\left[\frac{(i-1)}{\theta_1}+\tau_{i,i}\right],\tag{8}$$

$$\frac{\partial\ell}{\partial\theta_2}=\frac{m}{\theta_2}+\sum_{j=1}^{m}\left[\frac{(j-1)}{\theta_2}+\varphi_{j,j}\right],\tag{9}$$

$$\frac{\partial\ell}{\partial\theta_3}=\frac{w}{\theta_3}+\sum_{u=1}^{w}\left[\frac{(u-1)}{\theta_3}+\delta_{u,u}\right].\tag{10}$$

Then, the MLEs of θ_1,θ_2 and θ_3, denoted by $\hat{\theta}_1,\hat{\theta}_2$ and $\hat{\theta}_3$, are obtained by setting (8)–(10) to be zero and solving them numerically. Therefore, based on invariance property, we obtain the MLE of $\Re_{q_1,q_2,p_1,p_2}$, say $\hat{\Re}_{q_1,q_2,p_1,p_2}$, by inserting $\hat{\theta}_1,\hat{\theta}_2$ and $\hat{\theta}_3$ in (5) as follows

$$\hat{\Re}_{q_1,q_2,p_1,p_2},=\frac{M_{j_1,j_2,i_i,i_2}\hat{\theta}_3}{\hat{\theta}_1(i_i+j_1)+\hat{\theta}_2(i_2+j_2)+\hat{\theta}_3}.\tag{11}$$

2.3 Numerical Study

In this subsection, the ML estimates for the MSS variables are thoroughly numerically analysed. Absolute biases (**ABs**) and mean squared errors (**MSEs**) are used as measurements to evaluate the precision of estimates for various parameter values and record numbers. The numerical research is carried out in the manner described below:

- Create URRSS from BShD, where

(n, m, w) = (3,3,3), (3,3,5), (3,5,3), (5,5,5), (5,5,6), (6,5,5), (7,7,7) and (7,8,7).

- The parameters values of $(\theta_1,\theta_2,\theta_3)$ = (2,1,3), (1,1,3), (2,0.2,3) and (2,0.2, 2) for λ= 3 in all situations.

- The real values of $\Re_{q_1,q_2,p_1,p_2}$ have the following values at
  a. $(q_1, q_2, p_1, p_2) = (2, 2, 3, 3)$ are 0.563, 0.69, 0.615 and 0.495
  b. $(q_1, q_2, p_1, p_2) = (1, 2, 3, 3)$ are 0.725, 0.779, 0.838 and 0.737
  c. $(q_1, q_2, p_1, p_2) = (2, 1, 3, 3)$ are 0.611, 0.779, 0.618 and 0.499
  d. $(q_1, q_2, p_1, p_2) = (3, 3, 3, 3)$ are 0.25, 0.333, 0.559 and 0.233.
- 10000 repetitions are used to evaluate the ABs and MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$.
- The simulated outcomes are shown in Table 1 and illustrated in Figures 1–6.
- The MSEs and ABs of MSS reliability estimates at $(q_1, q_2, p_1, p_2) = (1, 2, 3, 3)$ are the smallest compared to others for different values of (q_1, q_2, p_1, p_2) (Table 1).The MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ at $(\theta_1,\theta_2,\theta_3) = (2, 1, 3)$ are the smallest in almost all cases (Table 1).
- As URRSS numbers increases, the MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for all values of $(\theta_1,\theta_2,\theta_3)$ decrease (Figure 1).
- Figure 2 demonstrates that the set of values (2, 1, 3) has the smallest MSEs at different sample sizes for $(q_1, q_2, p_1, p_2) = (2, 2, 3, 3)$.

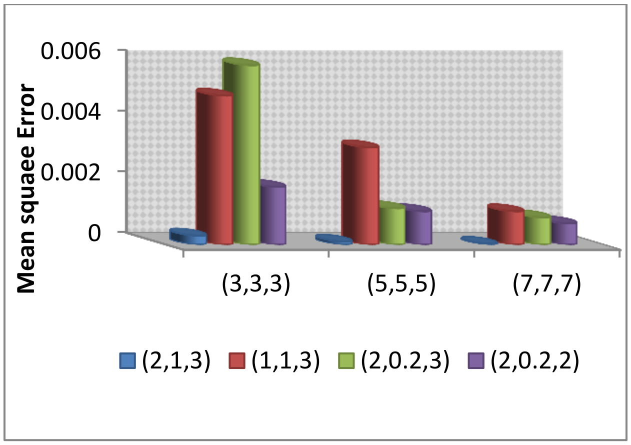

Figure1: MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for different $(\theta_1,\theta_2,\theta_3)$ values at (q_1, q_2, p_1, p_2) =(1,2,3,3) and $n = m = w$

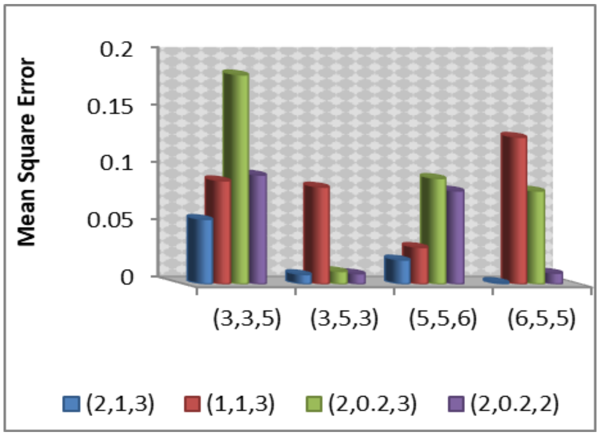

Figure 2: MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for different $(\theta_1,\theta_2,\theta_3)$ values at (q_1, q_2, p_1, p_2)=(2,2,3,3)

- Figure 3 indicates that as the number of n, m and w increases, the ABs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ decrease at all actual values of $(\theta_1,\theta_2,\theta_3)$.
- Figure 4 demonstrates that the MSEs of MSS reliability estimates at $(q_1, q_2, p_1, p_2) = (3, 3, 3, 3)$ are larger than the MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for other values of (q_1, q_2, p_1, p_2) at $(\theta_1,\theta_2,\theta_3) = (2, 1, 3)$.

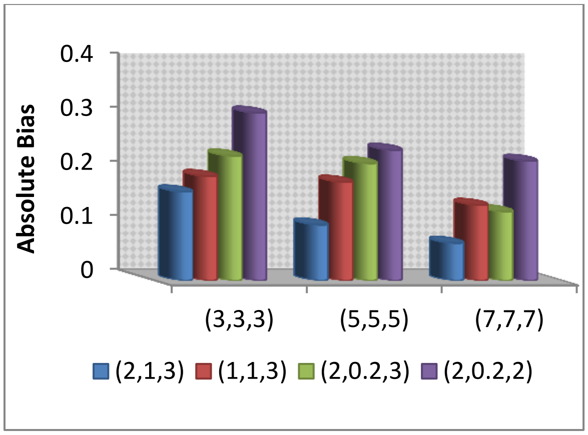

Figure 3: ABs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for different $(\theta_1,\theta_2,\theta_3)$ values at (q_1, q_2, p_1, p_2)=(3,3,3,3) and $n=m=w$

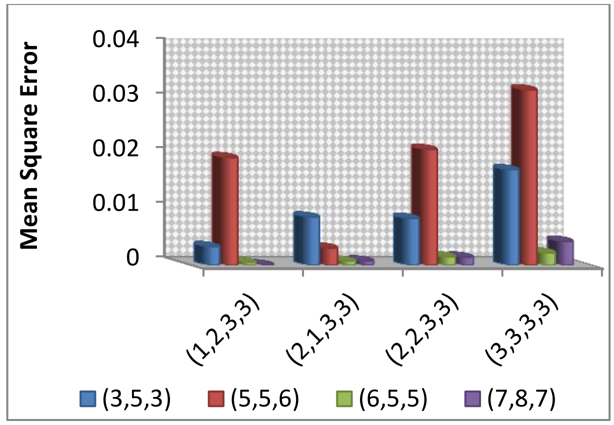

Figure 4: MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for different values of n, m, w at $(\theta_1,\theta_2,\theta_3)=(2,1,3)$

- Figure 5 illustrates that the ABs of MSS reliability estimates at (q_1, q_2, p_1, p_2) = (1, 2, 3, 3) are smaller than the ABs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for others values of (q_1, q_2, p_1, p_2).for all true parameter values.
- Figure 6 illustrates that the MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ decrease when the true value of $\Re_{q_1,q_2,p_1,p_2}$ increases for all values of (q_1, q_2, p_1, p_2).

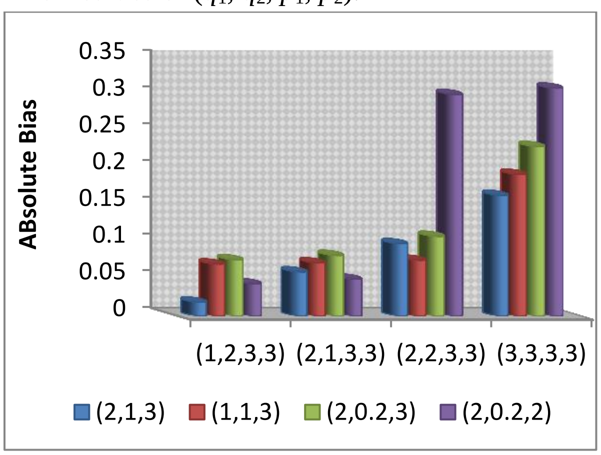

Figure 5: MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for different $(\theta_1,\theta_2,\theta_3)$ and (q_1, q_2, p_1, p_2) values at $n=m=w=3$

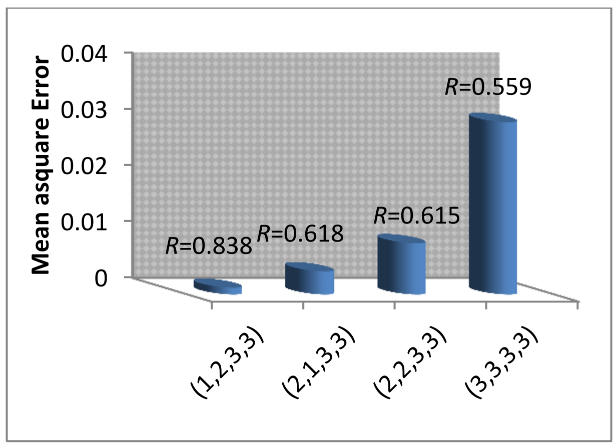

Figure 6: MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for $(\theta_1,\theta_2,\theta_3)=(2,0.2,3)$ at $n=m=w=5$

Table1: Numerical results of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ for different values of $(\theta_1,\theta_2,\theta_3)$

$(\theta_1,\theta_2,\theta_3)=(2,1,3)$					$(\theta_1,\theta_2,\theta_3)=(1,1,3)$				
(q_1,q_2, p_1,p_2)	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n,m,w)**	**AB**	**MSE**	**(q_1,q_2, p_1,p_2)**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n,m,w)**	**AB**	**MSE**
(1,2,3,3)	**0.725**	(3,3,3)	0.0193	0.0003	**(1,2,3,3)**	**0.779**	(3,3,3)	0.0711	0.0049
		(3,3,5)	0.2079	0.0432			(3,3,5)	0.2988	0.0893
		(3,5,3)	0.0585	0.0034			(3,5,3)	0.0191	0.0003
		(5,5,5)	0.0120	0.0001			(5,5,5)	0.0601	0.0032
		(5,5,6)	0.1406	0.0197			(5,5,6)	0.1147	0.0131
		(6,5,5)	0.0249	0.0006			(6,5,5)	0.0988	0.0097
		(7,7,7)	0.0054	0.0001			(7,7,7)	0.0323	0.0011
		(7,8,7)	0.0099	0.0001			(7,8,7)	0.1527	0.0233
(2,1,3,3)	**0.611**	(3,3,3)	0.0605	0.0036	**(2,1,3,3)**	**0.779**	(3,3,3)	0.0727	0.0052
		(3,3,5)	0.0720	0.0066			(3,3,5)	0.3010	0.0899
		(3,5,3)	0.0941	0.0088			(3,5,3)	0.0813	0.0066
		(5,5,5)	0.0323	0.0010			(5,5,5)	0.0641	0.0041

		(5,5,6)	0.0477	0.0032			(5,5,6)	0.1154	0.0142
		(6,5,5)	0.0289	0.0008			(6,5,5)	0.1985	0.0394
		(7,7,7)	0.0185	0.0003			(7,7,7)	0.0312	0.0009
		(7,8,7)	0.0284	0.0008			(7,8,7)	0.2616	0.0684
(2,2,3,3)	**0.563**	(3,3,3)	0.0988	0.0097	**(2,2,3,3)**	**0.69**	(3,3,3)	0.0763	0.0058
		(3,3,5)	0.2378	0.0565			(3,3,5)	0.3211	0.0901
		(3,5,3)	0.0926	0.0085			(3,5,3)	0.2909	0.0846
		(5,5,5)	0.0729	0.0053			(5,5,5)	0.0657	0.0043
		(5,5,6)	0.1456	0.0212			(5,5,6)	0.1871	0.0321
		(6,5,5)	0.0401	0.0016			(6,5,5)	0.3578	0.1280
		(7,7,7)	0.0307	0.0009			(7,7,7)	0.0451	0.0010
		(7,8,7)	0.0387	0.0014			(7,8,7)	0.2759	0.0761
(3,3,3,3)	**0.25**	(3,3,3)	0.1644	0.0270	**(3,3,3,3)**	**0.333**	(3,3,3)	0.1929	0.0372
		(3,3,5)	0.1746	0.0304			(3,3,5)	0.3471	0.1210
		(3,5,3)	0.1326	0.0175			(3,5,3)	0.3110	0.0871
		(5,5,5)	0.1033	0.0106			(5,5,5)	0.1824	0.0332
		(5,5,6)	0.2007	0.0321			(5,5,6)	0.1902	0.0394
		(6,5,5)	0.0481	0.0023			(6,5,5)	0.3672	0.1400
		(7,7,7)	0.0695	0.0048			(7,7,7)	0.1396	0.0195
		(7,8,7)	0.0663	0.0043			(7,8,7)	0.2804	0.0800
$(\theta_1,\theta_2,\theta_3)=(2,0.2,3)$					$(\theta_1,\theta_2,\theta_3)=(2,0.2,2)$				
(q_1,q_2,p_1,p_2)	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n,m,w)**	**AB**	**MSE**	**(q_1,q_2,p_1,p_2)**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n,m,w)**	**AB**	**MSE**
(1,2,3,3)	**0.838**	(3,3,3)	0.0770	0.0059	**(1,2,3,3)**	**0.737**	(3,3,3)	0.0444	0.0019
		(3,3,5)	0.2815	0.0792			(3,3,5)	0.3023	0.0131
		(3,5,3)	0.0968	0.0093			(3,5,3)	0.0188	0.0079
		(5,5,5)	0.0359	0.0012			(5,5,5)	0.0312	0.0011
		(5,5,6)	0.2560	0.0655			(5,5,6)	0.0510	0.0029
		(6,5,5)	0.2732	0.0746			(6,5,5)	0.2870	0.0813
		(7,7,7)	0.0176	0.0009			(7,7,7)	0.0199	0.0007
		(7,8,7)	0.2673	0.0714			(7,8,7)	0.2303	0.0651
(2,1,3,3)	**0.618**	(3,3,3)	0.0828	0.0068	**(2,1,3,3)**	**0.499**	(3,3,3)	0.0505	0.0025
		(3,3,5)	0.4055	0.1644			(3,3,5)	0.3030	0.0923
		(3,5,3)	0.0975	0.0098			(3,5,3)	0.0591	0.0071
		(5,5,5)	0.0712	0.0041			(5,5,5)	0.0432	0.0017
		(5,5,6)	0.2115	0.0701			(5,5,6)	0.0549	0.0041
		(6,5,5)	0.2750	0.0776			(6,5,5)	0.3001	0.0901
		(7,7,7)	0.0165	0.0012			(7,7,7)	0.0398	0.0010
		(7,8,7)	0.2801	0.0732			(7,8,7)	0.2413	0.0677
(2,2,3,3)	**0.615**	(3,3,3)	0.1088	0.0118	**(2,2,3,3)**	**0.495**	(3,3,3)	0.3014	0.0908
		(3,3,5)	0.4611	0.1823			(3,3,5)	0.3050	0.0951
		(3,5,3)	0.1113	0.0112			(3,5,3)	0.0611	0.0092
		(5,5,5)	0.1053	0.0091			(5,5,5)	0.2617	0.0712
		(5,5,6)	0.2981	0.0921			(5,5,6)	0.0560	0.0097
		(6,5,5)	0.3011	0.0812			(6,5,5)	0.3010	0.0923
		(7,7,7)	0.0859	0.0032			(7,7,7)	0.2501	0.0619
		(7,8,7)	0.2855	0.0749			(7,8,7)	0.2610	0.0710
(3,3,3,3)	**0.559**	(3,3,3)	0.2307	0.0532	**(3,3,3,3)**	**0.233**	(3,3,3)	0.3101	0.0968
		(3,3,5)	0.4801	0.1897			(3,3,5)	0.3270	0.1201
		(3,5,3)	0.1210	0.0134			(3,5,3)	0.200	0.0231
		(5,5,5)	0.2169	0.0306			(5,5,5)	0.2415	0.0065
		(5,5,6)	0.4101	0.1023			(5,5,6)	0.0760	0.0975
		(6,5,5)	0.4151	0.1249			(6,5,5)	0.3111	0.0618
		(7,7,7)	0.1279	0.0172			(7,7,7)	0.2211	0.0652
		(7,8,7)	0.3101	0.0802			(7,8,7)	0.4012	0.0135

3. Bayesian Estimator of $\Re_{q_1,q_2,p_1,p_2}$

We will look in this section at the Bayesian estimator of $\Re_{q_1,q_2,p_1,p_2}$ under the assumption that θ_1 , θ_2 , and θ_3 are random variables. It can be constructed that the suggested prior distributions for θ_1 , θ_2 and θ_3 are the gamma distribution with the following pdfs

$$\pi_i\left(\theta_i\right) \propto \theta_i^{a_i-1} e^{-b_i\theta_i},\ i=1,2,3,$$

where a_i, b_i are assumed to be constant. Assuming the independence of parameters, the joint prior distribution of parameters $\eta = (\theta_1, \theta_2, \theta_3)$ is as follows

$$\pi(\eta) \propto \theta_1^{a_1-1}\theta_2^{a_2-1}\theta_3^{a_3-1}e^{-(b_1\theta_1+b_2\theta_2+b_3\theta_3)}.$$

Based on URRSS samples, the joint pdf of $\eta = (\theta_1, \theta_2, \theta_3)$ and the data are

$$\pi^*(\eta \mid \underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}) \propto \theta_1^{n+a_1-1}\theta_2^{m+a_2-1}\theta_3^{w+a_3-1}\lambda^{n+m+w}e^{-(b_1\theta_1+b_2\theta_2+b_3\theta_3)}\prod_{i=1}^{n}\frac{\left[-\theta_1\tau_{i,i}\right]^{i-1}}{(i-1)!}e^{\theta_1\tau_{i,i}+(r_{i,i})^{\lambda}}(r_{i,i})^{\lambda-1}$$

$$\times\prod_{j=1}^{m}\frac{\left[-\theta_2\varphi_{j,j}\right]^{j-1}}{(i-1)!}e^{\theta_2\varphi_{j,j}+(t_{j,j})^{\lambda}}(t_{j,j})^{\lambda-1}\quad\prod_{u=1}^{w}\frac{\left[-\theta_3\delta_{u,u}\right]^{u-1}}{(u-1)!}e^{\theta_3\delta_{u,u}+(s_{u,u})^{\lambda}}(s_{u,u})^{\lambda-1}.$$

As a result, we may formulate the posterior density function of $\eta = (\theta_1, \theta_2, \theta_3)$ as:

$$\pi^*\left(\eta \mid \underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}\right) = \frac{L\left(\underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}} \mid \eta\right)\pi(\eta)}{\int_0^\infty\int_0^\infty\int_0^\infty L\left(\underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}} \mid \eta\right)\pi(\eta)\,d\theta_1 d\theta_2 d\theta_3}.$$

The Bayesian estimator of $\Re_{q_1,q_2,p_1,p_2}$, denoted by $\widehat{\Re}_{q_1,q_2,p_1,p_2}$, is its posterior mean, which results from the SE loss function assumption.

$$\widehat{\Re}_{q_1,q_2,p_1,p_2} = E(\Re_{q_1,q_2,p_1,p_2}) = \int_0^\infty\int_0^\infty\int_0^\infty \Re_{q_1,q_2,p_1,p_2}\pi^*\left(\eta \mid \underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}\right)d\theta_1 d\theta_2 d\theta_3. \qquad (12)$$

Additionally, the Bayesian estimator of $\Re_{q_1,q_2,p_1,p_2}$ for the LINEX loss function indicated by $\breve{\Re}_{q_1,q_2,p_1,p_2}$, is as follows:

$$\breve{\Re}_{q_1,q_2,p_1,p_2} = \frac{-1}{v}\log E(e^{-v\Re_{q_1,q_2,p_1,p_2}}) = \frac{-1}{v}\log\left[\int_0^\infty\int_0^\infty\int_0^\infty e^{-v\Re_{q_1,q_2,p_1,p_2}}\pi^*\left(\eta \mid \underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}\right)d\theta_1 d\theta_2 d\theta_3\right]. \qquad (13)$$

It is difficult to find an explicit formula for (12) and (13) because the posterior density function $\pi^*\left(\eta \mid \underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}\right)$ has a composite structure. In order to obtain Bayesian estimates (**BEs**), we calculate these integrations using the Metropolis-Hastings (M-H) technique using the MCMC algorithm.

4. MCMC Methodology

The MCMC simulation is implemented to investigate the behavior of $\Re_{q_1,q_2,p_1,p_2}$'s MSS. The BEs under different loss functions are produced using gamma priors. Using the ABs and MSEs, the $\Re_{q_1,q_2,p_1,p_2}$'s BE results were evaluated. The various URRSS options are (n, m, w) = (3,3,3), (3,3,5), (3,5,3), (5,5,5), (5,5,6), (6,5,5), (7,7,7) and (7,8,7). The different choices of (q_1, q_2, p_1, p_2) = (2,2,3,3), (1,2,3,3), (2,1,3,3) and (3,3,3,3).

The values of hyper-parameter are: Prior I: (1, 1, 1, 2, 2, 2) and Prior II: (3, 3, 3, 1, 1, 1).

We assume (v = –2, 2), and the results are based on 5,000 replications. The M-H process is a well-liked subgroup of the MCMC technique in Bayesian literature for modeling departures from the posterior density and producing accurate anticipated results. The key challenge with MCMC is obtaining the BEs from of $\Re_{q_1,q_2,p_1,p_2}$ specified loss functions using the M-H technique after simulating samples from the posterior density. It converges to the desired distribution using acceptance/rejection criteria. The M-H algorithm (see Lynch (2007) operates as follows:

a) Set the starting parameter value of $\Re^{0}_{q_1,q_2,p_1,p_2}$ and the sample number N.

b) For i = 2 to N, set $\Re_{q_1,q_2,p_1,p_2} = \Re^{i-1}_{q_1,q_2,p_1,p_2}$.

c) Create u using the uniform (0,1).

d) Choose a candidate parameter $\Re^{*}_{q_1,q_2,p_1,p_2}$ from the proposal density.

e) If $u \le \dfrac{\pi(\theta^{*})\, g(\theta \mid \theta^{*})}{\pi(\theta)\, g(\theta^{*} \mid \theta)}$, then set $\Re^{i}_{q_1,q_2,p_1,p_2} = \Re^{*}_{q_1,q_2,p_1,p_2}$; otherwise, set $\Re^{i}_{q_1,q_2,p_1,p_2} = \Re_{q_1,q_2,p_1,p_2}$.

f) Return to step (b) and perform the aforementioned actions N times using $i{=}i{+}1$.

Using the outputs of the study, which are shown in Tables 2 and 3 and illustrated by Figures 7–12, we come up with the following conclusions:

• The MSEs and ABs of BEs via the SE and LINEX decrease with increasing the record numbers n, m, w for all true values of (q_1, q_2, p_1, p_2) (Tables 2, 3).

• The ABs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ and $\overset{\frown}{\Re}_{q_1,q_2,p_1,p_2}$ via the SE and LINEX loss functions have the smallest values at (q_1, q_2, p_1, p_2) = (1, 2, 3, 3) (Tables 2, 3).

• At true value $\Re_{q_1,q_2,p_1,p_2} = 0.611$, the MSEs of BEs decrease as number of records increases (see Figure 7).

• At true value $\Re_{q_1,q_2,p_1,p_2} = 0.611$, the ABs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ and $\overset{\frown}{\Re}_{q_1,q_2,p_1,p_2}$ decrease when the number of records increases via prior I (see Figure 8).

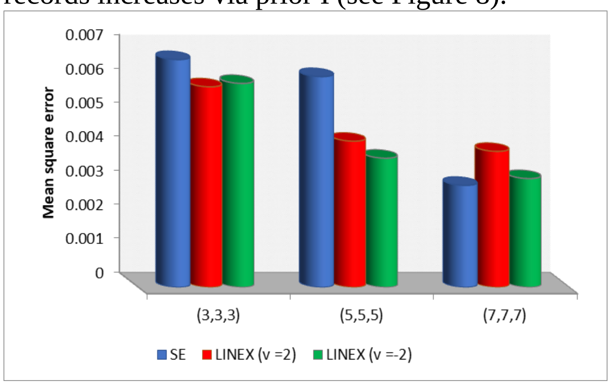

Figure 7: MSEs of BEs at (q_1,q_2,p_1,p_2)= (1,2,3,3) for prior I

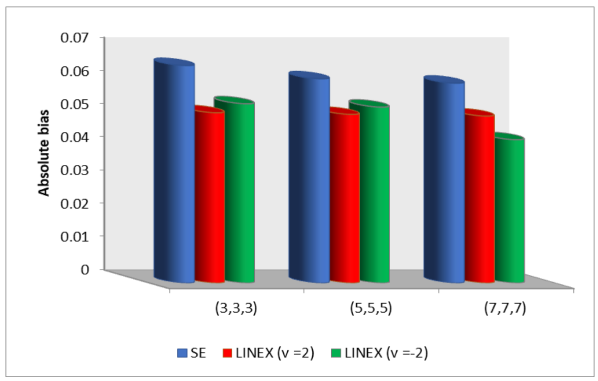

Figure 8: ABs of BEs at (q_1,q_2,p_1,p_2)=(2,1,3,3) for prior I

• At true value $\Re_{q_1,q_2,p_1,p_2} = 0.5639$, the MSEs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ under the LINEX (–2), take the smallest value via prior I (see Figure 9).

• At true value $\Re_{q_1,q_2,p_1,p_2} = 0.5639$, the ABs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ under LINEX (–2), are the least for a distinct number of records, except at (n, m, w) = (3, 3, 5) via prior I (see Figure 10).

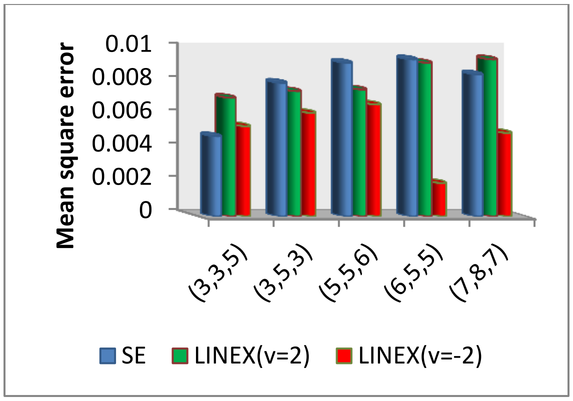

Figure 9: MSEs of BEs at (q_1,q_2,p_1,p_2)= (2, 2, 3, 3) for prior II

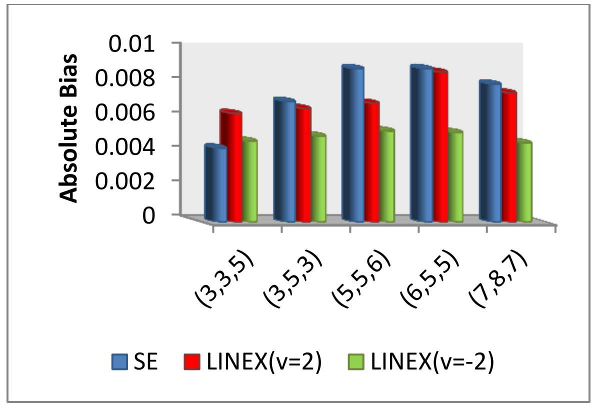

Figure 10: ABs of BEs at (q_1,q_2,p_1,p_2)=(2, 1, 3, 3) for prior II

• The MSEs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ inside the LINEX loss function, as opposed to the similar MSEs of $\overset{\frown}{\Re}_{q_1,q_2,p_1,p_2}$ within the SE loss function, possess the smallest values in nearly all situations (Figure 11).

• At true value $\Re_{q_1,q_2,p_1,p_2} = 0.25$, the ABs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ within v =–2 are the least for the records n=m=w via prior II (Figure 12).

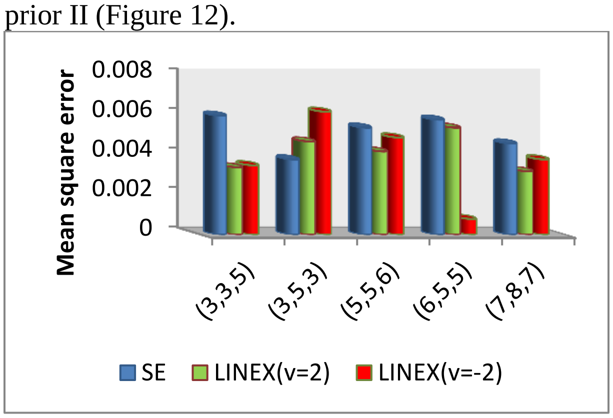

Figure 11: MSEs of BEs at (q_1,q_2,p_1,p_2)= (2,2,3,3) for prior I

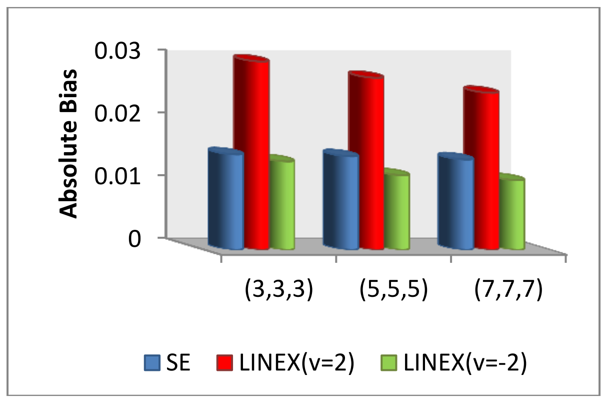

Figure 12: ABs of BEs at (q_1,q_2,p_1,p_2)=(3,3,3,3) for prior II

Table 2: Numerical results of BEs for prior I

	(q_1, q_2, p_1, p_2) =(1, 2, 3, 3)				**(q_1, q_2, p_1, p_2) =(2, 1, 3, 3)**			
Loss function	**Real $\Re_{q_1,q_2,p_1,p_2}$**	(n,m,w)	**AB**	**MSE**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	(n,m,w)	**AB**	**MSE**
SE	**0.725**	(3,3,3)	0.0324	0.0024	**0.611**	(3,3,3)	0.0157	0.0022
LINEX (2)			0.0214	0.0014			0.0147	0.0011
LINEX (–2)			0.0189	0.001			0.0134	0.0008
SE		(3,3,5)	0.0359	0.003		(3,3,5)	0.0143	0.0025
LINEX (2)			0.0245	0.0027			0.0121	0.0031
LINEX (–2)			0.0224	0.0026			0.0111	0.0023
SE		(3,5,3)	0.0131	0.0034		(3,5,3)	0.0121	0.003
LINEX (2)			0.0121	0.0028			0.0164	0.0026
LINEX (–2)			0.0341	0.0021			0.0147	0.0019
SE		(5,5,5)	0.0301	0.0021		(5,5,5)	0.0142	0.0019
LINEX (2)			0.0201	0.0011			0.0137	0.0007
LINEX (–2)			0.0177	0.0008			0.0129	0.0008

SE		(5,5,6)	0.0301	0.0041		(5,5,6)	0.014	0.0045
LINEX (2)			0.0288	0.0043			0.0112	0.0034
LINEX (–2)			0.0211	0.0033			0.0103	0.0032
SE		(6,5,5)	0.0471	0.0047		(6,5,5)	0.0131	0.004
LINEX (2)			0.0376	0.0041			0.0129	0.0038
LINEX (–2)			0.0478	0.0043			0.0111	0.004
SE		(7,7,7)	0.0271	0.0018		(7,7,7)	0.0135	0.0013
LINEX (2)			0.0187	0.0011			0.0132	0.0007
LINEX (–2)			0.0165	0.0007			0.0115	0.0005
SE		(7,8,7)	0.0132	0.0023		(7,8,7)	0.0321	0.0078
LINEX (2)			0.0214	0.0011			0.0137	0.0028
LINEX (–2)			0.0139	0.0025			0.0197	0.0041
(q_1, q_2, p_1, p_2) =(2, 2, 3, 3)					**(q_1, q_2, p_1, p_2) =(3, 3, 3, 3)**			
Loss function	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n,m,w)**	**AB**	**MSE**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n,m,w)**	**AB**	**MSE**
SE	**0.5636**	(3,3,3)	0.0514	0.0067	**0.25**	(3,3,3)	0.0654	0.0071
LINEX (2)			0.0501	0.0059			0.0512	0.0068
LINEX (–2)			0.0439	0.006			0.0539	0.0064
SE		(3,3,5)	0.0475	0.006		(3,3,5)	0.0499	0.0065
LINEX (2)			0.0361	0.0034			0.0487	0.0048
LINEX (–2)			0.035	0.0035			0.0412	0.0061
SE		(3,5,3)	0.0314	0.0038		(3,5,3)	0.0367	0.0044
LINEX (2)			0.0411	0.0047			0.0497	0.0058
LINEX (–2)			0.0377	0.0062			0.0391	0.0055
SE		(5,5,5)	0.0531	0.0062		(5,5,5)	0.0614	0.006
LINEX (2)			0.0477	0.0043			0.0507	0.0063
LINEX (–2)			0.0433	0.0038			0.0529	0.0057
SE		(5,5,6)	0.0399	0.0054		(5,5,6)	0.0415	0.0061
LINEX (2)			0.0378	0.0042			0.0412	0.0052
LINEX (–2)			0.0279	0.0049			0.03	0.0069
SE		(6,5,5)	0.0192	0.0028		(6,5,5)	0.0267	0.0037
LINEX (2)			0.0149	0.0054			0.0139	0.0064
LINEX (–2)			0.0113	0.0008			0.0412	0.0037
SE		(7,7,7)	0.0478	0.003		(7,7,7)	0.0601	0.0051
LINEX (2)			0.0314	0.004			0.0502	0.0049
LINEX (–2)			0.0217	0.0032			0.0431	0.0049
SE		(7,8,7)	0.0285	0.0046		(7,8,7)	0.0357	0.0058
LINEX (2)			0.0312	0.0032			0.0345	0.006
LINEX (–2)			0.0147	0.0068			0.0214	0.0037

Table 3: Numerical results of BEs for prior II

(q_1, q_2, p_1, p_2) =(1, 2, 3, 3)					**(q_1, q_2, p_1, p_2) =(2, 1, 3, 3)**			
Loss function	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n,m,w)**	**AB**	**MSE**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n,m,w)**	**AB**	**MSE**
SE	**0.725**	(3,3,3)	0.0137	0.0039	**0.61**	(3,3,3)	0.0144	0.005
LINEX (2)			0.0277	0.005			0.028	0.005
LINEX (–2)			0.0101	0.0024			0.0112	0.003
SE		(3,3,5)	0.0198	0.0041		(3,3,5)	0.02	0.004
LINEX (2)			0.0231	0.0058			0.024	0.006
LINEX (–2)			0.0181	0.004			0.019	0.005
SE		(3,5,3)	0.0327	0.0067		(3,5,3)	0.033	0.007
LINEX (2)			0.0258	0.0054			0.026	0.007
LINEX (–2)			0.0209	0.0043			0.021	0.005
SE		(5,5,5)	0.013	0.0034		(5,5,5)	0.0142	0.005

LINEX (2)			0.0241	0.0048			0.0245	0.006
LINEX (–2)			0.0097	0.0009			0.0107	0.002
SE		(5,5,6)	0.0408	0.0082		(5,5,6)	0.0411	0.009
LINEX (2)			0.0378	0.0067			0.039	0.007
LINEX (–2)			0.0214	0.005			0.0218	0.005
SE		(6,5,5)	0.0417	0.0086		(6,5,5)	0.042	0.009
LINEX (2)			0.04	0.0081			0.0406	0.009
LINEX (–2)			0.0312	0.0047			0.0319	0.005
SE		(7,7,7)	0.0127	0.0027		(7,7,7)	0.0135	0.003
LINEX (2)			0.0211	0.0041			0.0226	0.004
LINEX (–2)			0.0069	0.0007			0.0098	0.001
SE		(7,8,7)	0.0601	0.0074		(7,8,7)	0.0613	0.008
LINEX (2)			0.0517	0.0068			0.0523	0.008
LINEX (–2)			0.0411	0.0031			0.042	0.005
	(q_1, q_2, p_1, p_2) =(2,2,3,3)				**(q_1, q_2, p_1, p_2) =(3,3,3,3)**			
Loss function	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n, m, w)**	**AB**	**MSE**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	**(n, m, w)**	**AB**	**MSE**
SE	**0.5636**	(3,3,3)	0.0149	0.0053	**0.25**	(3,3,3)	0.0152	0.006
LINEX (2)			0.0291	0.0059			0.03	0.006
LINEX (–2)			0.013	0.0037			0.0141	0.005
SE		(3,3,5)	0.0213	0.0048		(3,3,5)	0.0246	0.005
LINEX (2)			0.0248	0.0071			0.0252	0.007
LINEX (–2)			0.0198	0.0054			0.0207	0.006
SE		(3,5,3)	0.0358	0.008		(3,5,3)	0.037	0.008
LINEX (2)			0.027	0.0075			0.0285	0.008
LINEX (–2)			0.0218	0.0062			0.0224	0.007
SE		(5,5,5)	0.0146	0.005		(5,5,5)	0.0149	0.005
LINEX (2)			0.0266	0.0059			0.0274	0.006
LINEX (–2)			0.0117	0.0023			0.0119	0.003
SE		(5,5,6)	0.0421	0.0092		(5,5,6)	0.0432	0.01
LINEX (2)			0.0397	0.0076			0.0411	0.008
LINEX (–2)			0.0224	0.0067			0.023	0.007
SE		(6,5,5)	0.0432	0.0094		(6,5,5)	0.0444	0.01
LINEX (2)			0.0412	0.0092			0.0439	0.01
LINEX (–2)			0.0321	0.002			0.0338	0.003
SE		(7,7,7)	0.0142	0.0037		(7,7,7)	0.0144	0.004
LINEX (2)			0.0237	0.0048			0.025	0.005
LINEX (–2)			0.0113	0.0016			0.0111	0.002
SE		(7,8,7)	0.0614	0.0085		(7,8,7)	0.0632	0.009
LINEX (2)			0.0536	0.0094			0.0539	0.01
LINEX (–2)			0.0431	0.005			0.0439	0.006

5. Actual Data Implementation

To demonstrate the procedures proposed in the preceding sections. We consider a data set on the time of successive failures of the air conditioning system of each member of a fleet of Boeing 720 jet airplanes. The data mentioned above can be found in Proschan (1963) and are listed in Table 4.

Table 4: Intervals between failures

7907	7908	7909	7910	7911	7912	7913	7914	7915
194	413	90	74	55	23	97	50	359
15	14	10	57	320	261	51	44	9
41	58	60	48	56	87	11	102	12
29	37	186	29	104	7	4	72	270
33	100	61	502	220	120	141	22	603

181	65	49	12	239	14	18	39	3
	9	14	70	47	62	142	3	104
	169	24	21	246	47	68	15	2
	447	56	29	176	225	77	197	438
	184	20	386	182	71	80	188	
	36	79	59	33	246	1	79	
	201	84	27		21	16	88	
	118	44			42	106	46	
		59			20	206	5	
		29			5	82	5	
		118			12	54	36	
		25			120	31	22	
		156			11	216	139	
		310			3	46	210	
		76			14	111	97	
		26			71	39	30	
		44			11	63	23	
		23			14	18	13	
		62			11	191	14	

The data sets are fitted separately with the BShD using the Kolmogorov–Smirnov goodness-of-fit test; based on the data provided in Table 4, we obtain the URRSS for these groups as follows:

Data group I			**Data group II**			**Data group III**		
194			487			438		
413	447		55	320		130	493	
90	186	310	102	209	230	50	254	283

According to the above, the URRSS of groups I, II III are respectively, as follows:
$\left(r_{1,1}, r_{2,2}, r_{3,3}\right) = (194, 447, 310)$, $\left(t_{1,1}, t_{2,2}, t_{3,3}\right) = (487, 320, 230)$, and $\left(s_{1,1}, s_{2,2}, s_{3,3}\right) = (438, 493, 283)$,
then we calculate the estimates of $\Re_{q_1,q_2,p_1,p_2}$ using the ML and Bayesian approaches within SE and LINEX loss functions for (q_1, q_2, p_1, p_2)= (1, 2, 3, 3). Using the above URRSS, the ML estimate and BEs of $\Re_{q_1,q_2,p_1,p_2}$, are calculated in Table 5.

Table 5: $\Re_{q_1,q_2,p_1,p_2}$ estimators based on real data

MLE	BE: SE	BE: LINEX (2)	BE: LINEX (−2)
0.4892	0.5603	0.5711	0.5534

6. Concluding Remarks

In the current work, where both the stress and strength variables are BShD, we investigate the reliability in a MSS system with non-identical component strengths. The reliability of MSS is examined using both estimating techniques. The measurements of the strength and stress distribution samples are shown in URRSS. In order to evaluate the validity of the proposed Bayesian estimates, we apply MCMC techniques. The simulation analysis shows that for four choices of (q_1,q_2, p_1, p_2), the MSEs and ABs decrease with the number of records increase, supporting the ML estimate's consistency characteristic of $\Re_{q_1,q_2,p_1,p_2}$ .Additionally, as the true value of $\Re_{q_1,q_2,p_1,p_2}$ increases, the MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$ decrease. Regarding the MCMC approach, we deduce that the MSEs and ABs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ via LINEX-LF generally hold the lowest values in majority of cases. The usage of real data shows that our model's reliability estimates are extremely close to one, proving its applicability.

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References

Ahmed, E. A. (2014). Bayesian estimation based on progressive Type-II censoring from two-parameter bathtub-shaped lifetime model: an Markov chain Monte Carlo approach. *Journal of Applied Statistics, 41*(4), 752–768.

Arnold, B. C., Balakrishnan, N., and Nagaraja, H. N. (2011). *Records*: John Wiley & Sons..

Arshad, M., Khandelwal, N., and Azhad, Q. J. (2022). Stress–strength reliability estimation in a system with p-type non-identical multicomponents from PRHR family based on records. *Journal of Statistical Computation and Simulation, 92*(13), 2740–2771 10.1080/00949655.2022.2045988.

Azhad, Q. J., Arshad, M., and Khandelwal, N. (2022). Statistical inference of reliability in multicomponent stress strength model for pareto distribution based on upper record values. *International Journal of Modelling and Simulation, 42*(2), 319–334.

Bhattacharyya, G. K., and Johnson, R. A. (1974). Estimation of reliability in a multicomponent stress-strength model. *Journal of the American Statistical Association, 69*(348), 966–970.

Birnbaum, Z. W. (1956). *On a use of the Mann-Whitney statistic.* Paper presented at the Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability. 1.

Birnbaum, Z. W., and McCarty, R. C. (1958). A distribution-free upper confidence bound for Pr(Y<X), based on independent samples of X and Y. *The Annals of Mathematical Statistics*, 558–562.

Çetinkaya, Ç. (2021). Reliability estimation of a stress-strength model with non-identical component strengths under generalized progressive hybrid censoring scheme. *Statistics, 55*(2), 250–275.

Chandler, K. N. (1952). The distribution and frequency of record values. *Journal of the Royal Statistical Society: Series B (Methodological), 14*(2), 220–228.

Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Statistics & Probability Letters, 49*(2), 155–161.

Dey, S., Mazucheli, J., and Anis, M. Z. (2017). Estimation of reliability of multicomponent stress–strength for a Kumaraswamy distribution. *Communications in Statistics-Theory and Methods, 46*(4), 1560–1572.

Hassan, A. S., Al-Omari, A., and Nagy, H. F. (2021). Stress–strength reliability for the generalized inverted exponential distribution using MRSS. *Iranian Journal of Science and Technology, Transactions A: Science*, *45*(2), 641–659.

Hassan, A. S., and Basheikh, H. M. (2012a). Estimation of reliability in multi-component stress-strength model following exponentiated Pareto distribution. *The Egyptian Statistical Journal, Faculty of Graduate Studies for Statistical Research, Cairo University, 56*(2), 82–95.

Hassan, A. S., and Basheikh, H. M. (2012b). Reliability estimation of stress-strength model with non-identical component strengths: the exponentiated Pareto case. *International Journal of Engineering Research and Applications (IJERA), 2*(3), 2774–2781.

Hassan, A. S., Ismail, D. M., and Nagy, H. F. (2022). Reliability Bayesian analysis in multicomponent stress–strength for generalized inverted exponential using upper record data. *IAENG International Journal of Applied Mathematics*, *52*(3), 555–567.

Hassan, A. S., Marwa, A. A., and Nagy, H. F. (2018). Estimation of P (Y<X) using record values from the generalized inverted exponential distribution. *Pakistan Journal of Statistics and Operation Research*, 645–660.

Hassan, A. S., Nagy, H. F., Muhammed, H. Z., and Saad, M. S. (2020). Estimation of multicomponent stress-strength reliability following Weibull distribution based on upper record values. *Journal of Taibah University for science*, *14*(1), 244–253.

Jana, N., and Bera, S. (2022). Estimation of parameters of inverse Weibull distribution and application to multi-component stress-strength model. *Journal of Applied Statistics, 49*(1), 169–194.

Johnson, R. A. (1988). Stress-strength models for reliability. *Handbook of statistics*, *7*, 27–54. https://doi.org/10.1016/S0169-7161(88)07005-1.

Karam, N. S., Yousif, S. M., and Tawfeeq, B. J. (2020). Multicomponent inverse Lomax stress-strength reliability. *Iraqi Journal of Science*, 72–80.

Kayal, T., Tripathi, Y. M., Dey, S., and Wu, S.-J. (2020). On estimating the reliability in a multicomponent stress-strength model based on Chen distribution. *Communications in Statistics-Theory and Methods, 49*(10), 2429–2447.

Kızılaslan, F. (2018). Classical and Bayesian estimation of reliability in a multicomponent stress–strength model based on a general class of inverse exponentiated distributions. *Statistical Papers, 59*(3), 1161–1192.

Kohansal, A., Fernández, A. J., and Pérez-González, C. J. (2021). Multi-component stress–strength parameter estimation of a non-identical-component strengths system under the adaptive hybrid progressive censoring samples. *Statistics*, *55*(4), 925–962.

Kotb, M. S., and Raqab, M. Z. (2021). Estimation of reliability for multi-component stress–strength model based on modified Weibull distribution. *Statistical Papers, 62*(6), 2763–2797.

Kotz, S., and Pensky, M. (2003). *The Stress-Strength Model and its Generalizations: Theory and Applications*: World Scientific..

Lynch, S. M. (2007). *Introduction to Applied Bayesian Statistics and Estimation for Social Scientists* (Vol. 1): Springer..

Pak, A., Jafari, A. A., and Khoolenjani, N. B. (2019). On reliability in a multicomponent stress-strength generalized Rayleigh model based on record values. *Journal of Testing and Evaluation, 48*(6), 4588–4607. 10.1520/JTE20170732.

Pandey, M., Uddin, M. B., and Ferdous, J. (1992). Reliability estimation of an s-out-of-k system with non-identical component strengths: the Weibull case. *Reliability Engineering & System Safety, 36*(2), 109–116.

Paul, R. K., and Uddin, M. B. (1997). Estimation of reliability of stress-strength model with non-identical component strengths. *Microelectronics Reliability, 37*(6), 923–927.

Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics, 5*(3), 375–383.

Rao, G. S., Aslam, M., and Arif, O. H. (2017). Estimation of reliability in multicomponent stress–strength based on two parameter exponentiated Weibull Distribution. *Communications in Statistics-Theory and Methods, 46*(15), 7495–7502.

Rao, G. S., Aslam, M., and Kundu, D. (2015). Burr-XII distribution parametric estimation and estimation of reliability of multicomponent stress-strength. *Communications in Statistics-Theory and Methods, 44*(23), 4953–4961.

Rasethuntsa, T. R., and Nadar, M. (2018). Stress–strength reliability of a non-identical-component-strengths system based on upper record values from the family of Kumaraswamy generalized distributions. *Statistics, 52*(3), 684–716.

Rastogi, M. K., Tripathi, Y. M., and Wu, S.-J. (2012). Estimating the parameters of a bathtub-shaped distribution under progressive type-II censoring. *Journal of Applied Statistics, 39*(11), 2389–2411.

Sadeghpour, A., Salehi, M., and Nezakati, A. (2020). Estimation of the stress–strength reliability using lower record ranked set sampling scheme under the generalized exponential distribution. *Journal of Statistical Computation and Simulation, 90*(1), 51–74.

Salehi, M., and Ahmadi, J. (2014). Record ranked set sampling scheme. *Metron, 72*(3), 351–365.

Sarhan, A. M., Hamilton, D. C., and Smith, B. (2012). Parameter estimation for a two-parameter bathtub-shaped lifetime distribution. *Applied Mathematical Modelling, 36*(11), 5380–5392.

Sharma, V. K. (2018). Bayesian analysis of head and neck cancer data using generalized inverse Lindley stress–strength reliability model. *Communications in Statistics-Theory and Methods, 47*(5), 1155–1180.

Wang, B. X., Geng, Y., and Zhou, J. X. (2018). Inference for the generalized exponential stress-strength model. *Applied Mathematical Modelling, 53*, 267–275.

Wu, S.-J. (2008). Estimation of the two-parameter bathtub-shaped lifetime distribution with progressive censoring. *Journal of Applied Statistics, 35*(10), 1139–1150.

Some Characterizations Based on Generalized Order Statistics from Weibull-Family of Life Distributions

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Abstract:

Characterizations of probability distributions based on recurrence relations for single and product moments of generalized order statistics has attracted the attention of many researchers. We present here, characterizations of Weibull -family of life distributions based on generalized order statistics.

Characterizations of distributions based on recurrence relations for single and product moments of generalized order statistics have been investigated quite extensively in the literature involving ordered random variables. As generalized order statistics (GOS) provide a unifying approach to models of ordered random variables.

Keywords: Generalized order statistics; Order statistics; Records ; Single and product moments; Recurrence relations ; Characterizations.

1. Introduction

The primary objective of this article is to offer characterizations for absolutely continuous distributions, utilizing recurrence relations for both single and product moments of GOS. It is important to note that our study does not seek to extend all characterization results in this context. However, our discoveries and mathematical approaches not only contribute novel characterization results for diverse models of ordered random variables but also hold potential applications in various aspects of GOS.

Bourguignon et al. (2014) introduced a new method of adding a parameter into a family of distributions. The resulting distribution, known as the Weibull generated distribution, includes the original distribution as a special case and gives more flexibility to model various types of data.

Let $G(x,\xi)$ be a continuous baseline distribution with density $g(x,\xi)$ depends on a parameter vector.

The Cumulative density function (cdf) of the Weibull-distribution

$$F(x,\alpha,\beta)=1-e^{-\alpha x^{\beta}};\qquad x,\alpha,\beta>0.$$

The cdf of the Weibull– G family is given by

$$F(x,\alpha,\beta,\xi)=\int_0^{\left(\frac{G(x)}{\bar{G}(x)}\right)}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}dx=1-e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}}.$$

The probability density function (pdf) of the Weibull– G family is given by

$$\bar{F}(x)=e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}};\qquad x,\beta,\alpha>0, \tag{1}$$

where α and β are the scale and shape parameters, respectively. The probability density function (*pdf*) corresponding to $\bar{F}(x)$ are:

$$f(x)=\alpha\beta g(x)\frac{G(x)^{\beta-1}}{\bar{G}(x)^{\beta+1}}e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}};\qquad x,\beta,\alpha>0 \tag{2}$$

where $\bar{G}(x)$ as

$$\bar{G}(x)=\mathrm{e}^{-\lambda(x)};\qquad x\geq 0,\ \theta>0, \tag{3}$$

where $\lambda(x)$ is a non-negative, continuous, monotone increasing, differentiable function of x such that $\lambda(x)\to 0$ as $x\to 0^{+}$ and $\lambda(x)\to\infty$ as $x\to\infty$ (see Mahmoud and Ghazal (2012)).

Substituting from (3) in (1), we get

$$\bar{F}(\mathrm{x})=e^{-\alpha\left(e^{\lambda(x)}-1\right)^{\beta}}\qquad x\geq 0 \tag{4}$$

The pdf corresponding to $\bar{F}(x)$

$$f(x)=\alpha\beta\lambda'(x)e^{\lambda(x)}\left(e^{\lambda(x)}-1\right)^{\beta-1}\mathrm{e}^{-\alpha\left(e^{\lambda(x)}-1\right)^{\beta}}\qquad x\geq 0 \tag{5}$$

The family of distributions in (5), is called Weibull-family of life distributions. Now in view of (4) and (5), we get

$$\bar{F}(x) = \frac{f(x)}{\alpha\beta\lambda'(x)} \sum_{u,v=0}^{\infty} \binom{1-\beta}{u} \frac{(-1)^{u+v} (\beta+u)^{v} \lambda^{v}(x)}{v!} \tag{6}$$

Then distributions can be obtained from (5), as illustrated.

Table (1) examples of *pdf* (5)

$\lambda(x)$	*pdf*	Distribution
λx; $\lambda > 0$ *and* $x \geq 0$	$\alpha\beta\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\beta-1} e^{-\alpha\left(e^{\lambda x}-1\right)^{\beta}}$	Weibull-exponential
$ax+\frac{b}{2}x^2$; a, x *and* $b > 0$	$ab\alpha\beta x e^{ax+\frac{b}{2}x^2}\left(e^{ax+\frac{b}{2}x^2}-1\right)^{\beta-1} e^{-\alpha\left(e^{ax+\frac{b}{2}x^2}-1\right)^{\beta}}$	Weibull-linear failure rate
λx^{θ}; $\lambda, \theta > 0$ *and* $x \geq 0$	$\alpha\beta\lambda\theta x^{\theta-1} e^{\lambda x^{\theta}}\left(e^{\lambda x^{\theta}}-1\right)^{\beta-1} e^{-\alpha\left(e^{\lambda x^{\theta}}-1\right)^{\beta}}$	Weibull-Weibull
λx^2 $\lambda \succ 0$ *and* $x \geq 0$	$2\alpha\beta\lambda x e^{\lambda x^2}\left(e^{\lambda x^2}-1\right)^{\beta-1} e^{-\alpha\left(e^{\lambda x^2}-1\right)^{\beta}}$	Weibull-Rayleigh
$\lambda x^{\theta} e^{\gamma x}$ $\lambda, \theta, \gamma > 0$ *and* $x \geq 0$	$\alpha\beta 2\lambda\theta\gamma e^{2\gamma x} x^{2\theta-1} e^{\lambda x^{\theta} e^{\gamma x}}\left(e^{\lambda x^{\theta} e^{\gamma x}}-1\right)^{\beta-1} e^{-\alpha\left(e^{\lambda x^{\theta} e^{\gamma x}}-1\right)^{\beta}}$	Modified - Weibull
$\frac{a}{c}\left(e^{cx}-1\right)$ $c \geq 0$ *and* $x, a > 0$	$\alpha\beta a e^{cx} e^{\frac{a}{c}\left(e^{cx}-1\right)}\left(e^{\frac{a}{c}\left(e^{cx}-1\right)}-1\right)^{\beta-1} e^{-\alpha\left(e^{\frac{a}{c}\left(e^{cx}-1\right)}-1\right)^{\beta}}$	Weibull - Gompertz
$a\ln\left(1+x^b\right)$; $a, b, x > 0$	$\frac{\alpha\beta ab x^{b-1}}{\left(1+x^b\right)} e^{a\ln\left(1+x^b\right)}\left(e^{a\ln\left(1+x^b\right)}-1\right)^{\beta-1} e^{-\alpha\left(e^{a\ln\left(1+x^b\right)}-1\right)^{\beta}}$	Weibull – Burr type XII
$a\ln(1+bx)$; $a, b, x > 0$	$\frac{ab\alpha\beta}{(1+bx)} e^{a\ln(1+bx)}\left(e^{a\ln(1+bx)}-1\right)^{\beta-1} e^{-\alpha\left(e^{a\ln(1+bx)}-1\right)^{\beta}}$	Weibull – Lomax
$a\ln(1+x)$ $a, x > 0$	$\frac{a\alpha\beta}{(1+x)} e^{a\ln(1+x)}\left(e^{a\ln(1+x)}-1\right)^{\beta-1} e^{-\alpha\left(e^{a\ln(1+x)}-1\right)^{\beta}}$	Weibull – Pareto

Kamps (1995) introduced the concept of generalized order statistics (GOS), encompassing a range of order models for random variables. For simplicity, let F consistently represent an absolutely continuous distribution function with a corresponding density function *f*.

The random variables $X(1,n,\tilde{m},k),\ldots, X(n,n,\tilde{m},k)$ are called generalized order statistics based on F, if their joint *pdf* of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i} f(x_i)\right)\left[\bar{F}(x_n)\right]^{k-1} f(x_n),$$

for $F^{-1}(0) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1)$. with parameters $n \in N,\ k \ge 2,\ k > 0,$

$\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in R^{n-1}, M_r = \sum_{i=r}^{n-1} m_i,$ such that $\gamma_r = k + n - r + M_r > 0,$ for all

$r \in \{1, 2, \ldots, n-1\}$. For $\gamma_i \ne \gamma_j$ for all $i, j \in (1, 2, \ldots, n-1)$ the pdf of $X(r,n,\tilde{m},k)$ is given by Cramer and Kamps (2000) in the following way

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r)\left[\bar{F}(x)\right]^{\gamma_i - 1}. \tag{7}$$

The joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$ is given as

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\left(\sum_{i=1}^{r} a_i^{(r)}\left[\bar{F}(x)\right]^{\gamma_i}\right)\frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)},$$

$$x < y \tag{8}$$

Where

$$a_i(r) = \prod_{\substack{j=1\\ j\ne i}}^{r}\frac{1}{\gamma_j - \gamma_i}, \qquad 1 \le i \le n$$

and $a_i^{(r)}(s) = \prod_{\substack{j=r+1\\ j\ne i}}^{s}\frac{1}{\gamma_j - \gamma_i}, \qquad r+1 \le i \le s \le n.$

It may be noted that for $m_1 = m_2 = \ldots = m_{n-1} = m \ne -1$

$$a_i(r) = \frac{(-1)^{r-i}\binom{r-1}{r-i}}{(m+1)^{r-1}(r-1)!}, \tag{9}$$

and $$a_i^{(r)}(s) = \frac{(-1)^{s-i}\binom{s-r-1}{s-i}}{(m+1)^{s-r-1}(s-r-1)!}, \tag{10}$$

Therefore *pdf* of $X(r,n,\tilde{m},k)$ given in (7) reduces to

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{\Gamma(r)}\left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left[F(x)\right], \quad x \in \chi \tag{11}$$

and joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$ is given in (8) reduced to

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\Gamma(s-r-1)!}\left[\bar{F}(x)\right]^{m} f(x) g_m^{r-1}\left[F(x)\right]$$

$$\left\{h_m\left[F(y)\right] - h_m\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s - 1} f(y), \quad x < y, \tag{12}$$

Where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} \dfrac{-(1-x)^{m+1}}{m+1}, & m = -1 \\ \log\left(\dfrac{1}{1-x}\right), & m \neq -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,\ 1).$$

We shall also take $X(0,n,m,k) = 0$. If m = 0, k = 1, then $X(r,n,m,k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when m = -1, then $X(r,n,m,k)$ reduces to the k^{th} record value (Pawlas and Szynal [2001]).

Numerous authors have incorporated generalized order statistics (GOS) into their research, including Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000, 2016), Pawlas and Szynal (2001), Ahmed (2007), Ahmed and Fawzy (2007), Khan et al. (2007), AL-Hussaini et al. (2005), Kumar (2011), Abdul-Moniem (2019), Nagwa (2020), and Alimohammadi (2022). These works explore various characterizations through conditional events of generalized order statistics.

In this paper, we present a characterization of the Weibull-Family distribution, employing a recurrence relation for both single and product moments of generalized order statistics (GOS).

2. Characterization Based on recurrence relation for single moments of gos

Theorem 2.1 Let X be a non-negative random variable having an absolutely continuous distribution function F(x) with F(0) = 0 and 0 < F(x) < 1 for all x > 0, then

$$E\left[X^{j}\left(r,n,m,k\right)\right]-E\left[X^{j}\left(r-1,n,m,k\right)\right]=\frac{j}{\alpha\beta\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\left(\beta+u\right)^{v}}{v\,!}E\left[\frac{X^{j-1}\left(r,n,m,k\right)\lambda^{v}\left(X\left(r,n,m,k\right)\right)}{\lambda'\left(X\left(r,n,\tilde{m},k\right)\right)}\right] \tag{13}$$

if and only if $\bar{F}\left(y\right)=e^{-\alpha\left(e^{\lambda\left(y\right)}-1\right)^{\beta}}$

Proof 2.1.

(i) Necessity

Building upon Lemma 2.3 (Athar and Islam [2004]) that

$$E\left[\xi\left\{X\left(r,n,\tilde{m},k\right)\right\}\right]-E\left[\xi\left\{X\left(r-1,n,\tilde{m},k\right)\right\}\right]=C_{r-2}\int_{\theta}^{\beta}\xi'\left(x\right)\sum_{i=1}^{r}a_{i}\left(x\right)\left[\bar{F}\left(x\right)\right]^{\gamma_{r}}dx$$

If we let $\xi\left(x\right)=x^{j}$, then

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{j}\left(r-1,n,\tilde{m},k\right)\right]=jC_{r-2}\int_{\theta}^{\beta}x^{j-1}\sum_{i=1}^{r}a_{i}\left(x\right)\left[\bar{F}\left(x\right)\right]^{\gamma_{r}}dx \tag{14}$$

On using (6) in (14), we get

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{j}\left(r-1,n,\tilde{m},k\right)\right]=\frac{jC_{r-1}}{\alpha\beta\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\left(\beta+u\right)^{v}}{v\,!}\int_{0}^{\infty}\frac{x^{j-1}\lambda^{v}\left(x\right)}{\lambda'\left(x\right)}\sum_{i=1}^{r}a_{i}\left(x\right)\left[\bar{F}\left(x\right)\right]^{\gamma_{r}-1}f\left(x\right)dx$$

The simplification process results in Eq (13).

(ii) Sufficiency

On the other hand if the recurrence relation in Eq (13) is satisfied, then by using Eq (11), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx-\frac{C_{r-2}}{(r-2)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_{r-1}-1}f(x)g_m^{r-2}\left[F(x)\right]dx$$

$$=\frac{jC_{r-1}}{\alpha\beta(r-1)!\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\int_0^\infty\frac{x^{j-1}\lambda^v(x)}{\lambda'(x)}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

By integrating the first term on the left-hand side using parts, we obtain:

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx$$

$$=\frac{jC_{r-1}}{\alpha\beta(r-1)!\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\int_0^\infty\frac{x^{j-1}\lambda^v(x)}{\lambda'(x)}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

This implies that

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]$$

$$\left\{\bar{F}(x)-\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\frac{\lambda^v(x)}{\alpha\beta\lambda'(x)}f(x)\right\}dx=0 \tag{15}$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [1984]) to (15), we get

$$\bar{F}(x)-\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\frac{\lambda^v(x)}{\alpha\beta\lambda'(x)}f(x)=0,$$

hence,

$$\bar{F}(x)=\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}\frac{\lambda^v(x)}{\alpha\beta\lambda'(x)}f(x)$$

$$=\sum_{u=0}^{\infty}\binom{1-\beta}{u}(-1)^u\frac{e^{-(\beta+u)\lambda(x)}}{\alpha\beta\lambda'(x)}f(x)$$

This implies that

$$\bar{F}(x)=\frac{e^{-\lambda(x)}f(x)}{\alpha\beta\lambda'(x)}\left(e^{\lambda(x)}-1\right)^{1-\beta}$$

Integrating both side from 0 to y, we get

$$\int_0^y\frac{f(x)}{\bar{F}(x)}dx=\alpha\beta\int_0^y\lambda'(x)e^{\lambda(x)}\left(e^{\lambda(x)}-1\right)^{\beta-1}dx$$

This implies that

$$-\ln\left[\bar{F}(y)\right]=\alpha\left(e^{\lambda(y)}-1\right)^{\beta}$$

which prove that

$$\bar{F}(y)=e^{-\alpha\left(e^{\lambda(y)}-1\right)^{\beta}}\quad ;\qquad y\geq 0$$

Corollary 2.2 for $m_1=m_2=...=m_{n-1}=m\neq -1$,the recurrence relations for single moment of gos for W−family of life distributions is given as

$$E\left[X^{j}(r,n,m,k)\right]-E\left[X^{j}(r-1,n,m,k)\right]=\frac{j}{\alpha\beta\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\,(\beta+u)^{v}}{v\,!}$$
$$E\left[\frac{X^{j-1}(r,n,m,k)\lambda^{v}\left(X\,(r,n,m,k)\right)}{\lambda'\left(X\,(r,n,\tilde{m},k)\right)}\right]$$

(16)

Proof. This can easy be deduced from (13) in view of the relation (9).

Note that: By considering and analyzing generalized order statistics (GOS), we can derive certain characterizations for the Weibull-Weibull distribution. $\lambda(x)=x^{\theta}$ in (3), established by A-Rahman et. al. (2023).

Remark 2.1 Putting *m = 0, k = 1* in Theorem 2.1., we obtain recurrence relations for single moments of order statistics as

$$E\left[X_{r:n}^{j}\right]-E\left[X_{r-1:n}^{j}\right]=\frac{j}{\alpha\beta(n-r+1)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\,(\beta+u)^{v}}{v\,!}E\left[\frac{X_{r:n}^{j-1}\lambda^{v}\left(X_{r:n}\right)}{\lambda'\left(X_{r:n}\right)}\right]$$

(17)

Remark 2.2 Setting *m = −1, k = 1* in Theorem 2.1.,we obtain the recurrence relations of upper record values as

$$E\left[X_{(r:n,-1,1)}^{j}\right]-E\left[X_{(r-1:n,-1,1)}^{j}\right]=\frac{j}{\alpha\beta(n-r+1)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\,(\beta+u)^{v}}{v\,!}$$
$$E\left[\frac{X_{(r:n,-1,1)}^{j-1}\lambda^{v}\left(X_{(r:n,-1,1)}\right)}{\lambda'\left(X_{(r:n,-1,1)}\right)}\right]$$

(18)

3. Characterization based on recurrence relation for product

moments of *gos*

Theorem 3.1 Let X be a non-negative random variable having an absolutely continuous distribution function F (x) with F (0) = 0 and 0 < F (x) < 1 for all $xy > 0$, then

$$E\left[X^i(r,n,m,k).X^j(s,n,m,k)\right]-E\left[X^i(r,n,m,k).X^j(s-1,n,m,k)\right]$$
$$=\frac{j}{\alpha\beta\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}E\left[\frac{X^{j-1}(r,n,m,k)\lambda^v\left(X(r,n,m,k)\right)}{\lambda'\left(X(r,n,\tilde{m},k)\right)}\right] \tag{19}$$

if and only if $\bar{F}(y)=e^{-\alpha\left(e^{\lambda(y)}-1\right)^{\beta}}$

Proof

(i) Necessity

Building upon Lemma 3.2 (Athar and Islam [7]) that

$$E\left[\xi\{X(r,n,m,k).X(s,n,m,k)\}\right]-E\left[\xi\{X(r,n,m,k).X(s-1,n,m,k)\}\right]=$$
$$\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi(x,y)\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$
$$\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s}dydx$$

where $\xi(x,y)=\xi_1(x)\xi_2(y)$

If let $\xi(x,y)=x^i y^j$, then

$$E\left[X^i(r,n,m,k).X^j(s,n,m,k)\right]-E\left[X^i(r,n,m,k).X^j(s-1,n,m,k)\right]=$$
$$\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\theta}^{\beta}\int_{x}^{\beta}x^i y^{j-1}\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$
$$\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s}dydx$$

On employing equation (6), it obtain:

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]=$$

$$\frac{C_{s-2}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^{v}\lambda^{v}}{\alpha\beta(r-1)!(s-r-1)!v!}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}\frac{y^{j-1}\lambda^{v}(y)}{\lambda'(y)}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]$$

$$\left[h_{m}(F(y))-h_{m}(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}-1}f(y)dydx$$

This, following simplification, results in equation (19).

(ii) Sufficiency

If the recurrence relation in equation (19) is satisfied, then by using equation (12), we have

$$\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-1}$$

$$\left[\bar{F}(y)\right]^{\gamma_{s}-1}f(y)dydx-\frac{C_{s-2}}{(r-1)!(s-r-2)!}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]$$

$$\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-2}\left[\bar{F}(y)\right]^{\gamma_{s-1}-1}f(y)dydx=\frac{jC_{s-1}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^{v}\lambda^{v}}{\alpha\beta\gamma_{s}\lambda(r-1)!(s-r-1)!v!}$$

$$\int_{0}^{\infty}\int_{x}^{\infty}x^{i}\frac{y^{j-1}\lambda^{v}(y)}{\lambda'(y)}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}-1}f(y)dydx$$

By integrating the first term on the left-hand side using parts, it obtain:

$$\frac{jC_{s-1}}{\gamma_{s}(r-1)!(s-r-1)!}\int_{0}^{\theta}\int_{x}^{\theta}x^{i}y^{j-1}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-1}$$

$$\left[\bar{F}(y)\right]^{\gamma_{s}}dydx=\frac{jC_{s-1}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^{v}\lambda^{v}}{\alpha\beta\gamma_{s}\lambda(r-1)!(s-r-1)!v!}\int_{0}^{\theta}\int_{x}^{\theta}x^{i}\frac{y^{j-1}\lambda^{v}(y)}{\lambda'(y)}\left[\bar{F}(x)\right]^{m}$$

$$f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-1}f(y)dydx$$

This implies that

$$\frac{jC_{s-1}}{\gamma_{s}(r-1)!(s-r-1)!}\int_{0}^{\theta}\int_{x}^{\theta}x^{i}y^{j-1}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-1}$$

$$\left[\bar{F}(y)\right]^{\gamma_{s}-1}\left\{\bar{F}(y)-\frac{1}{\alpha\theta\beta v!}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^{v}\lambda^{v}f(y)\right\}dydx=0$$

(20)

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [1984]) to (20), we get

$$\bar{F}(y)-\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}\frac{\lambda^{v}(y)}{\alpha\beta\lambda'(y)}f(y)=0,$$

hence,

$$\bar{F}(y)=\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}\frac{\lambda^{v}(y)}{\alpha\beta\lambda'(y)}f(y)$$

$$=\sum_{u=0}^{\infty}\binom{1-\beta}{u}(-1)^{u}\frac{e^{-(\beta+u)\lambda(y)}}{\alpha\beta\lambda'(y)}f(y)$$

This implies that

$$\bar{\mathrm{F}}(y)=\frac{e^{-\lambda(y)}f(y)}{\alpha\beta\lambda'(y)}\left(e^{\lambda(y)}-1\right)^{1-\beta}$$

Integrating both side from 0 to y, we get

$$\int_0^y\frac{f(x)}{\bar{F}(x)}dx=\alpha\beta\int_0^y\lambda'(x)e^{\lambda(x)}\left(e^{\lambda(x)}-1\right)^{\beta-1}dx$$

This implies that

$$-\ln\left[\bar{F}(y)\right]=\alpha\left(e^{\lambda(y)}-1\right)^{\beta}$$

which prove that

$$\bar{F}(y)=e^{-\alpha\left(e^{\lambda(y)}-1\right)^{\beta}}\quad;\qquad y\geq 0$$

Remark 3.1 Putting $m=0,\ k=1$ in (19), it obtain recurrence relations for product moments of order statistics as

$$E\left[X_{r,s:n}^{i,j}\right]-E\left[X_{r,s-1:n}^{i,j}\right]==\frac{j}{\alpha\beta(n-r+1)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}}{v!}E\left[\frac{X_{r,s:n}^{i,j}\lambda^{v}\left(X_{r,s:n}^{i,j}\right)}{\lambda'\left(X_{r,s:n}^{i,j}\right)}\right]$$

(21)

Remark 3.2 Setting $m=-1$ in (19), it obtain the recurrence relations for product moments of k^{th} record values as

$$E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^j\right]-E\left[\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j\right]$$

$$=\frac{j}{k\alpha\beta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}E\left[\frac{\left(X_r^{(k)}\right)^i\lambda^v\left(X_{s-1}^{(k)}\right)^j}{\lambda'\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j}\right] \tag{22}$$

4. Conclusion

In this paper we have studied the characterizations of a family called Weibull-Family of Life distributions based on recurrence relations for single and product moments of generalized order

References

[١] **Abdul-Moniem, I. B. (2019).** Recurrence relations for moments of generalized order statistics from Marshall–Olkin extended Kumaraswamy distribution and its characterization. American Journal of Applied Mathematics and Statistics, 2(5), 324-329

[2] **Ahmad, A. A. (2007).** Recurrence relations for single and product moments of generalized order statistics from doubly truncated Burr Type XII distribution. Journal of The Egyptian Mathematical Society, 15(1), 117-128.

[3] **Ahmad, A. A and Fawzy, A. M (2003)** Recurrence relations for single moments of generalized order statistics from doubly truncated distribution, Journal of Statistical planning and Inference. 117(2), 241-249.

[4] **Ahsanullah, M. (2000).** Generalized order statistics from exponential distribution. Journal of Statistical planning and Inference, 85(2), 85-91.

[5] **Ahsanullah, M and Hamedani, G.G. (2016).** Characterizations of Pareto, Weibull and power function distribution based on generalized order statistics. Journal of the Chungcheong Mathematical Society, Vol. 29, 385-396

[6] **Al-Hussaini, E. K., Abd EL-Baset, A. A., and Al-Kashif, M. A. (2005).** Recurrence relations for moment and conditional moment generating functions of generalized order statistics. Metrika, 61(2), 199-220.

[7] **Athar, H. and Islam, H. (2004).** Recurrence relations for single and product moments of generalized order statistics from a general class of distribution, Metron, LXII (3), 327-337.

[8] **Athar, H., Nayabuddin, S. K. K., and KHAWAJA, S. (2012).** Relations for moments of generalized order statistics from Marshall-Olkin extended Weibull distribution and its characterization. In ProbStat Forum, 5, 127-132.

[9] **Alimohammadi, M., Balakrishnan, N. & Cramer, E. (2022)** Some characterizations by conditional events of generalized order statistics. Ricerche di Mathematica, https://doi.org/10.1007/s11587-022-00741-1

[10] **Bourguignon, M., Silva, R.B., and Cordeiro, G.M.,(2014)**.The Weibull-G family of probability distributions, Journal of Data Science 12, 53-68

[11] **Cramer, E., and Kamps, U. (2000).** Relations for expectations of functions of generalized order statistics. Journal of Statistical Planning and Inference, 89(1-2), 79-89

[12] **Hwang, J.S., and Lin, G.D. (1984).** On a generalized moment problem II, Proceedings of the American Mathematical Society. 91, 577-580.

[13] **Kamps, U. (1995).** A concept of generalized order statistics. Journal of Statistical Planning and Inference, 48(1), 1-23.

[14] **Kamps, U. and Gather, U. (1997).** Characteristic properties of generalized order statistics from exponential distributions. Applications Mathematicae, 24, 383-391.

[15] **Keseling, C. (1999).** Conditional distributions of generalized order statistics and some characterizations. Metrika, 49, 27-40.

[16] **Khan, R. U., Anwar, Z. and Athar, H. (2007).** Recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution. Aligarh Journal of Statistics, 27, 69-79.

[17] **Kumar, D. (2011).** Generalized order statistics from Kumaraswamy distribution and its characterization. Tamsui Oxford Journal of Mathematical Sciences, 27, 463-476.

[18] **Mahmoud, M.A., and Ghazal, M.G., (2012)**. Characterization of exponentiated family of distributions based on recurrence relations for generalized order statistics, Journal of Mathematics and Computer Science, 2(6), 1894-1908.

[19] **Marshall, A.W., and Olkin, I., (1997)**. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, Biometrika 84(3), 641– 652.

[20] **Nagwa, M. M. (2020**). Characterization on modified Weibull distribution, Journal of Statistics Applications & Probability, 2, 333-345

[21] Pawlas, P. and Szynal, D. (2001). Recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto, and Burr distributions. Communications in Statistics: Theory and Methods, 30(4), 739-746.